

On triple intersections of three families of unit circles^{*†}

Orit E. Raz[‡]

Micha Sharir[§]

József Solymosi[¶]

July 25, 2014

Abstract

Let p_1, p_2, p_3 be three distinct points in the plane, and, for $i = 1, 2, 3$, let \mathcal{C}_i be a family of n unit circles that pass through p_i . We address a conjecture made by Székely, and show that the number of points incident to a circle of each family is $O(n^{11/6})$, improving an earlier bound for this problem due to Elekes, Simonovits, and Szabó [4]. The problem is a special instance of a more general problem studied by Elekes and Szabó [5] (and by Elekes and Rónyai [3]).

Keywords. Combinatorial geometry, incidences.

1 Introduction

In this paper we re-examine the following problem. Let p_1, p_2, p_3 be three distinct points in the plane, and, for $i = 1, 2, 3$, let \mathcal{C}_i be a family of n unit circles that pass through p_i . The goal is to obtain an upper bound on the number of *triple points*, which are points that are incident to a circle of each family. See Figures 1 and 2(a) for an illustration. Recently, Elekes et al. [4] have shown that the number of such points is $O(n^{2-\eta})$, for some constant parameter $\eta > 0$ (that they did not make concrete); by this they settled a conjecture of Székely (see [2, Conjecture 3.41]), stipulating that this number should be $o(n^2)$.

Using a different technique, which appears to be simpler than the one in [4], we improve the earlier bound, showing that the number of triple points is $O(n^{11/6})$.

The specific problem studied in this paper can be viewed as a special instance of a more general setup, which has been studied by Elekes and Rónyai [3] and by Elekes and Szabó [5] (see also [2]). From a high-level point of view the setup is as follows. We have

^{*}Work on this paper by Orit E. Raz and Micha Sharir was supported by Grant 892/13 from the Israel Science Foundation. Work by Micha Sharir was also supported by Grant 2012/229 from the U.S.–Israel Binational Science Foundation, by the Israeli Centers of Research Excellence (I-CORE) program (Center No. 4/11), and by the Hermann Minkowski-MINERVA Center for Geometry at Tel Aviv University. Work by József Solymosi was supported by NSERC, ERC-AdG 321104, and OTKA NK 104183 grants.

[†]A preliminary version of this paper appeared in *Proc. 30th Annu. ACM Sympos. Comput. Geom.*, 2014, pp. 198–205.

[‡]School of Computer Science, Tel Aviv University, Tel Aviv 69978, Israel. oritraz@post.tau.ac.il

[§]School of Computer Science, Tel Aviv University, Tel Aviv 69978, Israel. michas@post.tau.ac.il

[¶]Department of Mathematics, University of British Columbia, Vancouver, BC, V6T 1Z4, Canada. solymosi@math.ubc.ca

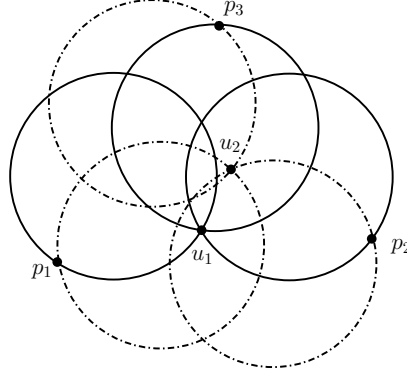


Figure 1: Two triple points u_1 and u_2 .

three sets A, B, C , each of n real numbers, and we have a trivariate real polynomial F of some constant degree d . Let $Z(F)$ denote the subset of $A \times B \times C$ where F vanishes. The goal is to show that, unless F and A, B, C have some very special structure, $|Z(F)|$ is subquadratic. (For a simple example where $|Z(F)|$ is quadratic in n , consider the case where $F(x, y, z) = x + y - z$, and where $A = B = C = \{1, 2, \dots, n\}$.)

Positive and significant results for this general problem have been obtained by Elekes and Rónyai [3] and by Elekes and Szabó [5], who showed that, unless F has a very restricted form, $|Z(F)|$ is indeed subquadratic in n . For example, in the case where F is of the form $z - f(x, y)$, if $|Z(F)|$ is quadratic in n , then f must be of one of the forms $p(q(x) + r(y))$ or $p(q(x) \cdot r(y))$, for suitable univariate polynomials p, q, r (see [3] and [2]). We have recently studied in [9] this specific problem (where $F(x, y, z) = z - f(x, y)$) and obtained improved bounds for $|Z(F)|$, when f does not have these special forms; see below. Related representations, somewhat more complicated to state, for a polynomial F with $|Z(F)| = \Theta(n^2)$, have also been obtained for the general case (see [5] and [2]).

The high-level approach used in this paper is similar to those used in several recent works that study problems in combinatorial geometry that are special instances of this general framework (see Sharir, Sheffer, and Solymosi [12] and Sharir and Solymosi [11]). However, the actual implementations of this approach in our paper, as well as in the other works just mentioned, are very problem-specific and exploit the special geometric structure of the relevant problem.

We will later detail the connection of our problem to the setup in [3, 5]. Roughly speaking, for each \mathcal{C}_i , its circles have one degree of freedom, and we parameterize them by a suitable single real parameter. Then, with a proper choice of these parameters, the condition that three circles, one from each family, have a common point can be expressed by an equation of the form $F(x, y, z) = 0$, where F is a real trivariate polynomial, and x, y, z are the parameters representing the three relevant circles.

In both cases, the specific problem studied in this paper (and the specific problems studied in [11, 12]), and the general one in [3, 5], the approach is to double count the number Q of quadruples (a, p, b, q) , such that, in our specific context, a, b represent two circles in \mathcal{C}_1 , p, q represent two circles in \mathcal{C}_2 , and there exists z , representing a circle through p_3 , such that $F(a, p, z) = 0$ and $F(b, q, z) = 0$. (In the general case, the quadruples to be considered are (a, p, b, q) such that $a, b \in A$, $p, q \in B$, and there exists $c \in C$ such that

$F(a, p, z) = F(b, q, z) = 0$.) A lower bound for Q , in terms of $|Z(f)|$, is easy to obtain (see below for details), and an upper bound is obtained by regarding each such quadruple (a, p, b, q) as an *incidence* between the point (p, q) , in a suitable parametric plane, and a curve $\gamma_{a,b}$ which is the locus of all points (p, q) that satisfy with a, b the above conditions. The comparison between the lower and upper bounds yields the asserted upper bound on $|Z(F)|$.

The main issue that arises in bounding the number of incidences is the possibility that many curves $\gamma_{a,b}$ overlap each other, in which case the standard techniques for analyzing point-curve incidences fail. A major part of the analysis in this paper is to show that the amount of overlap is bounded. In this case the standard incidence techniques do apply, and yield a sharp upper bound that leads to the aforementioned bound on $|Z(F)|$; see below for details.

In the general problem, the goal is to show that when there is a larger amount of overlap between the curves, the polynomial F must have a special form, as the ones mentioned above and established in [3], and to establish a subquadratic upper bound on $|Z(F)|$ when this is not the case. As mentioned, this indeed has recently been shown in a companion paper [9] for the special case where $F(x, y, z) = z - f(x, y)$, for any constant-degree bivariate polynomial f , with the same subquadratic bound $O(n^{11/6})$ as the one established in this paper. This argument has not yet been worked out for the case of a general trivariate polynomial F , except for the general (and weaker¹) analysis in [5]. In our problem, though, this part is not needed, and the argument that the overlap is bounded is an ad-hoc argument that exploits the geometric and algebraic structure of the problem.

2 Unit circles spanned by points on three unit circles

We begin by observing the following equivalent and, in our opinion, more convenient formulation of the problem. Let C_1, C_2, C_3 be three unit circles in \mathbb{R}^2 , and, for each $i = 1, 2, 3$, let S_i be a set of n points lying on C_i . The goal is to obtain a subquadratic upper bound on the number of unit circles, spanned by triples of points in $S_1 \times S_2 \times S_3$. (The equivalence between this formulation and the one in [4], as stated in the introduction, is indeed trivial: For each i , S_i is the set of centers of the circles of C_i , and the centers of the resulting “trichromatic” unit circles in the new formulation are the triple points in the previous one.) See Figures 2(a) and 2(b) for an illustration of this connection between the two setups. In what follows we use the new formulation.

We note that the condition that three points $p, q, r \in \mathbb{R}^2$ span a unit circle can be expressed as a polynomial equation in their coordinates. That is, there exists a 6-variate real polynomial F of degree 6, such that $F(p_1, p_2, q_1, q_2, r_1, r_2) = 0$ when $p = (p_1, p_2), q = (q_1, q_2), r = (r_1, r_2)$ span a unit circle. We delegate the explicit construction of F , which we believe might be of some interest, to Appendix A, and remark that our analysis will not make use of the explicit form of F .

For each $i = 1, 2, 3$, each point $p \in S_i$ can be parameterized by (an appropriate algebraic representation of) the orientation $v_p \in \mathbb{S}^1$ of p with respect to the center c_i of C_i (note that the centers c_i are the points p_i in the original formulation); denote the set of these n

¹For example, in the subquadratic bound obtained in [5], the exponent depends on the degree of F .

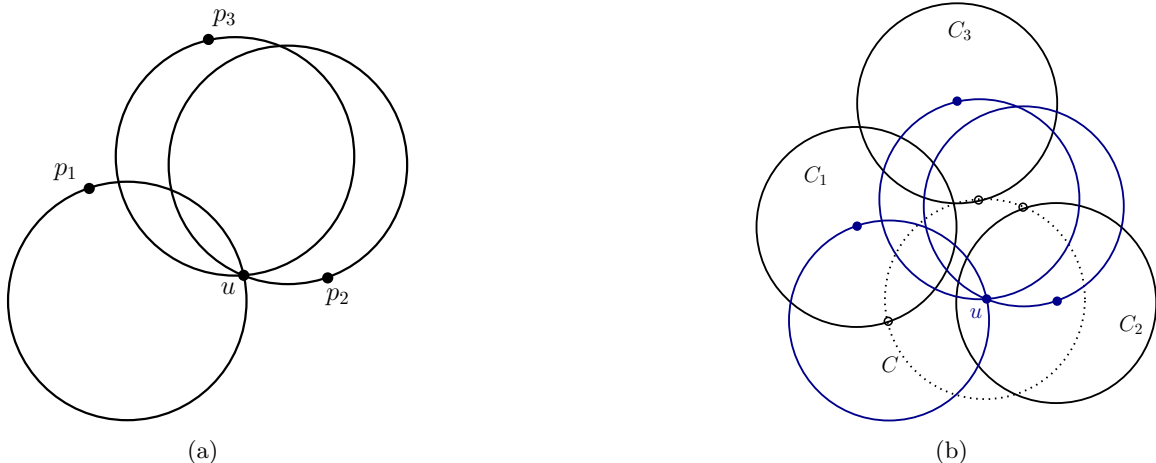


Figure 2: (a) A concurrent triple of unit circles and the corresponding triple point u . (b) The triple point u is mapped in the new setup to the unit circle C centered at u . The circles C_1, C_2, C_3 are centered at p_1, p_2, p_3 , respectively, and the hollow points on C are the centers of the three original circles.

orientations as Θ_i . In what follows we will interchangeably use both notations, referring to a point $p \in S_i$, for $i = 1, 2, 3$, either by its corresponding parameter $v_p \in \Theta_i$, when we want to stress the algebraic nature of the problem, or as p itself, when geometry is concerned.

We call a triple (v_1, v_2, v_3) , with $v_i \in \Theta_i$, $i = 1, 2, 3$, a *unit triple* if the three corresponding points $p_1 \in S_1, p_2 \in S_2, p_3 \in S_3$ span a unit circle. We use the standard algebraic representation of the v_i 's, where we replace v_i by $t_i = \tan \frac{v_i}{2}$, and the corresponding point on C_i then becomes $c_i + \left(\frac{1-t_i^2}{1+t_i^2}, \frac{2t_i}{1+t_i^2} \right)$. With these representations, the property of being a unit triple can be expressed by a polynomial equation $f(v_1, v_2, v_3) = 0$, obtained by the appropriate substitutions into the equation of F (see (7) in Appendix A). Clearly, f has constant (and small) degree. This illustrates how our problem is indeed a special instance of the general problem mentioned in the introduction. In what follows we will refer to the v_i 's as orientations, also when substituting them in f (the actual substitution should be of the corresponding parameters $\tan \frac{v_i}{2}$). This slightly incorrect treatment is made to simplify the presentation, and has no real consequences in the analysis.

We next argue that, without loss of generality (with a possible re-indexing of the input circles and points), we may assume that the points of S_1 all lie in the portion of C_1 that lies outside the closed disk circumscribed by C_2 (this property will become handy for the forthcoming analysis). To see this, let D_1, D_2, D_3 denote the three (closed) unit disks circumscribed by C_1, C_2, C_3 , respectively, and consider the intersection region $K = D_1 \cap D_2 \cap D_3$. Assume first that K has a nonempty interior. As is well known, the boundary ∂K of K is of the form $c_1 \cup c_2 \cup c_3$, where c_i is a single (possibly empty) connected arc of C_i , for $i = 1, 2, 3$. More generally, in the intersection K of any finite number of unit disks, each disk contributes at most one connected arc to ∂K . Let C be a unit circle in the plane, which is not one of C_1, C_2, C_3 , and let D denote the disk bounded by C . Then, as just mentioned, C contributes a single connected arc c to $\partial(K \cap D)$. It follows that C avoids the relative interior of at least one of the arcs c_1, c_2, c_3 , namely the arc not containing any endpoint of c . (If one of those arcs is empty, C trivially misses that arc.)

It follows that, for every triple $(p_1, p_2, p_3) \in S_1 \times S_2 \times S_3$ spanning a unit circle C , at least one of the points p_1, p_2, p_3 avoids K , because neither of these points can lie in the interior of K , and they lie on C , which meets ∂K in at most two points. So, for one of the indices $i_0 \in \{1, 2, 3\}$, and for at least a third of these triples (p_1, p_2, p_3) , the point $p_{i_0} \in S_{i_0}$ avoids K ; without loss of generality assume $i_0 = 1$. By discarding the other points of S_1 , we obtain a reduced configuration in which the points of S_1 lie outside K and the number of unit triples is at least one third of its original value. That is, each point in (the reduced) S_1 lies either outside D_2 or outside D_3 . One of these subsets of S_1 participates in at least half the (remaining) unit triples. To recap, by removing the points of the other subset, and by re-indexing if needed, we may assume that all the points of S_1 lie outside the disk D_2 , and that the number of unit triples is at least one sixth of the original number. This reasoning also applies when K is empty or is a singleton (with an empty interior), and in fact becomes much simpler in these cases.

We therefore continue the analysis under the assumption that the points of S_1 all lie outside D_2 .

Let M denote the number of unit circles spanned by $S_1 \times S_2 \times S_3$. Our strategy is to double count the quantity Q (mentioned in the introduction) that we are now going to define. For each $v_3 \in \Theta_3$, let P_{v_3} denote the set of pairs $(v_1, v_2) \in \Theta_1 \times \Theta_2$ such that (v_1, v_2, v_3) is a unit triple, so $f(v_1, v_2, v_3) = 0$. Note that we have $M \leq \sum_{v_3 \in \Theta_3} |P_{v_3}| \leq 8M$. Indeed, there are at most eight triples in $S_1 \times S_2 \times S_3$ that span the same unit circle C (C intersects each of C_1, C_2, C_3 in at most two points, and each triple of points, one from each pair, spans C), and clearly, by definition, at least one of these triples is counted in M .

We now define

$$Q := \sum_{v_3 \in \Theta_3} |P_{v_3}|^2.$$

The quantity Q may be interpreted as the number of ordered pairs of unit triples of the form $((v_1, v_2, v_3), (v'_1, v'_2, v_3))$, with a common third component v_3 . Using the Cauchy-Schwarz inequality, we have

$$M \leq \sum_{v_3 \in \Theta_3} |P_{v_3}| \leq \left(\sum_{v_3 \in \Theta_3} |P_{v_3}|^2 \right)^{1/2} n^{1/2} = Q^{1/2} n^{1/2}. \quad (1)$$

The curves $\gamma_{a,b}$. To obtain an upper bound for Q , we use the following approach. Fix two points $a, b \in S_1$, with orientations $v_a, v_b \in \Theta_1$, respectively, and define $\gamma_{a,b}$ to be the algebraic curve given by the polynomial equation

$$R(v_x, v_y) := \text{Res}_{v_3}(f(v_a, v_x, v_3), f(v_b, v_y, v_3)) = 0,$$

where $\text{Res}_{v_3}(f(v_a, v_x, v_3), f(v_b, v_y, v_3))$ is the *resultant* of the two polynomials $f(v_a, v_x, v_3)$, $f(v_b, v_y, v_3)$ with respect to v_3 (which is thus a real bivariate polynomial in v_x, v_y , independent of v_3). By the properties of the resultant, the curve $\gamma_{a,b}$ contains all points (v_x, v_y) , with corresponding points $x, y \in C_2$, for which there exists v_3 (not necessarily in Θ_3) such that

$$\begin{aligned} f(v_a, v_x, v_3) &= 0, \\ f(v_b, v_y, v_3) &= 0; \end{aligned} \quad (2)$$

for more details see, e.g., Cox et al. [1]. However, it might contain points (v_x, v_y) where there is no real point $z \in C_3$ that spans unit circles with both pairs $(a, x), (b, y)$. In general, the curve $R(v_x, v_y) = 0$ is partitioned into a constant number of connected arcs of two kinds: *real* arcs, over which (2) has a real solution v_3 , and *non-real* arcs, over which there are no such real solutions. We refer to the endpoints of these arcs as *transition points*. We will analyze and handle these points later on.²

Let Π denote the set $\Theta_2 \times \Theta_2$, represented as a set of points in the above parametric plane, let Γ' denote the (multi-)set of the curves $\gamma_{a,b}$, and let $I' = I'(\Pi, \Gamma')$ denote the number of incidences between the curves of Γ' and the points of Π .

Note that, for any fixed $v_3 \in \Theta_3$ and for any ordered pair of pairs $(v_a, v_c), (v_b, v_d)$ in P_{v_3} , we have $(v_c, v_d) \in \gamma_{a,b}$ and $(v_d, v_c) \in \gamma_{b,a}$. It follows that the number I' of point-curve incidences is at least $\frac{1}{4} \sum_{v_3 \in \Theta_3} |P_{v_3}|^2$. Indeed, there can be at most four values of v_3 that give rise to the same incidence (any of the pairs $(v_a, v_c), (v_b, v_d)$, say (v_a, v_c) , defines at most two unit circles that pass through the two corresponding points, and each of these circles can intersect C_3 in at most two points), and only those values among them that belong to Θ_3 are reflected in the above sum; also, the fact that each pair of pairs in P_{v_3} generate two incidences is “neutralized” by the fact that the same two incidences are generated for each of the two orderings of the pairs. That is, we have $Q \leq 4I'$, so it suffices to obtain an upper bound for I' .

The number of incidences between curves of Γ' that are of the form $\gamma_{a,a}$, with $a \in S_1$, and the points of Π , is $O(n^2)$. Indeed, let $a \in S_1$ and consider the curve $\gamma_{a,a}$. For $c \in S_2$, there exist at most two unit circles that pass through a and c , and these circles form at most four intersection points z with C_3 . Then for each such intersection point z , there exist at most two unit circles that pass through a and z , which form at most four intersection points d with C_2 . Thus there are at most 16 values d for which $(v_c, v_d) \in \gamma_{a,a}$. It follows that, for each $a \in S_1$, the curve $\gamma_{a,a}$ is incident to $O(n)$ points of Π , and hence the total number of incidences that curves of this form contribute is $O(n^2)$.

Therefore, letting $\Gamma \subset \Gamma'$ be the (multi-)set of the curves $\gamma_{a,b}$, with $a \neq b \in S_1$, and letting $I = I(\Pi, \Gamma)$ denote the number of incidences between the curves of Γ and the points of Π , we get $I' \leq I + O(n^2)$, and thus $Q \leq 4I + O(n^2)$.

We reiterate that I (and I') might include many irrelevant incidences, first, because the corresponding parameter v_3 does not belong to Θ_3 , and second, because v_3 is not real (the incidence occurs on a non-real arc of the relevant curve $\gamma_{a,b}$). Still, an upper bound on the (overestimate) I suffices for our purpose.

Hence the problem is reduced to obtaining an upper bound on I . This is an instance of a fairly standard point-curve incidence problem, which can be tackled using the well established machinery, such as the incidence bound of Pach and Sharir [8], or, more fundamentally, the crossing-lemma technique of Székely [13] (on which the analysis in [8] is based). However, to apply this machinery, it is essential that the curves of Γ have a constant bound on their multiplicity. More precisely, we need to know that no more than $O(1)$ curves of Γ can share a common irreducible component. In more detail, while the points of Π are clearly distinct, there might be potentially many pairs of curves of Γ that coincide or overlap in a common irreducible component, in which case the aforementioned incidence-

²In both real and non-real arcs, we only consider real values for v_ξ, v_η . Only z can assume non-real values for points on non-real arcs. In other words, ignoring its geometric interpretation, $\gamma_{a,b}$ is a real curve.

bounding techniques break down. Fortunately, this can be controlled through the following key proposition. (Recall that this arises as a key issue when applying this approach to the general setup of Elekes and Rónyai [3], as manifested in the companion paper [9], and it will certainly also be a key issue in further attempts to tackle the more general setup of Elekes and Szabó [5] in a similar manner.)

Proposition 1. *There exists a set U of $O(n)$ pairs (a, b) , such that, for any irreducible component γ' , there are at most $O(1)$ pairs (a, b) such that γ' contains a portion of a real arc of $\gamma_{a,b}$, and $(a, b) \notin U$.*

The proof of the proposition is given in Section 2.2. This allows us to derive an upper bound on the number of incidences, given in the following proposition.

Proposition 2. *Let Γ and Π be as above. Then the number I of incidences between Γ and Π is $O(|\Gamma|^{2/3}|\Pi|^{2/3} + |\Gamma| + |\Pi|)$.*

The proof of the proposition is given in Section 2.3. Since $|\Pi|, |\Gamma| = O(n^2)$, it follows that in this case $I = O(n^{8/3})$ and thus, recalling that $Q \leq 4I + O(n^2)$, $Q = O(n^{8/3})$ too, so we get $M = O(n^{11/6})$.

We thus obtain the following main result of the paper, returning to the original formulation of the problem.

Theorem 3. *Let p_1, p_2, p_3 be three distinct points in the plane, and, for $i = 1, 2, 3$, let C_i be a family of n unit circles that pass through p_i . Then the number of points incident to a circle of each family is $O(n^{11/6})$.*

2.1 Properties of the curves $\gamma_{a,b}$

In this section we provide a detailed analysis of the structure and properties of the curves $\gamma_{a,b}$, from both algebraic and geometric perspectives.

Explicit construction of the third point of a unit triple. We slightly change the notation temporarily, and let $a = (a_1, a_2)$ be a point on C_1 , and $x = (x_1, x_2)$ be a point on C_2 . We derive below an explicit expression for a point $z = (z_1, z_2)$ on C_3 such that (a, x, z) span a unit circle. This procedure will provide the basis, and will be used repeatedly in the forthcoming analysis. We note that the procedure consists of two similar substeps. Later on, we will formally break it into these substeps, and use them as the primitive building blocks for the analysis.

Let C be a unit circle that passes through a and x . The center w of C is the point

$$w = (w_1, w_2) = \left(\frac{a_1 + x_1}{2}, \frac{a_2 + x_2}{2} \right) \pm s \left(-\frac{x_2 - a_2}{2}, \frac{x_1 - a_1}{2} \right), \quad (3)$$

where $s = \frac{\sqrt{1 - \frac{\|x-a\|^2}{4}}}{\frac{\|x-a\|}{2}} = \sqrt{\frac{4}{\|x-a\|^2} - 1}$; see Figure 3(a). There can be zero, one, or two real solutions for w . Although w is written in terms of its two coordinates, it really has

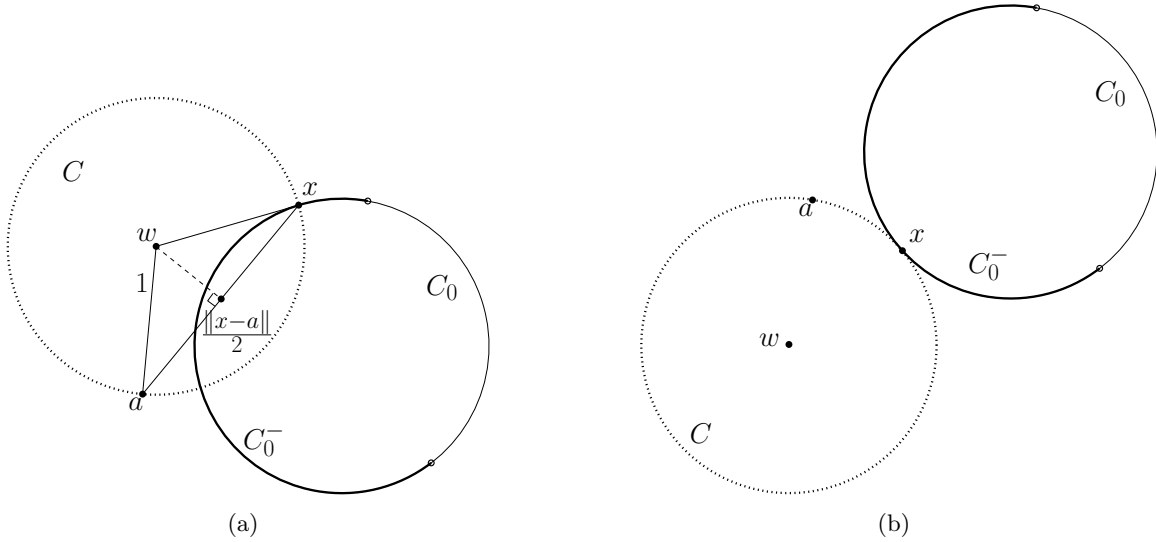


Figure 3: (a) Computing the center w of a unit circle C passing through a and x . Here x varies along C_0 and w exists when x lies in C_0^- (the highlighted arc). (b) The exceptional situation in Lemma 4, where C is tangent to C_0 at x .

only one degree of freedom, as a function of x , as it has to lie on the fixed unit circle C_a centered at a .

Let $c_3 = (q_1, q_2)$ be the center of C_3 . Then, similar to (3), the point z is given by

$$z = (z_1, z_2) = \left(\frac{w_1 + q_1}{2}, \frac{w_2 + q_2}{2} \right) \pm r \left(-\frac{q_2 - w_2}{2}, \frac{q_1 - w_1}{2} \right), \quad (4)$$

where $r = \sqrt{\frac{4}{\|w - c_3\|^2} - 1}$; again, for a given w , there can be zero, one, or two real solutions for z . Thus, the number of real values of the combined expression for z , obtained by substituting (3) into (4), is between 0 and 4.

Symmetric constructions give explicit expressions for the first or second point of a unit triple in terms of the two other points.

Remark. When a and x are given, the equation $F(a, x, z) = 0$, as given in (7) in Appendix A, is of degree 4 in the coordinates of z . The combination of (3) and (4) is in fact an explicit solution of this equation by radicals (with the standard, one-dimensional representations of the points involved).

The primitive steps of the procedure. Note that the two substeps of the above procedure, the one that computes w (in (3)) and the one that computes z (in (4)), are essentially identical. Each of them starts with a fixed point a and a point x that lies on some given (unit) circle C_0 , and constructs the center w of a unit circle that passes through a and x . (The fact that x lies on C_0 is not needed in the procedure itself, but will be important when we consider the functional dependence of w on x .)

Denote the output of the subprocedure as $\varphi_{a, C_0}(x)$. A couple of remarks are in order. First, similar to what has just been remarked, the roles of a and x in defining w are

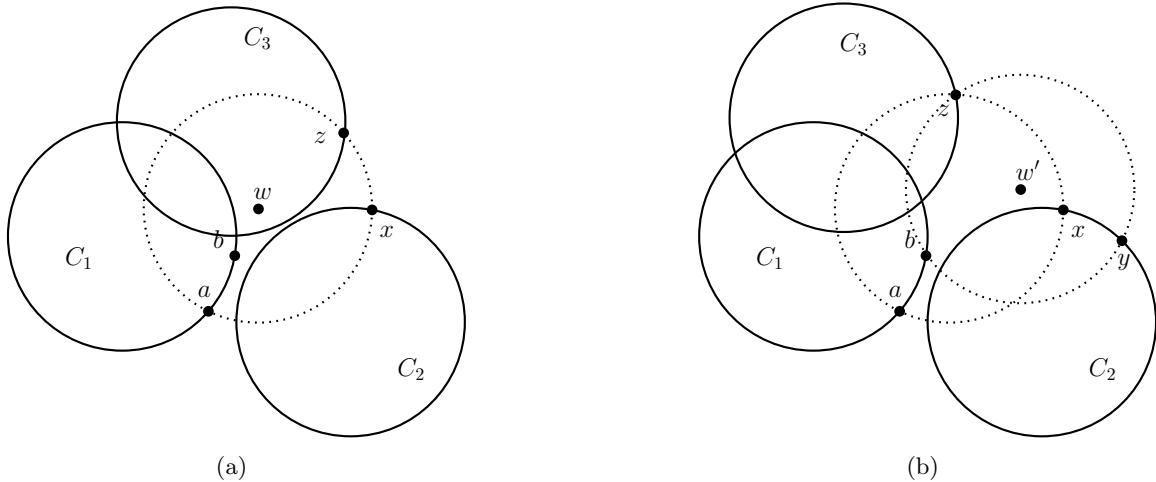


Figure 4: (a) Steps (i) and (ii) of the construction. (b) Steps (iii) and (iv) of the construction.

essentially identical. The above notation is supposed to signify that a is considered as a fixed parameter and x as a variable (along its circle C_0). Second, φ is in general 2-valued, unless $\|a - x\| = 2$ (it has complex values when $\|a - x\| > 2$). In what follows, we will always trace a single (real) branch of such a φ (over which $\|a - x\| < 2$), but stop when $\|a - x\|$ becomes 2.

Explicit construction of points on a curve $\gamma_{a,b}$. Let $\gamma_{a,b}$ be one of our curves. The preceding construction immediately leads to the following 4-step procedure for constructing points on $\gamma_{a,b}$. Specifically, given $a, b \in C_1$ and a point $x \in C_2$, we compute point(s) $y \in C_2$ such that $(v_x, v_y) \in \gamma_{a,b}$, as follows.

- (i) We start with a and x , and construct the point(s) $w = \varphi_{a,C_2}(x)$, each of which is the center of a unit circle passing through a and x . See Figure 4(a). We fix one of these points w . (We terminate the procedure, with failure, when there are no real solutions; this also applies to each of the following steps.)
- (ii) We then construct the point(s) $z = \varphi_{C_3,C_a}(w) \in C_3$, where C_a is the unit circle centered at a . By construction, we have $F(a, x, z) = 0$; again, there are (at most) two choices for z and we fix one of them. See Figure 4(a).
- (iii) We now construct the center(s) $w' = \varphi_{b,C_3}(z)$ of the unit circle(s) passing through b and z . See Figure 4(b). Fix w' to be one of these centers.
- (iv) Finally, we obtain the desired point(s) $y = \varphi_{C_2,C_b}(w') \in C_2$, where C_b is the unit circle centered at b . See Figure 4(b).

(Note that the circles appearing in the four applications of the φ -functions, namely, C_2 , C_a , C_3 , and C_b , are indeed fixed, regarding a and b themselves as fixed.)

The following lemma shows that the functions φ are well-behaved, in the sense made precise below, unless certain degenerate situations arise. In the lemma, the derivative of $w = \varphi_{a,C_0}(x)$ should have the expected interpretation. Concretely, let v_x (resp., v_w) denote

the orientation of x (resp., of w) with respect to the center of C_0 (resp., a). Interpret $w = \varphi_{a,C_0}(x)$ as the corresponding functional relationship between v_x and v_w , which, with a slight abuse of notation, we also write as $v_w = \varphi_{a,C_0}(v_x)$. Then $\varphi'_{a,C_0}(x)$ is simply the corresponding derivative of $\varphi_{a,C_0}(v_x)$ at the respective orientation v_x .

Lemma 4. *Let a be a fixed point, C_0 a fixed (unit) circle, and assume $a \notin C_0$. Put $C_0^- := \{x \in C_0 \mid \|a - x\| < 2\}$. Assume we are given a parameterization of C_0^- , $x(t) = (x_1(t), x_2(t))$, for $t \in I \subseteq \mathbb{S}^1$, such that each of the functions $x_1(t), x_2(t)$ is an analytic function on I . Then each branch of the function $\varphi_{a,C_0}(x)$ is continuous and analytic, and has non-zero derivative, at each point $x \in C_0^-$ for which the unit circle centered at $\varphi_{a,C_0}(x)$ is not tangent to C_0 at x .*

Proof. See Figure 3(a) for the general layout, and Figure 3(b) for the exceptional situation. First note that our assumption, combined with the explicit expression (3), implies that φ_{a,C_0} is analytic on I . Fix a point $x \in C_0^-$ and put $w = \varphi_{a,C_0}(x)$. Let v_x and v_w denote the corresponding orientations, let x' be the point with $v_{x'} = v_x + \Delta v_x$, for a small increment Δv_x , and put $w' = \varphi_{a,C_0}(x')$ and $\Delta v_w = v_{w'} - v_w$. Clearly, since $\varphi_{a,C_0}(x)$ is continuous over I , Δv_w is also small when Δv_x is small.

Put $\Delta x = x' - x$ and $\Delta w = w' - w$ (note that these are vector displacements, whereas $\Delta v_x, \Delta v_w$ are scalars, and we have $\|\Delta x\| < |\Delta v_x|, \|\Delta w\| < |\Delta v_w|$). We have

$$\begin{aligned} 1 &= \|w' - x'\|^2 = \|w + \Delta w - x - \Delta x\|^2 \\ &= \|w - x\|^2 + 2(w - x) \cdot (\Delta w - \Delta x) + o(|\Delta v_x| + |\Delta v_w|) \\ &= 1 + 2(w - x) \cdot (\Delta w - \Delta x) + o(|\Delta v_x| + |\Delta v_w|). \end{aligned}$$

Let τ_x and τ_w denote the unit tangent vectors to C_0 at x and to C_a at w , respectively. We then have

$$\begin{aligned} \|\Delta x - (\Delta v_x)\tau_x\| &= o(|\Delta v_x|) \\ \|\Delta w - (\Delta v_w)\tau_w\| &= o(|\Delta v_w|). \end{aligned}$$

We thus have

$$(w - x) \cdot ((\Delta v_w)\tau_w - (\Delta v_x)\tau_x) = o(|\Delta v_x| + |\Delta v_w|).$$

That is,

$$((w - x) \cdot \tau_w)\Delta v_w = ((w - x) \cdot \tau_x)\Delta v_x + o(|\Delta v_x| + |\Delta v_w|).$$

$w - x$ cannot be orthogonal to τ_w . Indeed, this is possible only when either a coincides with x , which has been ruled out in the lemma, or when $\|a - x\| = 2$, which is also excluded. Moreover, by assumption, $w - x$ is not orthogonal to τ_x (that is, the unit circle centered at w is not tangent to C_0 ; see Figure 3(b)). Hence both coefficients of $\Delta v_w, \Delta v_x$ are non-zero, so we have

$$\frac{\Delta v_w}{\Delta v_x} = \frac{(w - x) \cdot \tau_x}{(w - x) \cdot \tau_w} + o\left(1 + \frac{|\Delta v_w|}{|\Delta v_x|}\right),$$

so in the limit we get

$$\varphi'_{a,C_0}(x) = \frac{(w - x) \cdot \tau_x}{(w - x) \cdot \tau_w}, \tag{5}$$

which is well defined and nonzero, as asserted in the lemma. \square

Note that either the numerator or the denominator of (5) can become 0 as x varies along C_0 : The numerator becomes 0 when w is at distance 2 from the center of C_0 (the exceptional case depicted in Figure 3(b)), and the denominator becomes 0 when x is at distance 2 from a (the endpoints of C_0^- , if they exist). Note that both situations are ruled out in the lemma. These kinds of degeneracy will play a central role in what follows.

Let a, b be a pair of points on C_1 (outside the disk D_2), and let (v_x, v_y) be a point on $\gamma_{a,b}$, for a respective pair of points $x, y \in C_2$. Consider the 4-step procedure that produces y from x , as described above, and write the outputs of its steps as $w = \varphi_{a,C_2}(x)$, $z = \varphi_{c_3,C_a}(w)$, $w' = \varphi_{b,C_3}(z)$, and $y = \varphi_{c_2,C_b}(w')$. In each step we pick an appropriate branch of the relevant function, and assume that the degeneracies ruled out in Lemma 4 (and summarized in the preceding paragraph) do not arise (in suitable neighborhoods of the four respective points) in any of these four steps. With these notation, assumptions, and conventions, we obtain the composite function

$$v_y = \Phi_{a,b}(v_x) := \varphi_{c_2,C_b} \circ \varphi_{b,C_3} \circ \varphi_{c_3,C_a} \circ \varphi_{a,C_2}(v_x),$$

which is well defined and analytic in a suitable neighborhood N of x , its graph over N is a portion of $\gamma_{a,b}$, and (v_x, v_y) is not a local x -extremum or y -extremum of that portion. The non-extremality properties are consequences of the chain rule, combined with a four-way application of (5) in the proof of Lemma 4. (See below for a more explicit repetition of this argument.)

2.2 Proof of Proposition 1

Let γ' be an irreducible component of potentially many curves $\gamma_{a,b}$. The strategy is to identify (few) points along γ' , from which we can reconstruct all the values of a and b in only a constant number of ways.

Fix a generic point $q_0 = (v_{x_0}, v_{y_0})$ on γ' that is non-singular and non-extremal for γ' , and which does not lie on any other irreducible component of any other curve. Let $\Xi(q_0)$ denote the subset of all pairs (a, b) so that $\gamma_{a,b}$ contains γ' , and q_0 lies on a real arc of $\gamma_{a,b} \cap \gamma'$. The preceding discussion implies that, for each such curve $\gamma_{a,b}$, there is a unique way to define the corresponding function $\Phi_{a,b}$ in a suitable neighborhood of q_0 (which may depend on a and b), and the graph of each of these functions near q_0 (in the intersection of all these neighborhoods) is γ' itself.

Now trace γ' , along with all these functions, from q_0 in, say, increasing v_x -direction, and stop at the first point (v_ξ, v_η) at which one of the assumptions made in Lemma 4 is violated, for some pair $(a, b) \in \Xi(q_0)$, for one of the corresponding functions $w = \varphi_{a,C_2}(\xi)$, $z = \varphi_{c_3,C_a}(w)$, $w' = \varphi_{b,C_3}(z)$, or $\eta = \varphi_{c_2,C_b}(w')$. The definition of the curves $\gamma_{a,b}$, and the property that all the points of S_1 lie outside the disk D_2 , are easily seen to imply that γ' does indeed contain such a point (v_ξ, v_η) . As we will shortly argue, there are no transition points of any curve $\gamma_{a,b}$, for $(a, b) \in \Xi(q_0)$, between q_0 and (v_ξ, v_η) , so (v_ξ, v_η) still belongs to the same real subarcs of all the curves with pairs in $\Xi(q_0)$ (possibly being a delimiting point of some of these arcs).

Applying explicitly the chain rule, we have, for an arbitrary point (v_x, v_y) on the relatively open portion $(\gamma')^-$ of γ' between q_0 and (v_ξ, v_η) , with corresponding points $x, y \in C_2$,

$$\Phi'_{a,b}(x) = \varphi'_{c_2,C_b}(w') \varphi'_{b,C_3}(z) \varphi'_{c_3,C_a}(w) \varphi'_{a,C_2}(x),$$

where w , z , and w' are as defined above. By (5), this is a product of four fractions, and the above assumption about (v_ξ, v_η) means that the numerator or denominator of at least one of these fractions is zero at ξ , but they all remain nonzero before reaching ξ .

For technical reasons that will become clearer later on, we trace γ' from q_0 in both increasing and decreasing v_x -direction.

Call (a, b) an *ultra-degenerate* pair if (at least) one numerator and one denominator vanish simultaneously at ξ . We will shortly show that ultra-degenerate pairs are sparse, and handle them all via a global argument that does not depend on γ' . For the time being, we ignore all such pairs. (The exceptional set U in the lemma will be the set of these ultra-degenerate pairs.)

In other words, excluding ultra-degenerate pairs, $\Phi'_{a,b}(v_x)$, which is equal to the slope of the tangent at the corresponding point on γ' (assuming that point to be non-singular), becomes zero or tends to ∞ at v_ξ , so γ' has a (one-sided) horizontal or vertical tangent at (v_ξ, v_η) . That is, (v_ξ, v_η) is a local (one-sided) x - or y -extremum of γ' , in the above weak sense.

It is important to note that the curve $v_y = \Phi_{a,b}(v_x)$ does indeed trace γ' between $q_0 = (v_{x_0}, \Phi_{a,b}(v_{x_0}))$ and (v_ξ, v_η) . Indeed, let $h(t, s) = 0$ be the (irreducible) polynomial equation defining γ' . Let $t_1 \in (v_{x_0}, v_\xi)$ be such that the graph of $\Phi_{a,b}$, restricted to the subinterval $[v_{x_0}, t_1]$, is an arc of γ' , but this does not hold for $[v_{x_0}, t_1^+]$, for any $t_1^+ > t_1$, arbitrarily close to t_1 . We thus have, $h(t, \Phi_{a,b}(t)) = 0$, for every $t \in [v_{x_0}, t_1]$. By Lemma 4, the function $\Phi_{a,b}(t)$ is analytic in some sufficiently small neighborhood N of t_1 inside (v_{x_0}, v_ξ) . Thus, letting $H(t) = h(t, \Phi_{a,b}(t))$, and since h is a polynomial, we have that H is analytic and $H(t) = 0$, for every $t \in [v_{x_0}, t_1] \cap N$. In particular, the derivatives of H at $t = t_1$, of any order, are all zero (because H is identically zero in a one-sided neighborhood of t_1). Thus, since H is analytic, the Taylor series of H at t_1 is identically zero, which means that H is identically zero in some suitable neighborhood $N' \subset N$ of t_1 (this time on both sides of t_1). In other words, the graph of $\Phi_{a,b}$ continues to coincide with γ' on the other side of $(t_1, \Phi_{a,b}(t_1))$ too, contradicting our assumption on t_1 .

We note that the preceding argument also holds when the tracing of γ' encounters a singular point (v_x, v_y) of γ' (before reaching (v_ξ, v_η)). Even if two branches of γ' meet tangentially at (v_x, v_y) , the graph of $\Phi_{a,b}$ remains well defined, and follows a unique branch of γ' , on either side of (v_x, v_y) .

Transition points. We next argue that at a transition point we must have one of the degeneracies ruled out in Lemma 4, for one of the four φ -functions. Specifically, let (v_ξ, v_η) be a point on γ' which is a transition point along a containing curve $\gamma_{a,b}$. Each of the four functions φ_{a,C_2} , φ_{c_3,C_a} , φ_{b,C_3} , φ_{c_2,C_b} , whose composition yields $\Phi_{a,b}$, involves a square root (with a fixed sign). As long as none of these roots vanish, the functions continue to be defined (as real functions) and produce real values, so the two unit circles that are spanned by the corresponding triples (a, ξ, z) , (b, η, z) , for a suitable point $z \in C_3$, are such that z is real and the circles are real too, and this continues to hold in a suitable neighborhood of (ξ, η) . Since this does not occur at a transition point, one of the square roots has to vanish at (ξ, η) , and when this occurs one of the degenerate conditions in the lemma occurs for the corresponding function (where the denominator of one of the fractions in (5) vanishes). This establishes the promised claim.

Recap. The preceding analysis leads to the following overall treatment of γ' . We partition γ' into maximal connected subarcs, each delimited by points with horizontal or vertical tangency (and does not contain any such point in its relative interior). Since γ' has constant degree, there are only $O(1)$ subarcs of this kind. For each of the containing curves $\gamma_{a,b}$, each such subarc γ'' is fully contained either in a real arc of $\gamma_{a,b}$ or in a non-real arc of $\gamma_{a,b}$, and at least one subarc γ'' is contained in a real arc of $\gamma_{a,b}$.

In the next step of the analysis, we show that, for each of the $O(1)$ locally extremal points $(v_\xi, v_\eta) \in \gamma'$, there are only $O(1)$ pairs (a, b) , for which (v_ξ, v_η) is a (possibly delimiting) point of a real arc of $\gamma_{a,b}$. Altogether, we conclude that γ' can be an irreducible component of only $O(1)$ curves $\gamma_{a,b}$ (with the additional requirements that γ' overlaps at least one real arc of $\gamma_{a,b}$ and that the pair (a, b) is not ultra-degenerate).

The possible geometric scenarios near a locally extremal point of γ' . Let (v_ξ, v_η) be a locally x -extremal point of γ' (locally y -extremal points will be handled in a fully symmetric manner; see below), and let ξ, η be the points in C_2 with orientations v_ξ, v_η , respectively. Let $a, b \in C_1$ be a fixed pair for which at least one of the subarcs of γ' delimited by (v_ξ, v_η) is a (portion of a) real arc of $\gamma_{a,b}$, and assume that (a, b) is not an ultra-degenerate pair. Here we do not know a and b , and our goal is to reconstruct them from v_ξ, v_η . This is done as follows.

We first note that, as argued earlier, the x -extremality of (v_ξ, v_η) means that $\Phi'_{a,b}(v_\xi) = \infty$ for every such pair (a, b) . That is, one of the denominators in the four expressions, as in (5), vanishes. We thus have the following four respective situations.

Case (i) $\|a - \xi\| = 2$. In this case we can reconstruct a in at most two possible ways, as an intersection point of C_1 with the circle of radius 2 centered at ξ ; see Figure 5(a). We can then retrieve the corresponding point $z \in C_3$ in two possible ways, as an intersection point of C_3 with the (unique) unit circle that passes through a and ξ . Since η is also given, we can compute b , as one of the intersection points of C_1 with one of the at most two unit circles that pass through z and η . Altogether, there are (at most) two ways to choose a , two for z , and four for b , so in the present case we can reconstruct (a, b) in at most 16 possible ways.

Case (ii) $\|c_3 - w\| = 2$. In this case, there is a unit circle that passes through a and ξ and is tangent to C_3 at z ; see Figure 5(b). Hence z is a tangency point of C_3 with one of the at most two unit circles that are incident to ξ and tangent to C_3 . This allows us to reconstruct a , as an intersection point of C_1 with one of these two unit circles. We then retrieve b from z and η as in the preceding case. Altogether, there are (at most) two ways to choose z , two for a , and four for b , so here too we can reconstruct (a, b) in at most 16 possible ways.

Case (iii) $\|b - z\| = 2$. This is geometrically more challenging to analyze, because the points z, b , at distance 2 apart, are both unknown. See Figure 6(a). We handle this case by observing that the lengths of the edges of the quadrilateral $R = c_1 b z c_3$ are fixed—they are 1, 2, 1 and $|c_1 c_3|$, respectively, but this does not determine R , because it can flex (with one degree of freedom) about its fixed edge $c_1 c_3$. As R flexes, the midpoint w' of bz traces an

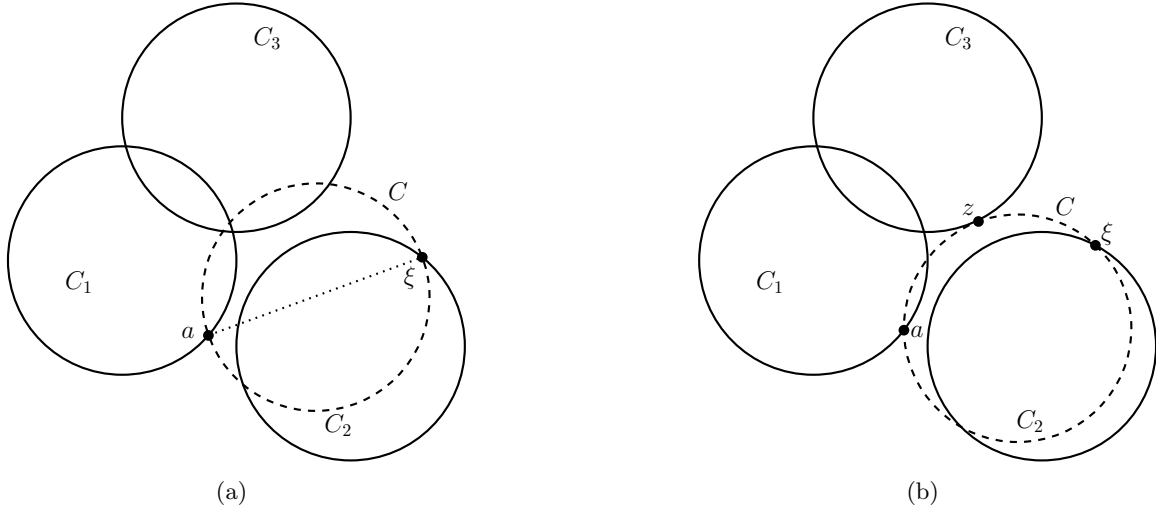


Figure 5: (a) The situation in Case (i) of the reconstruction ($a\xi$ is a diameter of C). (b) The situation in Case (ii) (C is tangent to C_3 at z).

algebraic curve τ of some constant degree d . Note that the unit circle that passes through b, η and z , has its center w' (which is the midpoint of bz) on τ . Since the point η is known, we can find w' , by computing the intersection points of τ with the unit circle C_η centered at η , and then retrieve b , as the intersection point of C_1 with the unit circle centered at w' . We claim that, in general, there are at most $2d$ intersection points of τ with C_η , and hence at most a constant number of ways to reconstruct b . Indeed, if this were not the case, then, by Bézout's theorem (see, e.g., [1]), τ would have to contain C_η as one of its components. This situation is controlled by the following simple claim.

Claim. *The curve τ does not contain any unit circle as one of its components, unless C_1 and C_3 are tangent to each other, in which case τ does indeed contain the unit circle centered at the point of tangency.*

Proof. See Figure 6(a) for the exceptional situation in the claim. Let C be a unit circle, centered at a point c , such that $C \subseteq \tau$. By the construction of τ , every point $p \in C$ is the midpoint of a segment whose endpoints lie on C_1 and C_3 , respectively. This implies, in particular, that C is contained in $K := \text{conv}(C_1 \cup C_3)$, the convex hull of $C_1 \cup C_3$, and since the three circles C, C_1, C_3 are of the same radius, it follows that the center c of C lies on c_1c_3 . Moreover, as is easily checked (cf. Figure 6(a)), in this case c must be the midpoint of c_1c_3 , and we must have $|c_1c_3| = 2$, and then C_1 and C_3 are tangent to each other at c , implying that C_1, C_3 , and C must indeed be of the exceptional kind stated in the claim. \square

To complete the analysis, we recall that the only problematic case is when the unit circle centered at η is contained in τ . By the claim, η must be the midpoint of c_1c_3 , so in particular C_2 is not tangent to C_1 (or else it would coincide with C_3). Note that in this special situation, the points of S_1 , with the possible exception of η , clearly lie outside the disk circumscribed by C_3 . We can discard the tangency point η of C_1 and C_3 from S_1 , if needed, losing at most $O(n)$ unit triples spanned by the original sets S_1, S_2, S_3 . We now restart the whole analysis, switching the roles of C_2 and C_3 , and are now guaranteed that the exceptional situation described in the above claim does not occur.

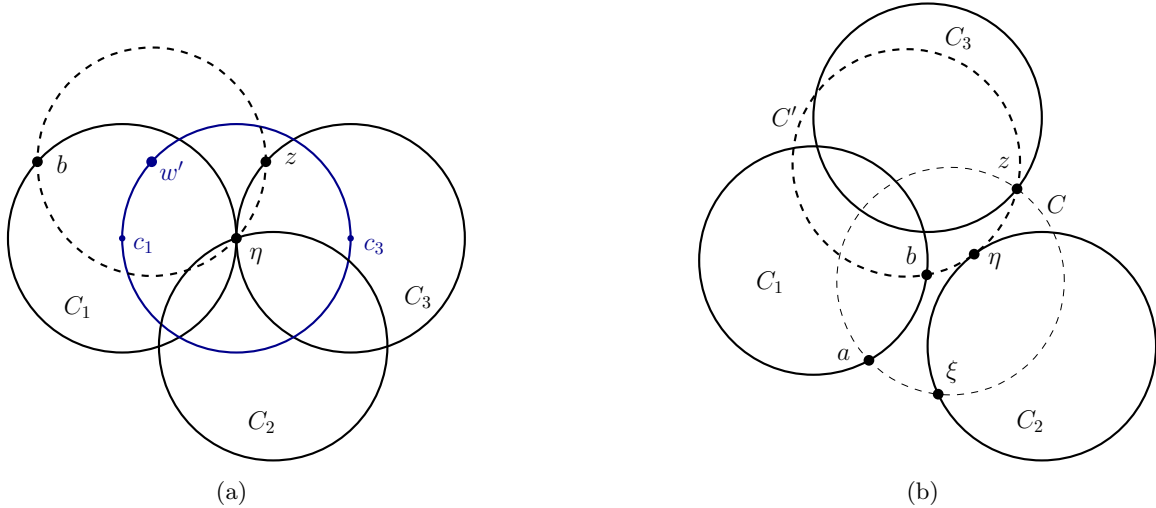


Figure 6: (a) The exceptional situation in the claim in Case (iii) (C_1 and C_3 are tangent at η). (b) The situation in Case (iv) (C' is tangent to C_2 at η).

We can then retrieve z , as an intersection point of C_3 with a unit circle that passes through b and η , and, since ξ is also given, compute a , as one of the intersection points of C_1 with one of unit circles that pass through z and ξ .

Case (iv) $\|c_2 - w'\| = 2$. In this case, depicted in Figure 6(b), the unit circle that passes through b, η and z is tangent to C_2 at η . Hence b is one of the intersection points of C_1 and the unique unit circle (externally) tangent to C_2 at η , and then a can be reconstructed from b, ξ and η , as in the previous cases.

Handling y -extremal points of γ' . The cases where (v_ξ, v_η) is a y -extremal point of γ' are handled in a fully symmetric manner, as follows. Let $\hat{\gamma}'$ denote the “transpose” of γ' , that is,

$$\hat{\gamma}' = \{(v_q, v_p) \mid (v_p, v_q) \in \gamma'\}.$$

Clearly, γ' is an irreducible component of $\gamma_{a,b}$ if and only if $\hat{\gamma}'$ is an irreducible component of $\gamma_{b,a}$. Moreover, (v_ξ, v_η) is a locally y -extremal point of γ' if and only if (v_η, v_ξ) is a locally x -extremal point of $\hat{\gamma}'$. We can therefore apply the preceding analysis, essentially verbatim, to $\hat{\gamma}'$ and the “transposed” containing curves $\gamma_{b,a}$, and conclude that, from each locally y -extremal point (v_ξ, v_η) of γ' , there are only $O(1)$ curves $\gamma_{a,b}$ such that $\gamma' \subset \gamma_{a,b}$ and (v_ξ, v_η) lies on a real arc of $\gamma_{a,b}$.

Ultra-degenerate pairs. Let U denote the set of all ultra-degenerate pairs; recall that a pair (a, b) is said to be *ultra-degenerate* if there exists a point (ξ, η) at which at least one numerator and one denominator of the four fractions that define $\Phi'_{a,b}$ vanish simultaneously at ξ . The reason for singling out these pairs is that it is not clear what happens to the slope of the tangent to $\gamma_{a,b}$ (that is, to γ') at this point.

Fortunately, the overall size of U , over all possible components γ' , is only $O(n)$. The somewhat tedious case analysis that establishes this claim is given in Appendix B. Since

each such curve has only $O(n)$ incidences with the points of $S_2 \times S_2$ (an easy property which is a special case of the Schwartz–Zippel Lemma [10, 15]), we get a total of $O(n^2)$ incidences that correspond to such pairs. This is a small bound, subsumed by the overall bound on Q that we derive.

Combining all the steps of the analysis, and excluding the $O(n)$ pairs in U , we finally conclude that γ' overlap (a portion of a real arc of) $\gamma_{a,b}$, for at most a constant number of pairs (a, b) . This completes the proof of Proposition 1. \square

2.3 Proof of Proposition 2

We apply Székely’s technique [13], which is based on the crossing lemma (see also Pach and Agarwal [7]). As noted, this is also the approach used in [8], but the possible overlap of curves requires some extra (and more explicit) care in the application of the technique. A similar argument is given in the companion paper [9], but we repeat it here to make the paper more self-contained.

Throughout this subsection, we completely ignore curves $\gamma_{a,b}$ for which (a, b) is an ultra-degenerate pair; as argued in Section 2.2, these curves contribute only $O(n^2)$ to the incidence bound.

We begin by constructing a plane embedding of a multigraph G , whose vertices are the points of Π , and each of whose edges connects a pair $\pi_1 = (\xi_1, \eta_1)$, $\pi_2 = (\xi_2, \eta_2)$ of points that lie on the same curve $\gamma_{a,b}$ and are consecutive along (some connected component of) $\gamma_{a,b}$; the edge is drawn along the portion of the curve between the points. One edge for each such curve (connecting π_1 and π_2) is generated, even when the curves coincide or overlap. Thus there might potentially be many edges of G connecting the same pair of points, whose drawings coincide. Nevertheless, by Proposition 1, this number is at most $O(1)$.

In spite of this control on the number of mutually overlapping (or, rather, coinciding) edges, we still face the potential problem that the edge multiplicity in G (over all curves, overlapping or not, that connect the same pair of vertices) may not be bounded (by a constant). More concretely, we want to avoid edges (π_1, π_2) whose multiplicity exceeds d^2 , where here $d = O(1)$ denotes the degree of the curves of Γ .

To follow this strategy, we pass to a dual parametric plane, in which the roles of Θ_1 and Θ_2 are interchanged, so curves $\gamma_{a,b}$ of Γ become dual points (v_a, v_b) , and points (v_ξ, v_η) of Π become dual curves $\gamma_{\xi,\eta}^*$, defined as the locus of all points (v_x, v_y) , each corresponding to a pair of points $x, y \in C_1$, for which there exists v_3 (not necessarily in Θ_3) such that (compare with (2))

$$\begin{aligned} f(v_x, v_\xi, v_3) &= 0 \\ f(v_y, v_\eta, v_3) &= 0; \end{aligned} \tag{6}$$

we denote by Γ^* and Π^* the sets of the dual points and of the dual curves, respectively. Clearly, we have $(v_\xi, v_\eta) \in \gamma_{a,b}$ if and only if $(v_a, v_b) \in \gamma_{\xi,\eta}^*$. We recall our assumption that, for $(v_a, v_b) \in \Gamma^*$, the corresponding points a and b lie on the portion of C_1 which is outside D_2 . In view of this, we can ignore irreducible components of curves of Π^* which contain only “irrelevant” points (v_a, v_b) , that is, only points (v_a, v_b) for which one of a or b lies in D_2 .

Claim. Let $\pi_0 = (\xi_0, \eta_0) \in \Pi$ be a vertex of G . Then π_0 has at most $d^3 - 1$ neighbors $\pi \in \Pi$ in G , for which the multiplicity of the edge (π_0, π) is larger than d^2 .

Proof. For contradiction, assume there exist $\pi_i = (\xi_i, \eta_i)$, $i = 1, \dots, d^3$, such that (π_0, π_i) is of multiplicity at least $d^2 + 1$. By construction (and using duality), every edge connecting π_0 and π_i in G corresponds to a dual point of Γ^* that lies on both dual curves γ_{ξ_0, η_0}^* and γ_{ξ_i, η_i}^* . Thus, by the assumption on the multiplicity of (π_0, π_i) , for each $i = 1, \dots, d^3$, the curves γ_{ξ_0, η_0}^* and γ_{ξ_i, η_i}^* have at least $d^2 + 1$ points in common, and hence, by Bézout's theorem (see, e.g., [1]), the two curves share a common irreducible component. Note that γ_{ξ_0, η_0}^* , having degree d , has at most d irreducible components, and thus, by the pigeonhole principle, there exists an irreducible component γ_0^* of γ_{ξ_0, η_0}^* that is shared by at least d^2 curves γ_{ξ_i, η_i}^* ; by reindexing, if needed, assume these are γ_{ξ_i, η_i}^* , $i = 1, \dots, d^2$.

Let $\kappa = O(1)$ be the multiplicity bound obtained in Proposition 1. Let (v_{a_j}, v_{b_j}) , $j = 0, \dots, \kappa d$, be $\kappa d + 1$ (distinct) points on γ_0^* , having the property that, for each $j = 0, \dots, \kappa d$, the points a_j, b_j lie on the portion of C_1 which is outside D_2 (but not necessarily in S_1); as already noted, we may assume that γ_0^* contains at least one such point, but then, by continuity, it contains infinitely many such points. We have $(v_{a_j}, v_{b_j}) \in \gamma_{\xi_i, \eta_i}^*$, or, by duality, $(v_{\xi_i}, v_{\eta_i}) \in \gamma_{a_j, b_j}$, for every $i = 1, \dots, d^2$, $j = 0, \dots, \kappa d$. Similarly, we also have $(v_{a_j}, v_{b_j}) \in \gamma_{\xi_0, \eta_0}^*$ and so $(v_{\xi_0}, v_{\eta_0}) \in \gamma_{a_j, b_j}$, for $j = 0, \dots, \kappa d$. Using Bézout's theorem once again, we have that γ_{a_0, b_0} and γ_{a_j, b_j} , which intersect in at least $d^2 + 1$ points, share a common irreducible component, for each $j = 1, \dots, \kappa d$. Since γ_{a_0, b_0} is of degree d , and thus has at most d irreducible components, we conclude that there exists an irreducible component of γ_{a_0, b_0} that is shared by κ other curves γ_{a_j, b_j} . This however contradicts Propositions 1, and hence the claim follows. \square

Consider a point π_1 and one of its bad neighbors³ π_2 . Let $\gamma_{a,b}$ be one of the curves along which π_1 and π_2 are neighbors. Then, rather than connecting π_1 to π_2 along $\gamma_{a,b}$, we continue along the curve from π_1 past π_2 until we reach a good point for π_1 , and then connect π_1 to that point (along $\gamma_{a,b}$). We skip over at most $d^3 - 1$ points in the process, but now, having applied this “stretching” to each pair of bad neighbors, each of the modified edges has multiplicity at most $2d^2$ (the factor 2 comes from the fact that a new edge e can be obtained by stretching an original edge from either endpoint of e).

Note that this edge stretching does not always succeed: It will fail when the connected component γ' of $\gamma_{a,b}$ along which we connect the points contains fewer than $d^3 + 1$ points of Π , or when there are fewer than $d^3 - 1$ points of Π between π_1, π_2 , and the “end” of γ' (recall the constraint in the definition of the curves $\gamma_{a,b}$). Still, the number of new edges in G is at least $I(\Pi, \Gamma) - \lambda|\Gamma|$, for a suitable constant λ , where the term $\lambda|\Gamma|$ accounts for missing edges on connected components of the curves, for the reasons just discussed. By what have just been argued, the number of edges lost on any single component is at most $O(d^3)$.

The final ingredient needed for this technique is an upper bound on the number of crossings between the (new) edges of G . Each such crossing is a crossing between two curves of Γ . Even though the two curves might overlap in a common irreducible component (where they have infinitely many intersection points, none of which is a crossing), the

³We make the pessimistic assumption that they are (consecutive) neighbors along all these curves, which of course does not have to be the case in general.

number of proper crossings between them is $O(d^2) = O(1)$, as follows, for example, from the Milnor–Thom theorem (see [6, 14]), or from Bézout’s theorem. Finally, because of the way the drawn edges have been stretched, the edges, even those drawn along the same original curve $\gamma_{a,b}$, may now overlap one another, and then a crossing between two curves may be claimed by more than one pair of (stretched) crossing edges. Nevertheless, since no edge straddles more than $d^3 - 1$ points, the number of pairs that claim a specific crossing is still a constant (that depends on d). Hence, we conclude that the total number of edge crossings in G is $O(|\Gamma|^2)$.

We can now continue by applying the crossing lemma argument, exactly as done by Székely and in other works (e.g., see [8, 13]), and conclude that

$$I(\Pi, \Gamma) = O\left(|\Pi|^{2/3}|\Gamma|^{2/3} + |\Pi| + |\Gamma|\right),$$

with the constant of proportionality depending on d . This completes the proof of Proposition 2. \square

3 Conclusion

We do not know whether the bound in Theorem 3 is tight in the worst case, and suspect that it is not. Resolving this question is a major problem for further research, especially since it arises in all the related specific and general problems in [3, 5, 9, 11, 12].

The specific problem studied in Elekes et al. [4] and in this paper is well motivated in [4], because it yields a combinatorial distinction between unit circles and lines. That is, there exist three families of lines passing through three respective points, which determine $\Theta(n^2)$ triple points. Nevertheless, this problem is clearly only one special instance of several related problems, in which we have three sets S_1, S_2, S_3 of points, each contained in some curve, and we want to bound the number of triples in $S_1 \times S_2 \times S_3$ that satisfy some property (that can be specified by polynomial equation). As a simple example, consider the case where each S_i is contained in some respective line ℓ_i , for $i = 1, 2, 3$, and the property is that the triple span a triangle of unit area. In a companion work in progress we show that in this case the number of triples can be $\Theta(n^2)$, but the bound is likely to drop when the sets S_i are contained in other curves.

Acknowledgment. We would like to thank an anonymous reviewer of a previous version of the paper, whose comments helped a lot to improve the paper. Part of this research was performed while the authors were visiting the Institute for Pure and Applied Mathematics (IPAM), which is supported by the National Science Foundation. The authors deeply appreciate the stimulating environment and facilities provided by IPAM.

References

- [1] D. A. Cox, J. Little and D. O’Shea, *Using Algebraic Geometry*, Springer-Verlag, 2nd Edition, Heidelberg, 2005.

- [2] G. Elekes, Sums versus products in number theory, algebra and Erdős geometry—A survey, in *Paul Erdős and his Mathematics II*, Bolyai Math. Soc., Stud. 11, Budapest, 2002, pp. 241–290.
- [3] G. Elekes and L. Rónyai, A combinatorial problem on polynomials and rational functions, *J. Combin. Theory Ser. A*, 89 (2000), 1–20.
- [4] G. Elekes, M. Simonovits and E. Szabó, A combinatorial distinction between unit circles and straight lines: How many coincidences can they have? *Combinat. Probab. Comput.* 18 (2009), 691–705.
- [5] G. Elekes and E. Szabó, How to find groups? (And how to use them in Erdős geometry?), *Combinatorica* 32 (2012), 537–571.
- [6] J. Milnor, On the Betti numbers of real varieties, *Proc. Amer. Math. Soc.*, 15(2) (1964), 275–280.
- [7] J. Pach and P. K. Agarwal, *Combinatorial Geometry*, Wiley-Interscience, New York, 1995.
- [8] J. Pach and M. Sharir, On the number of incidences between points and curves, *Combinat. Probab. Comput.* 7 (1998), 121–127.
- [9] O. E. Raz, M. Sharir and J. Solymosi, Polynomials vanishing on grids: The Elekes-Rónyai problem revisited, in arXiv:1401.7419 (2014).
- [10] J. Schwartz, Fast probabilistic algorithms for verification of polynomial identities, *J. ACM* 27(4) (1980), 701–717.
- [11] M. Sharir and J. Solymosi, Distinct distances from three points, in arXiv:1308.0814.
- [12] M. Sharir, A. Sheffer, and J. Solymosi, Distinct distances on two lines, *J. Combin. Theory, Ser. A* 20 (2013), 1732–1736.
- [13] L. Székely, Crossing numbers and hard Erdős problems in discrete geometry, *Combinat. Probab. Comput.* 6 (1997), 353–358.
- [14] R. Thom, Sur l’homologie des variétés algébriques réelles, in: S.S. Cairns (ed.), *Differential and Combinatorial Topology*, Princeton University Press, Princeton, NJ, 1965, 255–265.
- [15] R. Zippel, An explicit separation of relativised random polynomial time and relativised deterministic polynomial time, *Inform. Process. Lett.* 33(4) (1989), 207–212.

A Polynomial representation of unit triples

In this appendix we derive the explicit form of the polynomial $F(p_1, p_2, q_1, q_2, r_1, r_2)$ whose vanishing represents triples $p = (p_1, p_2)$, $q = (q_1, q_2)$, $r = (r_1, r_2)$ that span a unit circle.

Put

$$\begin{aligned}x &= \|p - q\|, & X &= x^2 \\y &= \|p - r\|, & Y &= y^2 \\z &= \|q - r\|, & Z &= z^2.\end{aligned}$$

Let S denote the area of the triangle Δpqr . Then the circumradius $R = 1$ of this triangle is given by the formula

$$1 = R = \frac{xyz}{4S}.$$

The area S can be expressed by Heron's formula, written as

$$16S^2 = (x + y + z)(-x + y + z)(x - y + z)(x + y - z).$$

That is, we have $(x + y + z)(-x + y + z)(x - y + z)(x + y - z) = x^2y^2z^2$. With some algebraic manipulations, this can be expressed in terms of the squared distances X, Y, Z , as

$$X^2 + Y^2 + Z^2 - 2XY - 2XZ - 2YZ + XYZ = 0. \quad (7)$$

The left side of (7) is the desired polynomial F in the coordinates of p, q, r . It is of degree 6 in these six variables. In the analysis, with a slight abuse of notation, we write $F(p, q, r)$ for $F(p_1, p_2, q_1, q_2, r_1, r_2)$, interpreting each of p, q, r as a 2-variable aggregate.

B Ultra-degenerate pairs

Recall that a pair (a, b) is said to be *ultra-degenerate* if there exists a point (ξ, η) at which at least one numerator and one denominator of the four fractions that define $\Phi'_{a,b}$ vanish simultaneously at ξ . The reason for singling out these pairs is that it is not clear what happens to the slope of the tangent to $\gamma_{a,b}$ at this point.

There are 16 cases of such a simultaneous vanishing of a numerator and a denominator. The four numerators are (where $\tau_{u,C}$ denotes the tangent vector to the circle C at the point u)

$$(w - \xi) \cdot \tau_{\xi, C_2}, \quad (z - w) \cdot \tau_{w, C_a}, \quad (w' - z) \cdot \tau_{z, C_3}, \quad (\eta - w') \cdot \tau_{w', C_b},$$

and the four denominators are

$$(w - \xi) \cdot \tau_{w, C_a}, \quad (z - w) \cdot \tau_{z, C_3}, \quad (w' - z) \cdot \tau_{w', C_b}, \quad (\eta - w') \cdot \tau_{\eta, C_2}.$$

As noted earlier, the vanishing of any of these eight expressions means that a corresponding pair of points lie at distance 2 from each other. Specifically, for the numerators, the corresponding constraints are, respectively,

$$\|wc_2\| = 2, \quad \|az\| = 2, \quad \|w'c_3\| = 2, \quad \|\eta b\| = 2,$$

and for the denominators, the corresponding constraints are, respectively,

$$\|a\xi\| = 2, \quad \|wc_3\| = 2, \quad \|zb\| = 2, \quad \|w'c_2\| = 2.$$

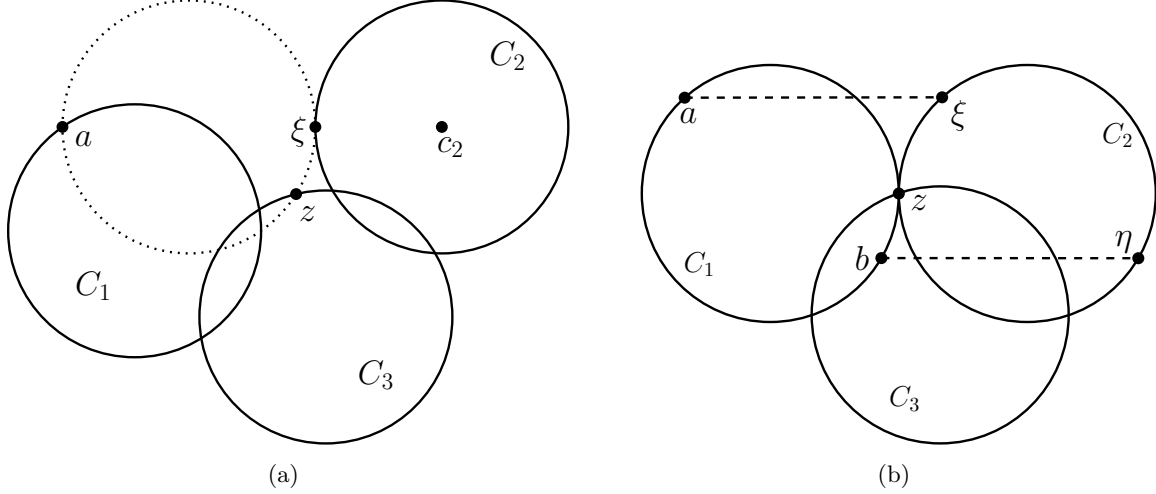


Figure 7: (a) The numerator and denominator of the first fraction both vanish (the distance between the point a and the center c_2 is 3). (b) The exceptional situation in case N4D1 (C_1 is tangent to C_2 at z , and $\|a\xi\| = \|b\eta\| = 2$).

Consider first the cases where the numerator and denominator of the same fraction both vanish. Consider for specificity the first fraction, so we have

$$(w - \xi) \cdot \tau_{\xi, C_2} = (w - \xi) \cdot \tau_{w, C_a} = 0, \quad \text{or} \quad \|a\xi\| = \|wc_2\| = 2.$$

In this case the four points a , w , ξ and c_2 must be collinear, with $\|a\xi\| = \|wc_2\| = 2$. This is easily seen to imply that $\|ac_2\| = 3$. In this case, a is an intersection point of C_1 with the circle of radius 3 centered at c_2 . Hence there are at most two choices for a , for a total of $O(n)$ pairs (a, b) that fall into this subcase. A similar argument applies when the numerator and denominator of any of the three other fractions both vanish: The corresponding constraints are $\|ac_3\| = 3$, $\|bc_3\| = 3$, and $\|bc_2\| = 3$, and in each of these cases it follows that there are at most two choices for either a or b , for a total of $O(n)$ pairs of these types. (Note that in these cases the curves $\gamma_{a,b}$ are singletons; see Figure 7(a).)

Therefore, in what follows, we assume that the vanishing numerator and denominator belong to distinct fractions. We use the mnemonic notation NtDs to mean that the numerator of the t -th fraction and the denominator of the s -th fraction both vanish, and consider the following 12 possible cases.

N1D2: Here we have $\|wc_3\| = \|wc_2\| = 2$. In this case, w is an intersection point of the two circles of radius 2 centered at c_2 and at c_3 . Once w is known, a is an intersection point of C_1 with the unit circle centered at w . Hence there are only $O(1)$ ways to choose a , for a total of $O(n)$ ultra-degenerate pairs of this type.

N1D3: Here we have $\|wc_2\| = \|zb\| = 2$. We claim that each $b \in S_1$ can be coupled with only $O(1)$ a 's to form an ultra-degenerate pair of this type. Indeed, given b , we find z , as an intersection point of C_3 and a circle of radius 2 centered at b . Given z , we find w , as an intersection point of a unit circle centered at z and a circle of radius 2 centered at c_2 . From w we can find a , as an intersection point of C_1 and a unit circle centered at w . Altogether, there are only $O(1)$ choices for a , as claimed.

N1D4: Here we have $\|wc_2\| = \|w'c_2\| = 2$. Here each $a \in S_1$ can be coupled with only

$O(1)$ b 's. Indeed, given a , we find w , as an intersection point of the unit circle centered at a and a circle of radius 2 centered at c_2 . From w we find z , as in the standard procedure, and from z we find w' , as an intersection point of the unit circle centered at z and a circle of radius 2 centered at c_2 . From w' we find b , as an intersection point of C_1 with the unit circle centered at w' . Altogether, there are only $O(1)$ choices for b , as claimed.

N2D1: Here we have $\|a\xi\| = \|az\| = 2$. Since (a, ξ, z) span a unit circle, we must have $\xi = z$, which is thus an intersection point of C_2 and C_3 . This allows us to reconstruct a , essentially as in case N1D2.

N2D3: Here we have $\|az\| = \|zb\| = 2$. This case is easy: Given a , we find z , as an intersection point of C_3 and a circle of radius 2 centered at a , and from z we find b , as an intersection point of C_1 and a circle of radius 2 centered at z .

N2D4: Here we have $\|az\| = \|w'c_2\| = 2$. This case is symmetric to case N1D3, and is treated in a fully symmetric manner, starting from a and reconstructing b in $O(1)$ ways.

N3D1: Here we have $\|w'c_3\| = \|a\xi\| = 2$. Given a , we find ξ , as an intersection point of C_2 and a circle of radius 2 centered at a . From a and ξ we find z , as an intersection point of C_3 and the diametral (unit) circle determined by $a\xi$. From z we find w' , as the point that lies on the line through c_3 and z at distance 1 from z (and 2 from c_3). From w' we find b , as an intersection point of C_1 and the unit circle centered at w' .

N3D2: Here we have $\|w'c_3\| = \|wc_3\| = 2$. This case is treated exactly as case N1D4, except that here c_3 plays the role that was played there by c_2 .

N3D4: Here we have $\|w'c_3\| = \|w'c_2\| = 2$, so this is a symmetric version of case N1D2, with an essentially identical reconstruction process.

N4D1: Here we have $\|b\eta\| = \|a\xi\| = 2$. Given a , we find ξ , as an intersection point of C_2 with a circle of radius 2 centered at a . We then find z , as an intersection of C_3 with the diametral (unit) circle determined by $a\xi$. In complete analogy with the treatment of case (iii) of the standard reconstruction process (at an extremal point of γ'), we note that the quadrilateral $R = c_1b\eta c_2$ has edges of fixed lengths, namely, 1, 2, 1, and $\|c_1c_2\|$, and that it can flex around its fixed edge c_1c_2 . As R flexes, the midpoint of $b\eta$ traces an algebraic curve τ of some constant degree, and the intersection(s) of τ with the unit circle C_z centered at z gives us the center(s) w' of the unit circle spanned by (b, η, z) , from which b is readily obtained, as in several preceding cases. If C_z and τ do not overlap, the number of intersection points between them is finite and bounded by a constant, and there are only $O(1)$ way to reconstruct b .

As in case (iii), the situation where C_z and τ do overlap can happen only when C_1 and C_2 are tangent to each other, and z is this tangency point (see Figure 7(b)). In this case it is possible to have a superlinear (in fact, any) number of pairs (a, b) , for which there exists $(\xi, \eta) \in C_2 \times C_2$, such that $\|a\xi\| = \|b\eta\| = 2$, and (a, ξ, z) , (b, η, z) are unit triples. We therefore do not exclude those pairs as ultra-degenerate. Instead, we claim that any such pair can be reconstructed from γ' , using the reconstruction process described in Section 2.2. For this, we note that, for (a, b) fixed, there exists at most one point (ξ, η) , such that $\|a\xi\| = \|b\eta\| = 2$ (and (a, ξ, z) , (b, η, z) are unit triples). Since, in the proof of Proposition 1, we trace γ' from the point q_0 , in both increasing and decreasing v_x -direction, we will not encounter this degeneracy in at least one of these traversals, and reach a locally x - or y -extremal point $(v_\xi, v_\eta) \in \gamma'$, from which (a, b) can be reconstructed (if not excluded

in one of the other ultra-degenerate cases), in at most a constant number of ways.

N4D2: Here we have $\|b\eta\| = \|wc_3\| = 2$. This case is symmetric to case N3D1, and is treated in an analogous manner, starting from b and reconstructing a in $O(1)$ ways.

N4D3: Here we have $\|b\eta\| = \|bz\| = 2$. This is a symmetric variant of case N2D1, with an essentially identical treatment.

In summary, we have shown that the overall number of ultra-degenerate pairs is $O(n)$, as claimed.