

Incidences between points and lines in \mathbb{R}^{4*}

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Abstract

We show that the number of incidences between m distinct points and n distinct lines in \mathbb{R}^4 is $O\left(2^{c\sqrt{\log m}}(m^{2/5}n^{4/5} + m) + m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + n\right)$, for a suitable absolute constant c , provided that no 2-plane contains more than s input lines, and no hyperplane or quadric contains more than q lines. The bound holds without the factor $2^{c\sqrt{\log m}}$ when $m \leq n^{6/7}$ or $m \geq n^{5/3}$. Except for this factor, the bound is tight in the worst case.

Keywords. Combinatorial geometry, incidences, the polynomial method, algebraic geometry, ruled surfaces.

1 Introduction

Let P be a set of m distinct points in \mathbb{R}^4 and let L be a set of n distinct lines in \mathbb{R}^4 . Let $I(P, L)$ denote the number of incidences between the points of P and the lines of L ; that is, the number of pairs (p, ℓ) with $p \in P$, $\ell \in L$, and $p \in \ell$. If all the points of P and all the lines of L lie in a common plane, then the classical Szemerédi–Trotter theorem [40] yields the worst-case tight bound

$$I(P, L) = O\left(m^{2/3}n^{2/3} + m + n\right). \quad (1)$$

This bound clearly also holds in \mathbb{R}^4 (or in any other dimension), by projecting the given lines and points onto some generic plane. Moreover, the bound will continue to be worst-case tight by placing all the points and lines in a common plane, in a configuration that yields the planar lower bound.

In the recent groundbreaking paper of Guth and Katz [14], an improved bound has been derived for $I(P, L)$, for a set P of m points and a set L of n lines in \mathbb{R}^3 , provided that not too many lines of L lie in a common plane¹. Specifically, they showed:

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¹The additional requirement in [14], that no regulus contains too many lines, is not needed for the incidence bound given below.

Theorem 1.1 (Guth and Katz [14]). *Let P be a set of m distinct points and L a set of n distinct lines in \mathbb{R}^3 , and let $s \leq n$ be a parameter, such that no plane contains more than s lines of L . Then*

$$I(P, L) = O\left(m^{1/2}n^{3/4} + m^{2/3}n^{1/3}s^{1/3} + m + n\right). \quad (2)$$

This bound is tight in the worst case.

In this paper, we establish the following analogous and sharper result in four dimensions.

Theorem 1.2. *Let P be a set of m distinct points and L a set of n distinct lines in \mathbb{R}^4 , and let $q, s \leq n$ be parameters, such that (i) each hyperplane or quadric contains at most q lines of L , and (ii) each 2-flat contains at most s lines of L . Then*

$$I(P, L) \leq 2^{c\sqrt{\log m}} \left(m^{2/5}n^{4/5} + m\right) + A \left(m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + n\right), \quad (3)$$

where A and c are suitable absolute constants. When $m \leq n^{6/7}$ or $m \geq n^{5/3}$, we get the sharper bound

$$I(P, L) \leq A \left(m^{2/5}n^{4/5} + m + m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + n\right). \quad (4)$$

In general, except for the factor $2^{c\sqrt{\log m}}$, the bound is tight in the worst case, for any values of m, n , with corresponding suitable ranges of q and s .

The proof of Theorem 1.2 will be by induction on m . To facilitate the inductive process, we extend the theorem as follows. We say that a hyperplane or a quadric H in \mathbb{R}^4 is q -restricted for a set of lines L and for an integer parameter q , if there exists a (trivariate) polynomial g_H , defined on H and of degree at most $O(\sqrt{q})$, such that each of the lines of L that is contained in H , except for at most q lines, is contained in some irreducible component of $Z(g_H)$ (possibly a 2-flat) that is ruled by lines (see below for details). In other words, a q -restricted hyperplane or quadric contains in principle at most q lines of L , but it can also contain an unspecified number of additional lines, all fully contained in ruled components of the zero set of some polynomial of degree $O(\sqrt{q})$. We then have the following more general result.

Theorem 1.3. *Let P be a set of m distinct points and L a set of n distinct lines in \mathbb{R}^4 , and let q and $s \leq n$ be parameters, such that (i') each hyperplane or quadric is q -restricted, and (ii) each 2-flat contains at most s lines of L . Then,*

$$I(P, L) \leq 2^{c\sqrt{\log m}} \left(m^{2/5}n^{4/5} + m\right) + A \left(m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + n\right), \quad (5)$$

where the parameters A and c are as in Theorem 1.2. As in the preceding theorem, when $m \leq n^{6/7}$ or $m \geq n^{5/3}$, we get the sharper bound

$$I(P, L) \leq A \left(m^{2/5}n^{4/5} + m + m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + n\right). \quad (6)$$

Moreover, except for the factor $2^{c\sqrt{\log m}}$, the bound is tight in the worst case, as above.

The requirement that a hyperplane or quadric H be q -restricted extends (i.e., is a weaker condition than) the simpler requirement that H contain at most q lines of L . Hence, Theorem 1.2 is an immediate corollary of Theorem 1.3.

A few remarks are in order.

(a) Only the range $\sqrt{n} \leq m \leq n^2$ is of interest; outside this range, regardless of the dimension of the ambient space, we have the well known and trivial upper bound $I(P, L) = O(m + n)$, an immediate consequence of (1).

(b) The term $m^{1/2}n^{1/2}q^{1/4}$ comes from the bound of Guth and Katz [14] in three dimensions (as in Theorem 1.1), and is unavoidable, as it can be attained if we densely “pack” points and lines into hyperplanes, in patterns that realize the bound in three dimensions within each hyperplane; see Section 4 for details.

(c) Likewise, the term $m^{2/3}n^{1/3}s^{1/3}$ comes from the planar Szemerédi–Trotter bound (1), and is too unavoidable, as it can be attained if we densely pack points and lines into 2-planes, in patterns that realize the bound in (1); again, see Section 4.

(d) Ignoring these terms, and the term n , which is included only to cater for the case $m < \sqrt{n}$, the two terms $m^{2/5}n^{4/5}$ and m “compete” for dominance; the former dominates when $m = O(n^{4/3})$ and the latter when $m = \Omega(n^{4/3})$. Thus the bound in (5) is qualitatively different within these two ranges.

(e) The threshold $m = n^{4/3}$ also arises in the related problem of *joints* (points incident to at least four lines not in a common hyperplane) in a set of n lines in 4-space; see [21, 27], and a remark below.

By a standard argument, the theorem implies the following corollary.

Corollary 1.4. *Let L be a set of n lines in \mathbb{R}^4 , satisfying the assumptions (i') and (ii) in Theorem 1.3, for given parameters q and s . Then, for any $k = \Omega(2^{c\sqrt{\log n}})$, the number $m_{\geq k}$ of points incident to at least k lines of L satisfies*

$$m_{\geq k} = O\left(\frac{2^{\frac{4}{3}c\sqrt{\log n}}n^{4/3}}{k^{5/3}} + \frac{nq^{1/2}}{k^2} + \frac{ns}{k^3} + \frac{n}{k}\right).$$

Remarks. (i) It is instructive to compare Corollary 1.4 with the analysis of *joints* in a set L of n lines. As just mentioned, in \mathbb{R}^d , a joint of L is a point incident to at least d lines of L , not all in a common hyperplane. As shown in [21, 27], the maximum number of joints of such a set is $O(n^{d/(d-1)})$, and this bound is worst-case tight. In four dimensions, this bound is $O(n^{4/3})$, which corresponds to the numerator of the first term of the bound in Corollary 1.4.

(ii) The other terms cater to configurations involving co-hyperplanar or coplanar lines. For example, when $q = n$, the second term is $O(n^{3/2}/k^2)$, in accordance with the bound obtained in Guth and Katz [14] in three dimensions, and when $s = n$, the third and fourth terms comprise (an equivalent formulation of) the bound (1) of Szemerédi and Trotter [40] for the planar case.

(iii) A major interesting and challenging problem is to extend the bound of Corollary 1.4 for any value of k . In particular, is it true that the number of intersection points of the lines (this is the case $k = 2$) is $O(n^{4/3} + nq^{1/2} + ns)$? We conjecture that this is indeed the case.

(iv) Before the question in (iii) can be tackled, we first need to improve our bound, so as to get rid of, or at least reduce the factor $2^{c\sqrt{\log m}}$. As stated in the theorems, this can be achieved when $m \leq n^{6/7}$ or $m \geq n^{5/3}$.

Additional remarks and open issues are given in the concluding Section 5.

Background. Incidence problems have been a major topic in combinatorial and computational geometry for the past thirty years, starting with the Szemerédi–Trotter bound [40] back in 1983. Several techniques, interesting in their own right, have been developed, or adapted, for the analysis of incidences, including the crossing-lemma technique of Székely [39], and the use of cuttings as a divide-and-conquer mechanism (e.g., see [3]). Connections with range searching and related problems in computational geometry have also been noted, and studies of the Kakeya problem

(see, e.g., [41]) indicate the connection between this problem and incidence problems. See Pach and Sharir [25] for a comprehensive (albeit a bit outdated) survey of the topic.

The landscape of incidence geometry has dramatically changed in the past six years, due to the infusion, in two groundbreaking papers by Guth and Katz [13, 14] (the first of which was inspired by a similar result of Dvir [6] for finite fields), of new tools and techniques drawn from algebraic geometry. Although their two direct goals have been to obtain a tight upper bound on the number of joints in a set of lines in three dimensions [13], and an almost tight lower bound for the classical distinct distances problem of Erdős [14], the new tools have quickly been recognized as useful for incidence bounds of various sorts. See [9, 19, 20, 33, 38, 44, 45] for a sample of recent works on incidence problems that use the new algebraic machinery.

The simplest instances of incidence problems involve points and lines. Szemerédi and Trotter completely solved this special case in the plane [40]. Guth and Katz’s second paper [14] provides a worst-case tight bound in three dimensions, under the assumption that no plane contains too many lines; see Theorem 1.1. Under this assumption, the bound in three dimensions is significantly smaller than the planar bound (unless one of m, n is significantly smaller than the other), and the intuition is that this phenomenon should also show up as we move to higher dimensions. Unfortunately, the analysis becomes more involved in higher dimensions, and requires the development or adaptation of progressively more complex tools from algebraic geometry. Most of these tools still appear to be unavailable, and their absence leads either to interesting (new) open problems in the area, or to the need to adapt existing machinery to fit into the new context.

The present paper is a first step in this direction, which considers the four-dimensional case. It does indeed derive a sharper bound, assuming that the configuration of points and lines is “truly four-dimensional”, in the precise sense spelled out in Theorems 1.2 and 1.3.

We also note that studying incidence problems in four (or higher) dimensions has already taken place in several contemporary works, such as in Solymosi and Tao [38], Zahl [45], and Basu and Sombra [1] (and in work in progress by Solymosi and de Zeeuw). These works, though, consider incidences with higher-dimensional varieties, and the study of incidences involving lines, presented in this paper, is new. (There are several ongoing studies, including a companion work joint with Sheffer, that aim to derive weaker but more general bounds involving incidences between points and curves in higher dimensions.)

Our study of point-line incidences in four dimensions has lead us to adapt more advanced tools in algebraic geometry, such as tools involving surfaces that are ruled by lines or by flats, including Severi’s 1901 work [32], as well as the more recent works of Landsberg [17, 23] on osculating lines and flats to algebraic surfaces in higher dimensions.

In a preliminary version of this study [34], we have obtained a weaker and more constrained bound. A discussion of the significant differences between this preliminary work and the present one is given in the overview of the proof, which comes next.

Overview of the proof. ² The analysis follows the general approach of Guth and Katz [14], albeit with many significant adaptations and modifications. We use induction on $m = |P|$, but we begin the description by ignoring this aspect (for a while). We apply the polynomial partitioning technique of Guth and Katz [14], with some polynomial $f \in \mathbb{R}[x, y, z, w]$ of suitable degree D , and obtain a partition of \mathbb{R}^4 into $O(D^4)$ cells, each containing at most $O(m/D^4)$ points of P .

²In this overview we assume some familiarity of the reader with the new “polynomial method” of Guth and Katz, and with subsequent applications thereof. Otherwise, the overview can be skipped on first reading.

In our first phase, we use

$$D = O(m^{2/5}/n^{1/5}), \quad \text{for } m = O(n^{4/3}), \quad \text{and} \quad D = O(n/m^{1/2}), \quad \text{for } m = \Omega(n^{4/3}). \quad (7)$$

There are three types of incidences that may arise: an incidence between a point in some cell of the partition and a line crossing that cell, an incidence between a point on the zero set $Z(f)$ of f and a line not fully contained in $Z(f)$, and an incidence between a point on $Z(f)$ and a line fully contained in $Z(f)$. The above choices of D make it a fairly easy task to bound the number of incidences of the first two types, and the hard part is to estimate the number of incidences of the third kind, as we have no control on the number of points and lines contained in $Z(f)$ —in the worst case all the points and lines could be of this kind.

At the “other end of the spectrum,” choosing D to be a constant (as done in our preliminary aforementioned study of this problem [34] and in other recent studies of related problems [12, 33, 38]) simplifies considerably the handling of incidences on $Z(f)$, but then the analysis of incidences within the cells of the partition becomes more involved, as the sizes of the subproblems within each cell are too large. In the works just cited (as well as in this paper), this is handled via induction, but the price of a naive inductive approach is three-fold: First, the bound becomes weaker, involving additional factors of the form $O(m^\varepsilon)$, for any $\varepsilon > 0$ (with a constant of proportionality that depends on ε). Second, the requirement that no hyperplane or quadric contains more than q lines of L has to be replaced by the much more restrictive requirement, that no variety of degree at most c_ε contains more than q input lines, where c_ε is a (fairly large) constant that depends on ε (and becomes larger as ε gets smaller). Finally, the sharp “lower-dimensional” terms, such as $m^{1/2}n^{1/2}q^{1/4}$ and $m^{2/3}n^{1/3}s^{1/3}$ in our case (recall that both are worst-case tight), do not pass through the induction successfully, so they have to be replaced by weaker terms; see the preliminary version [34] for such weaker terms, and [33] for a similar phenomenon in a different incidence problem in three dimensions.

We note that a recent study by Guth [12] reexamines the point-line incidence problem in \mathbb{R}^3 and presents an alternative and simpler analysis (than the original one in [14]), in which he uses a constant-degree partitioning polynomial, and manages to handle successfully the relevant lower-dimensional term $m^{2/3}n^{1/3}s^{1/3}$ through the induction, but the analysis still incurs the extra m^ε factors in the bound, and needs the restrictive assumption that no algebraic surface of some large constant maximum degree c_ε contains too many lines. In a companion ongoing work [35], we provide yet another simpler derivation (which is somewhat sharper than Guth’s) of such an incidence bound in three dimensions.

Our approach is to use two different choices of the degree of the partitioning polynomial. We first choose the large value of D specified above, and show that the bound in the right-hand side of (5) accounts for the incidences within the partition cells, and for most of the cases involving incidences between points and lines on the zero set $Z(f)$. We are then left with “problematic” subsets of points and lines on $Z(f)$, which are difficult to analyze when the degree is large. (Informally, this happens when the lines lie in certain ruled two-dimensional subvarieties of $Z(f)$.) To handle them, we retain only these subsets, discard the partitioning, and start afresh with a new partitioning polynomial of a much smaller, albeit still non-constant degree. As the degree is now too small, we need induction to bound the number of incidences within the partition cells. A major feature that makes the induction work well is that the first partitioning step ensures that the surviving set of lines that is passed to the induction is such that each hyperplane or quadric is now $O(D^2)$ -restricted, with respect to the set of surviving lines, and each 2-flat contains at most $O(D)$ lines of that set (where D is the large degree used in the first step). As a consequence, the induction works better, and “retains” the lower-dimensional terms $m^{1/2}n^{1/2}q^{1/4}$ and $m^{2/3}n^{1/3}s^{1/3}$. (In fact, it does not touch them at all, because q and s are not passed to the induction step.) We still pay a small price for this approach, involving the extra factor $2^{c\sqrt{\log m}}$ in the “leading terms” $m^{2/5}n^{4/5} + m$ (but not in

the “lower-dimensional” terms). When m is “not too close to” $n^{4/3}$, as specified in the theorems, induction is not needed, and a direct analysis yields the sharper bound in (6), without this extra factor.

The idea of using a “small” degree for the partitioning polynomial is not new, and has been applied also in [33, 45]. However, the induction process in [33] results in weaker lower-dimensional terms, which we avoid here with the use of two different partitionings. We note that we have recently applied this approach in the aforementioned “warm-up” study of point-line incidences in three dimensions [35], with a simpler analysis (than that in [12, 14]) and an improved bound (than the one in [12]).

The main part of the analysis is still in handling incidences within $Z(f)$ in the first partitioning step, where the degree of f is large. (Similar issues arise in the second step too, but the bounds there are generally sharper than those obtained in the first step.) This is done as follows. We first ignore the singular points on $Z(f)$. They will be handled separately, as points lying on the zero sets of polynomials of smaller degree (namely, partial derivatives of f). We also assume that f is irreducible, by considering each irreducible factor of the original f separately (see Section 3 for details). This step results in a *partition* of the points of P and the lines of L among several varieties, each defined by an irreducible factor of f or of some derivative of f , so that it suffices to bound the number of incidences between points and lines assigned to the *same* variety. The number of “cross-variety” incidences is shown to be only $O(nD)$.

We next define (a four-dimensional variant of) the *flecnode polynomial* $g := \text{FL}_f^4$ of f (see Salmon [30] for the more classical three-dimensional variant which is used in Guth and Katz [13, 14]), which vanishes at those points $p \in Z(f)$ that are incident to a line that *osculates* to $Z(f)$ (i.e., agrees with $Z(f)$ near p) up to order four (and in particular to lines that are fully contained in $Z(f)$); see below for precise definitions. We show that $g = \text{FL}_f^4$ is a polynomial of degree $O(D)$. If $g \equiv 0$ on $Z(f)$ then $Z(f)$ is ruled by lines (as follows from Landsberg’s work [23], which provides a generalization of the classical Cayley–Salmon theorem [14, 30]). We handle this case by first reducing it to the case where $Z(f)$ is “infinitely ruled” by lines, meaning that most of its points are incident to infinitely many lines that are contained in $Z(f)$ (otherwise, we can show, using Bézout’s theorem, that most points are incident to at most 6 lines, for a total of $O(m)$ incidences), and then by using the aforementioned result of Severi [32] from 1901, which shows that in this case $Z(f)$ is ruled by 2-flats (each point on $Z(f)$ is incident to a 2-flat that is fully contained in $Z(f)$), unless $Z(f)$ is a hyperplane or a quadric. This allows us to reduce the problem to several planar incidence problems, which are reasonably easier to handle.

The other case is where the common zero set $Z(f, g)$ of f and g is two-dimensional. In this case, we decompose $Z(f, g)$ into its irreducible components, and show that the number of incidences between points of P and lines fully contained in irreducible components that are not 2-flats is

$$\min \{O(mD^2 + nD), O(m + nD^4)\}. \quad (8)$$

Both terms are too large for the standard “large” values of D , but they are non-trivial to establish, and are useful tools for slightly improving the bound and simplifying the analysis considerably when D is not too large—see below. The derivation of these bounds is based on a new study of point-line incidences within ruled two-dimensional varieties in 3-space, provided in a companion paper [36].

The irreducible components that are 2-flats are harder to handle, because their number can be $O(D^2)$ (as follows from Fulton’s generalization of Bézout’s theorem [11]), a number that turns out to be too large for the purpose of our incidence bound, when a naive analysis (with a large value of D) is used, so some care is needed in this case. The difficult step in this part is when there are many points, each contained in at least three (and in general many) 2-flats fully contained in

$Z(f, g)$ (and thus in $Z(f)$). Non-singular points of this kind are called *linearly flat* points of $Z(f)$, naturally generalizing Guth and Katz’s notion of linearly flat points in \mathbb{R}^3 [14] (see also Kaplan et al. [9]). Linearly flat points are also *flat* points, i.e., points where the *second fundamental form* of $Z(f)$ vanishes (e.g., see Pressley [26]). Flatness of a point p can be expressed, again by a suitable generalization to four dimensions of the techniques in [9, 14], by the vanishing of nine polynomials, each of degree $\leq 3D - 4$, at p , which are constructed from f and from its first and second-order derivatives. The problem can then be reduced to the case where all the points and lines are flat (a line is flat, when not all of its points are singular points of $Z(f)$, and all of its non-singular points are flat). With a careful (and somewhat intricate) probing into the geometric properties of flat lines, we can bound the number of incidences with flat lines by reducing the problem into several incidence problems in three dimensions (specifically, within hyperplanes tangent to $Z(f)$ at the flat points), and then using an extension of Guth and Katz’s bound (2) for each of these problems, where, in this application, we exploit the fact that each hyperplane contains at most q lines, to obtain a better, q -dependent bound.

However, as noted, the terms $O(mD^2)$ (when $n^{6/7} \leq m \leq n^{4/3}$) and $O(nD^4)$ (when $n^{4/3} \leq m \leq n^{5/3}$) are too large (for the choices of our “large” values of D in (7)). We retain them for the second partitioning step, when the degree of the partitioning polynomial is smaller, but finesse them, for the large D , by showing that, after pruning away points and lines whose incidences can be estimated directly (within the bound (6), not using the weaker bounds of (8)), we are left with subsets for which every hyperplane or quadric is $O(D^2)$ -restricted, and each 2-flat contains at most $O(D)$ lines. However, when $m \leq n^{6/7}$ or $m \geq n^{5/3}$, there is no need for this part of the analysis, and a direct application of the bounds in (8) yields the sharper bound in (6) and simplifies the proof considerably.

For the remaining range of m and n , we apply these weaker bounds in the second partitioning step, when the degree of the partitioning polynomial is smaller, but finesse them, for the large D , by showing that, after pruning away points and lines whose incidences can be estimated directly (within the bound in (5)), we are left with subsets for which every hyperplane or quadric is $O(D^2)$ -restricted, and each 2-flat contains at most $O(D)$ lines.

In the general case, we then go on to our second partitioning step. We discard f and start afresh with a new partitioning polynomial h of degree $E \ll D$. As already noted, bounding incidences within the partition cells becomes non-trivial, and we use induction, exploiting the fact that now the parameters q and s are replaced by $O(D^2)$ and $O(D)$, respectively. On the flip side of the coin, bounding incidences within $Z(h)$ is now simpler, because E is smaller, and we can use the bounds in (8) (i.e., $O(mE^2 + nE)$ or $O(m + nE^4)$) to establish the bound in (5) for the “problematic” incidences.

The reason for using the weaker requirement that each hyperplane and quadric be q -restricted, instead of just requiring that no hyperplane or quadric contain more than q lines of L , is that we do not know how to bound the overall number of lines in a hyperplane or quadric H by $O(D^2)$, because of the potential existence of ruled components of $Z(f, g)$ within H , which can accommodate any number of lines. A major difference between this case and the analysis of ruled components in Guth and Katz’s study [14] is that here the overall degree of $Z(f, g)$ is $O(D^2)$, as opposed to the degree of $Z(f)$ being only D in [14]. This precludes the application of the techniques of Guth and Katz to our scenario—they would lead to bounds that are too large.

We also note that our analysis of incidences within $Z(f)$ is actually carried out (in the projective 4-space) over the complex field, which makes it simpler, and facilitates the application of numerous tools from algebraic geometry that are developed in this setting. The passage from the complex projective setup back to the real affine one is straightforward—the former is a generalization of the latter. The real affine setup is needed only for the construction of a polynomial partitioning,

which is meaningless over \mathbb{C} . Once we are within the variety $Z(f)$, we can switch to the complex projective setup, which facilitates the application of many techniques from algebraic geometry.

Note that, in spite of these improvements, Theorem 1.3 still has the peculiar feature, which is not needed in Guth and Katz [14] (for the incidence bound of Theorem 1.1), that also requires that every *quadric* be q -restricted (or, in the simpler version in Theorem 1.2, contains at most q lines of L).³ In a recent work in progress, Solomon and Zhang [37] show that this requirement cannot be dropped, by providing a construction of a quadric that contains many points and lines, where the number of incidences between them is significantly larger than the bound in (5).

2 Algebraic Preliminaries

In this section we collect and adapt a large part of the machinery from algebraic geometry that we need for our analysis.

In what follows, to facilitate the application of standard techniques in algebraic geometry, it will be more convenient to work over the complex field \mathbb{C} , and in complex projective spaces. We do so even though Theorem 1.3 is stated (and will be proved) for the real affine case. The passage between the two scenarios, in the proof of the theorem, will be straightforward, as discussed in the preceding overview. Concretely, the realness of the underlying field is needed only for the partitioning step itself, which has no (simple) parallel over \mathbb{C} . However, after reducing the problem to points and lines contained in $Z(f)$, it is more convenient to carry out the analysis over \mathbb{C} , to allow us to apply the algebraic machinery that we are going to present next.

2.1 Lines on varieties

We begin with several basic notions and results in differential and algebraic geometry that we will need (see, e.g., Ivey and Landsberg [17], and Landsberg [23] for more details). For a vector space V (over \mathbb{R} or \mathbb{C}), let $\mathbb{P}V$ denote its projectivization. That is, $\mathbb{P}V = V \setminus \{0\} / \sim$, where $v \sim w$ iff $w = \alpha v$ for some non-zero constant α .

An algebraic variety is the common zero set of a finite collection of polynomials. We call it affine, if it is defined in the affine space, or projective, if it is defined in the projective space, in terms of homogeneous polynomials. For an (affine) algebraic variety X , and a point $p \in X$, let $T_p X$ denote the (affine) tangent space of X at the point p . A point p is *non-singular* if $\dim T_p X = \dim X$ (see Hartshorne [16, Definition I.5] and Hartshorne [16, Theorem I.5.1]). For a point $p \in X$, let Ξ_p denote the union of the (complex) lines passing through p and contained in X , and let Σ_p denote the set of the directions (considered as points in $\mathbb{P}T_p X$) of these lines (here X is implicit in these notations). In Hartshorne [16, Ex.I.2.10], Ξ_p is also called the (*affine*) cone over Σ_p . Clearly, $\Xi_p \subseteq T_p X$.

Consider the special case where X is a hypersurface in \mathbb{P}^4 , i.e., $X = Z(f)$, for a non-linear polynomial $f \in \mathbb{C}[x, y, z, w]$, which we assume to be irreducible, where

$$Z(f) = \{p \in \mathbb{P}^4 \mid f(p) = 0\}$$

is the *zero set* of $Z(f)$. A line $\ell_v = \{p + tv \mid t \in \mathbb{C}\}$ passing through p in direction v is said to *osculate* to $Z(f)$ to order k at p , if the Taylor expansion of f around p in direction v vanishes to

³This is not quite the case: Guth and Katz also require that no *regulus* contains more than s (actually, \sqrt{n}) lines, but this is made to bound the number of points incident to just two lines, and is not needed for the incidence bound in Theorem 1.1.

order k , i.e., if $f(p) = 0$, and

$$\nabla_v f(p) = 0, \quad \nabla_v^2 f(p) = 0, \quad \dots, \quad \nabla_v^k f(p) = 0, \quad (9)$$

where $\nabla_v f$ (which for uniformity we also denote as $\nabla_v^1 f$), $\nabla_v^2 f, \dots, \nabla_v^k f$ are, respectively, the first, second, and higher order derivatives of f , up to order k , in direction v (where v is regarded as a vector in projective 3-space, and the derivatives are interpreted in a scale-invariant manner—we only care whether they vanish or not). That is, $\nabla_v f = \nabla f \cdot v$, $\nabla_v^2 f = v^T H_f v$, where H_f is the *Hessian matrix* of f , and $\nabla_v^i f$ is similarly defined, for $i > 2$, albeit with more complicated explicit expressions. For simplicity of notation, put $F_i(p; v) := \nabla_v^i f(p)$, for $i \geq 1$.

In fact, one can extend the definition of osculation of lines to arbitrary varieties in any dimension (see, e.g., Ivey and Landsberg [17]). For a variety X , a point $p \in X$, and an integer $k \geq 1$, let $\Sigma_p^k \subset \mathbb{P}T_p X$ denote the variety of the directions of the lines that pass through p and osculate to X to order k at p ; we represent the directions in Σ_p^k as points in the corresponding projective space. For each $k \in \mathbb{N}$, there is a natural inclusion $\Sigma_p \subseteq \Sigma_p^k$. In analogy with the previous notation, we denote by Ξ_p^k the union of the lines that pass through p with directions in Σ_p^k . We let $F(X)$ denote the variety of lines (fully) contained in X ; this is known as the *Fano variety* of X , and it is a subvariety of the $(2d - 2)$ -dimensional *Grassmannian manifold* of lines in $\mathbb{P}^d(\mathbb{C})$; see [15, Lecture 6, page 63] for details, and [15, Example 6.19] for an illustration, and for a proof that this is indeed a variety. We will sometimes denote $F(X)$ also as Σ (or $\Sigma(X)$), to conform with the notation involving osculating lines. We also let Σ^k denote the variety of the lines osculating to order k at some point of X . Here too Σ^k can be shown to be a variety (within the Grassmannian manifold) and $F(X) \subseteq \Sigma^k$ for each k . We also have, for any $p \in Z(f)$, $\Sigma_p \subseteq F(X)$ and $\Sigma_p^k \subseteq \Sigma^k$.

Genericity. We recall that a property is said to hold *generically* (or *generally*) for polynomials f_1, \dots, f_n , of some prescribed degrees, if there are nonzero polynomials g_1, \dots, g_k in the coefficients of the f_i 's, such that the property holds for all f_1, \dots, f_n for which none of the polynomials g_j is zero (see, e.g., [5, Definition 3.6]). In this case we say that the collection f_1, \dots, f_n is *general* or *generic*, with respect to the property in question, namely, with respect to the vanishing of the polynomials g_1, \dots, g_k that define that property.

2.2 Generalized Bézout's theorem

An affine variety $X \subset \mathbb{C}^d$ is called *irreducible* if, whenever V is written in the form $V = V_1 \cup V_2$, where V_1 and V_2 are affine varieties, then either $V_1 = V$ or $V_2 = V$.

Theorem 2.1 (Cox et al. [4, Theorem 4.6.2]). *Let V be an affine variety. Then V can be written as a finite union*

$$V = V_1 \cup \dots \cup V_m,$$

where V_i is an irreducible variety, for $i = 1, \dots, m$.

If one also requires that $V_i \not\subseteq V_j$ for $i \neq j$, then this decomposition is unique, up to a permutation (see, e.g., [4, Theorem 4.6.4]), and is called the *minimal decomposition* of V into irreducible components.

We next state a generalized version of Bézout's theorem, as given in Fulton [11]. It will be a major technical tool in our analysis.

Theorem 2.2 (Fulton [11, Proposition 2.3]). *Let V_1, \dots, V_s be subvarieties of \mathbb{P}^d , and let Z_1, \dots, Z_r be the irreducible components of $\bigcap_{i=1}^s V_i$. Then*

$$\sum_{i=1}^r \deg(Z_i) \leq \prod_{j=1}^s \deg(V_j).$$

A simple application of Theorem 2.2 yields the following useful result.

Lemma 2.3. *A curve $\mathcal{C} \subset \mathbb{P}^4$ of degree D can contain at most D lines.*

Proof. Let t denote the number of these lines, and let $\mathcal{C}_0 \subset \mathcal{C}$ denote their union. Intersect \mathcal{C}_0 with a generic hyperplane H . By Theorem 2.2, the number of intersection points satisfies

$$t \leq \deg(\mathcal{C}_0) \cdot \deg(H) \leq \deg(\mathcal{C}) \cdot 1 = D,$$

as asserted. □

This immediately yields the following result, derived in Guth and Katz [13] (see also [9]) in a somewhat different manner.

Corollary 2.4. *Let f and g be two trivariate polynomials without a common factor. Then $Z(f, g) := Z(f) \cap Z(g)$ contains at most $\deg(f) \cdot \deg(g)$ lines.*

Proof. This follows since $Z(f, g)$ is a curve of degree at most $\deg(f) \cdot \deg(g)$. □

2.3 Generically finite morphisms and the Theorem of the Fibers

The following results can be found, e.g., in Harris [15, Chapter 11].

For a map $\pi : X \rightarrow Y$ of projective varieties, and for $y \in Y$, the variety $\pi^{-1}(y)$ is called the *fiber* of π over y .

The following result is a slight paraphrasing of Harris [15, Proposition 7.16].

Theorem 2.5 (Harris [15, Proposition 7.16]). *Let $\pi : X \rightarrow Y$ be the restriction of the projection $p : \mathbb{P}^d \rightarrow \mathbb{P}^{d-1}$ to the respective projective varieties $X \subset \mathbb{P}^d$, $Y \subset \mathbb{P}^{d-1}$. For a general point $y \in Y$, the fiber $\pi^{-1}(y)$ is finite if and only if $\dim(X) = \dim(Y)$. In this case, we say that π is generically finite.*

An important technical tool for our analysis is the following so-called Theorem of the Fibers.

Theorem 2.6 (Harris [15, Corollary 11.13]). *Let X be a projective variety and $\pi : X \rightarrow \mathbb{P}^d$ be a polynomial map (i.e., the coordinate functions $x_0 \circ \pi, \dots, x_d \circ \pi$ are homogeneous polynomials); let $Y = \pi(X)$ denote its image. For any $p \in Y$, let $\lambda(p) = \dim(\pi^{-1}(p))$. Then $\lambda(p)$ is an upper semi-continuous function of p in the Zariski topology⁴ on Y ; that is, for any m , the locus of points $p \in Y$ such that $\lambda(p) \geq m$ is closed in Y . Moreover, if $X_0 \subset X$ is any irreducible component, $Y_0 = \pi(X_0)$ its image, and λ_0 the minimum value of $\lambda(p)$ on Y_0 , then*

$$\dim(X_0) = \dim(Y_0) + \lambda_0.$$

⁴The Zariski closure of a set Y is the intersection of all varieties X that contain Y . Y is Zariski closed if it is equal to its closure (and is therefore a variety), and is Zariski open if its complement is Zariski closed. See [16] for further details.

2.4 Flecnode polynomials and ruled surfaces in four dimensions

Ruled surfaces in three dimensions. We first review several basic properties of ruled two-dimensional surfaces in \mathbb{R}^3 or in \mathbb{C}^3 . Most of these results are considered folklore in the literature, although we have been unable to find concrete rigorous proofs (in the “modern” jargon of algebraic geometry). For the sake of completeness we provide such proofs in a companion paper [36].

For a modern approach to ruled surfaces, there are many references; see, e.g., Hartshorne [16, Section V.2], or Beauville [2, Chapter III]. We say that a real (resp., complex) surface X is *ruled by real* (resp., *complex*) *lines* if every point $p \in X$ in a Zariski-open dense set is incident to a real (complex) line that is fully contained in X ; see, e.g., [30] or [7] for details on ruled surfaces. This definition is slightly weaker than the classical definition, where it is required that *every* point of X be incident to a line contained in X (e.g., as in [30]). It has been used in recent works, see, e.g., [14, 18]. Similarly to the proof of Lemma 3.4 in Guth and Katz [14], a limiting argument implies that the two definitions are equivalent.

We note that some care has to be exercised when dealing with ruled surfaces, because ruledness may depend on the underlying field. Specifically, it is possible for a surface defined by real polynomials to be ruled by complex lines, but not by real lines. For example, the sphere defined by $x^2 + y^2 + z^2 - 1 = 0$, regarded as a real variety, is certainly not ruled by lines, but as a complex variety it is ruled by (complex) lines. (Indeed, each point (x_0, y_0, z_0) on the sphere is incident to the (complex) line $(x_0 + \alpha t, y_0 + \beta t, z_0 + \gamma t)$, for $t \in \mathbb{C}$, where $\alpha^2 + \beta^2 + \gamma^2 = 0$ and $\alpha x_0 + \beta y_0 + \gamma z_0 = 0$, which is fully contained in the sphere.)

In three dimensions, a two-dimensional irreducible ruled surface can be either *singly ruled*, or *doubly ruled* (notions that are elaborated below), or a plane. As the following lemma shows, the only doubly ruled surfaces are *reguli*, where a regulus is the union of all lines that meet three pairwise skew lines. There are only two kinds of reguli, both of which are quadrics—hyperbolic paraboloids and hyperboloids of one sheet; see, e.g., Fuchs and Tabachnikov [10] for more details.

The following (folklore) lemma provides a (somewhat stronger than usual) characterization of doubly ruled surfaces; see [36] for a proof.

Lemma 2.7. *Let V be an irreducible ruled surface in \mathbb{R}^3 or in \mathbb{C}^3 which is not a plane, and let $\mathcal{C} \subset V$ be an algebraic curve, such that every non-singular point $p \in V \setminus \mathcal{C}$ is incident to exactly two lines that are fully contained in V . Then V is a regulus.*

When V is an irreducible ruled surface which is neither a plane nor a regulus, it must be *singly ruled*, in the precise sense spelled out in the following lemma (see also [14]); again, see [36] for a proof.

Lemma 2.8. *(a) Let V be an irreducible ruled two-dimensional surface of degree $D > 1$ in \mathbb{R}^3 or in \mathbb{C}^3 , which is not a regulus. Then, except for at most two exceptional lines, the lines that are fully contained in V form a 1-parameter family of generator lines $\ell(t)$, that depend continuously on the (real or complex) parameter t . Moreover, if $t_1 \neq t_2$, and $\ell(t_1) \neq \ell(t_2)$, then there exist sufficiently small and disjoint neighborhoods Δ_1 of t_1 and Δ_2 of t_2 , such that all the lines $\ell(t)$, for $t \in \Delta_1 \cup \Delta_2$, are distinct.*

(b) The set of points of V that are incident to (at least) two generator lines of V is at most one-dimensional.

Following the lemma, we refer to irreducible ruled surfaces that are neither planes nor reguli as *singly ruled*. A line ℓ , fully contained in an irreducible singly ruled surface V , such that any point of ℓ is incident to another line fully contained in V , is called an *exceptional line* of V (these

are the lines mentioned in Lemma 2.8(a)). If there exists a point $p_V \in V$, which is incident to infinitely many lines fully contained in V , then p_V is called an *exceptional point* of V . By Guth and Katz [14], V can contain at most one exceptional point p_V (in which case V is a cone with p_V as its apex), and (as also asserted in the lemma) at most two exceptional lines.

The flecnode polynomial in four dimensions. Let $f \in \mathbb{C}[x, y, z, w]$ be a polynomial of degree $D \geq 4$. A *flecnode* of f is a point $p \in Z(f)$ for which there exists a line that passes through p and osculates to $Z(f)$ to order four at p . Therefore, if the direction of the line is $v = (v_0, v_1, v_2, v_3)$, then it osculates to $Z(f)$ to order four at p if $f(p) = 0$ and

$$F_i(p; v) = 0, \quad \text{for } i = 1, 2, 3, 4. \quad (10)$$

The four-dimensional *flecnode polynomial* of f , denoted FL_f^4 , is the polynomial obtained by eliminating v from the four equations in the system (10). (See Salmon [30], and the relevant applications thereof in [9, 14], for details concerning flecnode polynomials in three dimensions; see also Ivey and Landsberg [17] for a more modern generalization of this concept.) Note that these four polynomials are homogeneous in v (of respective degrees 1, 2, 3, and 4). We thus have a system of four equations in eight variables, which is homogeneous in the four variables v_0, v_1, v_2, v_3 . Eliminating those variables results in a single polynomial equation in $p = (x, y, z, w)$. Using standard techniques, as in Cox et al. [5], the resulting polynomial FL_f^4 is the *multipolynomial resultant* $\text{Res}_4(F_1, F_2, F_3, F_4)$ of F_1, F_2, F_3, F_4 , regarding these as polynomials in v . By definition, FL_f^4 vanishes at all the flecnodes of f . The following results are immediate consequences of the theory of multipolynomial resultants, presented in Cox et al. [5].

Lemma 2.9. *Given a polynomial $f \in \mathbb{C}[x, y, z, w]$ of degree $D \geq 4$, its flecnode polynomial FL_f^4 has degree $O(D)$.*

Proof. The polynomial F_i , for $i = 1, \dots, 4$, is a homogeneous polynomial in v of degree $d_i = i$ over $\mathbb{C}[x, y, z, w]$. By [5, Theorem 4.9], putting $d := \left(\sum_{i=1}^4 d_i\right) - 3 = 7$, the multipolynomial resultant $\text{FL}_f^4 = \text{Res}_4(F_1, F_2, F_3, F_4)$ is equal to $\frac{D_3^3}{D_3^3}$, where D_3 is a polynomial of degree $\binom{d+3}{3} = \binom{10}{3} = 120$ in the coefficients of the polynomials F_i , and D_3' is a polynomial of degree $d_1 d_2 d_3 + d_1 d_2 d_4 + d_1 d_3 d_4 + d_2 d_3 d_4 = 6 + 8 + 12 + 24 = 50$ in these coefficients (see [5, Chapter 3.4, exercises 1,3,6,12,19]). Since each coefficient of any of the polynomials F_i is of degree at most $D - 1$, we deduce that FL_f^4 is of degree at most $O(D)$. \square

Lemma 2.10. *Given a polynomial $f \in \mathbb{C}[x, y, z, w]$ of degree $D \geq 4$, every line that is fully contained in $Z(f)$ is also fully contained in $Z(\text{FL}_f^4)$.*

Proof. Every point on any such line is a flecnode of f , so FL_f^4 vanishes identically on the line. \square

Ruled Surfaces in four dimensions. Flecnode polynomials are a major tool for characterizing ruled surfaces. This is manifested in the following theorem of Landsberg [23], which is a crucial tool for our analysis. It is established in [23] as a considerably more general result, but we formulate here a special instance that suffices for our needs. We note that it extends the classical Cayley–Salmon theorem in three dimensions (see Salmon [30]).

Theorem 2.11 (Landsberg [23]). *Let $f \in \mathbb{C}[x, y, z, w]$ be a polynomial of degree $D \geq 4$. Then $Z(f)$ is ruled by (complex) lines if and only if $Z(f) \subseteq Z(\text{FL}_f^4)$.*

When f is of degree ≤ 3 , we have the following simpler situation.

Lemma 2.12. *For every polynomial $f \in \mathbb{C}[x, y, z, w]$ of degree ≤ 3 , $Z(f)$ is ruled by (possibly complex) lines.*

Proof. Let $v = (v_0, v_1, v_2, v_3) \in \mathbb{C}^4$ be a direction. First notice that for a point $p \in \mathbb{C}^4$, the line through p in direction v is contained in $Z(f)$ if and only if the first three equations in (10) are satisfied, because all the other terms in the Taylor expansion of $f(p + tv)$ always vanish for a polynomial f of degree ≤ 3 . This is a system of three homogeneous polynomials in v_0, v_1, v_2, v_3 , of degrees 1, 2, 3, respectively. By Bézout’s theorem, as stated in Theorem 2.2 below, the number of solutions (complex projective, counted with multiplicities) of this system is either six or infinite, so there is at least one (possibly complex) line that passes through p and is contained in $Z(f)$. \square

Back to three dimensions. In three dimensions the analysis is somewhat simpler, and goes back to the 19th century, in Salmon’s work [30] and others. The flecnode polynomial FL_f of f , defined in an analogous manner, is of degree $11 \deg(f) - 24$ [30]. Theorem 2.11) is replaced by the Cayley–Salmon theorem [30], with the analogous assertion that $Z(f)$ is ruled by lines if and only if $Z(f) \subseteq Z(\text{FL}_f)$. A simple proof of the Cayley–Salmon theorem can be found in Terry Tao’s blog.

We will be using the following result, established by Guth and Katz [13]; see also [9].

Proposition 2.13. *Let f be a trivariate irreducible polynomial of degree D . If $Z(f)$ fully contains more than $11D^2 - 24D$ lines then $Z(f)$ is ruled by (possibly complex) lines.*

Proof. Apply Corollary 2.4 to FL_f and f , to conclude that FL_f and f must have a common factor. Since f is irreducible, this factor must be f itself, and then the Cayley–Salmon theorem implies that $Z(f)$ is ruled. \square

2.5 Flat points and the second fundamental form

Extending the notation in Guth and Katz [13] (see also [9], and also Pressley [26] and Ivey and Landsberg [17] for more basic references), we call a non-singular point p of $Z(f)$ *linearly flat*, if it is incident to at least three distinct 2-flats that are fully contained in $Z(f)$ (and thus also in the tangent hyperplane $T_p Z(f)$). The condition for a point p to be linearly flat can be worked out as follows, suitably extending the technique used in three dimensions in [9, 13]. Although this extension is fairly routine, we are not aware of any previous concrete reference, so we spell out the details for the sake of completeness.

Let p be a non-singular point of $Z(f)$, and let $f^{(2)}$ denote the second-order Taylor expansion of f at p . That is, we have, for any direction vector v and $t \in \mathbb{C}$,

$$f^{(2)}(p + tv) = t \nabla f(p) \cdot v + \frac{1}{2} t^2 v^T H_f(p) v. \quad (11)$$

If p is linearly flat, there exist three 2-flats π_1, π_2, π_3 , contained in the tangent hyperplane $T_p Z(f)$, such that $v^T H_f(p) v = 0$, for all $v \in \pi_1, \pi_2, \pi_3$ (clearly, the first term $\nabla f(p) \cdot v$ also vanishes for any such v). Using a suitable coordinate frame within $T_p Z(f)$, we can regard $v^T H_f(p) v$ as a quadratic trivariate homogeneous polynomial.

Since $v^T H_f(p) v$ vanishes on three 2-flats inside $T_p Z(f)$, a (generic) line ℓ , fully contained in $T_p Z(f)$ and not passing through p , intersects these 2-flats at three distinct points, at which $v^T H_f(p) v$ vanishes. Since this is a quadratic polynomial, it must vanish identically on ℓ . Thus, $v^T H_f(p) v$ is zero for all vectors $v \in T_p Z(f)$, and thus $f^{(2)}$ vanishes identically on $T_p Z(f)$. In this case, we say that p is a *flat point* of $Z(f)$. Therefore, every linearly flat point of $Z(f)$ is also a flat point of $Z(f)$ (albeit not necessarily vice versa).

We next express the set of linearly flat points of $Z(f)$ as the zero set of a certain collection of polynomials. To do so, we define three canonical 2-flats, on which we test the vanishing of the quadratic form $v^T H_f v$. (The preceding analysis shows that, for a linearly flat point, it does not matter which triple of 2-flats is used for testing the linear flatness, as long as they are distinct.) These will be the 2-flats

$$\pi_p^x := T_p Z(f) \cap \{x = x_p\}, \quad \pi_p^y := T_p Z(f) \cap \{y = y_p\}, \quad \text{and} \quad \pi_p^z := T_p Z(f) \cap \{z = z_p\}. \quad (12)$$

These are indeed distinct 2-flats, unless $T_p Z(f)$ is orthogonal to the x -, y -, or z -axis. Denote by $Z(f)_{axis}$ the subset of non-singular points $p \in Z(f)$, for which $T_p Z(f)$ is orthogonal to one of the axes, and assume in what follows that $p \in Z(f) \setminus Z(f)_{axis}$. We can ignore points in $Z(f)_{axis}$ by assuming that the coordinate frame of the ambient space is generic, to ensure that none of our (finitely many) input points has a tangent hyperplane that is orthogonal to any of the axes.

Lemma 2.14. *Let p be a non-singular point of $Z(f) \setminus Z(f)_{axis}$. Then p is a flat point of $Z(f)$ if and only if p is a flat point of each of the varieties $Z(f)|_{\{x=x_p\}}$, $Z(f)|_{\{y=y_p\}}$, $Z(f)|_{\{z=z_p\}}$.*

Proof. Note that the three varieties in the lemma are two-dimensional varieties within the corresponding three-dimensional cross-sections $x = x_p$, $y = y_p$, and $z = z_p$, of 4-space.

If p is a flat point of $Z(f) \setminus Z(f)_{axis}$, then the second-order Taylor expansion $f^{(2)}$ vanishes identically on $T_p Z(f)$. By the assumption on p , we have

$$\begin{aligned} T_p Z(f)|_{\{x=x_p\}} &= T_p Z(f) \cap \{x = x_p\}, \\ T_p Z(f)|_{\{y=y_p\}} &= T_p Z(f) \cap \{y = y_p\}, \quad \text{and} \\ T_p Z(f)|_{\{z=z_p\}} &= T_p Z(f) \cap \{z = z_p\}, \end{aligned}$$

and these are three distinct 2-flats. Therefore, $(f|_{\{x=x_p\}})^{(2)}$ vanishes identically on $T_p Z(f)|_{\{x=x_p\}}$, implying that p is a flat point of $Z(f)|_{\{x=x_p\}}$; similarly p is a flat point of $Z(f)|_{\{y=y_p\}}$ and of $Z(f)|_{\{z=z_p\}}$. For the other direction, notice that if p satisfies the assumptions in the lemma, and is a flat point of each of $Z(f)|_{\{x=x_p\}}$, $Z(f)|_{\{y=y_p\}}$, and $Z(f)|_{\{z=z_p\}}$, then $f^{(2)}$ vanishes on three distinct 2-flats contained in $T_p Z(f)$ (namely, the intersection of $T_p Z(f)$ with $\{x = x_p\}$, $\{y = y_p\}$ and $\{z = z_p\}$). Since $f^{(2)}$ is quadratic, the argument given above implies that it is identically 0 on $T_p Z(f)$. \square

Recall from Elekes et al. [9] that p is flat for $f|_{\{x=x_p\}}$ if and only if $\Pi_j^1 := \Pi_j(f|_{\{x=x_p\}})$ vanishes at p , for $j = 1, 2, 3$, where $\Pi_j(h) = (\nabla h \times e_j)^T H_h (\nabla h \times e_j)$, e_1, e_2, e_3 denote the unit vectors in the respective y -, z -, and w -directions, and the symbol \times stands for the vector product in $\{x = x_p\}$, regarded as a copy of \mathbb{C}^3 . In fact, when x_p is also considered as a variable (call it x then), we get that, as in the three-dimensional case, each of Π_j^1 , for $j = 1, 2, 3$, is a polynomial in x, y, z, w of (total) degree $3D - 4$. Similarly, the analogously defined polynomials $\Pi_j^2 := \Pi_j(f|_{\{y=y_p\}})$, $\Pi_j^3 := \Pi_j(f|_{\{z=z_p\}})$, for $j = 1, 2, 3$, vanish at p if and only if p is a flat point of $f|_{\{y=y_p\}}$ and $f|_{\{z=z_p\}}$. By Lemma 2.14, we conclude that a non-singular point $p \in Z(f) \setminus Z(f)_{axis}$ is flat if and only if $\Pi_j^i(p) = 0$, for $1 \leq i, j \leq 3$.

We say that a line $\ell \subset Z(f)$ is a *singular* line of $Z(f)$, if all of its points are singular. We say that a line $\ell \subset Z(f)$ is a *flat* line of $Z(f)$ if it is not a singular line of $Z(f)$, and all of its non-singular points are flat. An easy observation is that a flat line can contain at most $D - 1$ singular points of $Z(f)$ (these are the points on ℓ where all four first-order partial derivatives of f vanish). Similarly, a non-singular line is flat if (and only if) it is incident to at least $3D - 3$ flat points.

The second fundamental form. We use the following notations and results from differential geometry; see Pressley [26] and Ivey and Landsberg [17] for details. For a variety X , the differential

$d\gamma$ of the Gauss morphism γ that maps each point $p \in X$ to its tangent space $T_p X$, is called the *second fundamental form* of X . For $X = Z(f)$, and for any non-singular point $p \in Z(f)$, the second fundamental form, locally near p , can be written as (see [17])

$$\sum_{1 \leq i, j \leq 3} a_{ij} du_i du_j,$$

where $x = x(u_1, u_2, u_3)$ is a parametrization of $Z(f)$, locally near p , and $a_{ij} = x_{u_i u_j} \cdot \mathbf{n}$, where $\mathbf{n} = \mathbf{n}(p) = \nabla f(p) / \|\nabla f(p)\|$ is the unit normal to $Z(f)$ at p . Since the second fundamental form is the differential of the Gauss mapping, it does not depend on the specific local parametrization of f near p . An important property of the second fundamental form is that it vanishes at every non-singular flat point $p \in Z(f)$ (see, e.g., Pressley [26] and Ivey and Landsberg [17]).

Lemma 2.15. *If a line $\ell \subset Z(f)$ is flat, then the tangent space $T_p Z(f)$ is fixed for all the non-singular points $p \in \ell$.*

Proof. The proof applies a fairly standard argument in differential geometry (see, e.g., Pressley [26]); see also a proof of a similar claim for the three-dimensional case in [9, Appendix]. Fix a non-singular point $p \in \ell$, and assume that $x = x(u_1, u_2, u_3)$ is a parametrization of $Z(f)$, locally near p . We assume, as we may, that the relevant neighborhood N_p of p consists only of non-singular points. For any point (a, b, c) in the corresponding parameter domain, $x_{u_1}, x_{u_2}, x_{u_3}$ span the tangent space to $Z(f)$ at $x(a, b, c)$. Indeed, since $x(u_1, u_2, u_3)$ is a local parametrization, its differential $(dx)_{(a,b,c)} : T_{(a,b,c)} \mathbb{C}^3 \rightarrow T_{x(a,b,c)} Z(f)$ is an isomorphism. Hence, the image of this latter map is spanned by $x_{u_1}, x_{u_2}, x_{u_3}$ at $x(a, b, c)$. In particular, we have

$$x_{u_i} \cdot \mathbf{n} = 0, \quad i = 1, 2, 3,$$

over N_p . We now differentiate these equations with respect to u_j , for $j = 1, 2, 3$, and obtain

$$x_{u_i u_j} \cdot \mathbf{n} + x_{u_i} \cdot \mathbf{n}_{u_j} \equiv 0 \quad \text{on } \ell \cap N_p, \text{ for } 1 \leq i, j \leq 3.$$

The first term vanishes because ℓ is flat, so, as noted above, the second fundamental form vanishes at each non-singular point of ℓ . We therefore have

$$x_{u_i} \cdot \mathbf{n}_{u_j} \equiv 0 \quad \text{on } \ell \cap N_p, \text{ for } i, j = 1, 2, 3.$$

Since $x_{u_1}, x_{u_2}, x_{u_3}$ span the tangent space $T_q Z(f)$, for each $q \in N_p$, it follows that $\mathbf{n}_{u_j}(q)$ is orthogonal to $T_q Z(f)$ for $q \in \ell \cap N_p$, and thus must be parallel to $\mathbf{n}(q)$ in this neighborhood. However, since \mathbf{n} is of unit length, we have $\mathbf{n} \cdot \mathbf{n} \equiv 1$, and differentiating this equation yields

$$\mathbf{n}_{u_j} \cdot \mathbf{n} \equiv 0 \quad \text{on } \ell \cap N_p, \text{ for } j = 1, 2, 3.$$

Since $\mathbf{n}_{u_j}(q)$ is both parallel and orthogonal to $\mathbf{n}(q)$, it must be identically zero on $\ell \cap N_p$, for $j = 1, 2, 3$.

Write $\ell = p + tv$, $t \in \mathbb{C}$, and define $\mathbf{h}(t) := \mathbf{n}(p + tv)$, for $t \in \mathbb{C}$. Then, in a suitable tensor notation,

$$\mathbf{h}'(t) = (\mathbf{n}_{u_1}(p + tv), \mathbf{n}_{u_2}(p + tv), \mathbf{n}_{u_3}(p + tv)) \cdot v \equiv 0,$$

locally near $t = 0$. Thus, $\mathbf{n}(p + tv)$ is constant locally near $t = 0$, implying that \mathbf{n} is constant along ℓ , locally near p .

It still remains to show that \mathbf{n} is constant on the set of all the non-singular points of $Z(f)$ contained in ℓ . Set

$$Z_s(\ell) := \{t \in \mathbb{C} \mid p + tv \text{ is a singular point of } Z(f)\}.$$

As ℓ is not critical, $|Z_s(\ell)| \leq D - 1$ (as already observed). The map $t \mapsto \mathbf{n}(p + tv)$ is constant in a neighborhood of every point t of $Z_{ns}(\ell) := \mathbb{C} \setminus Z_s(\ell)$. Since $Z_{ns}(\ell)$ is a connected set,⁵ \mathbf{n} has a fixed value at all the non-singular points on ℓ , as asserted. \square

2.6 Finitely and infinitely ruled surfaces in four dimensions, and u-resultants

Recall again the definition of Ξ_p , for a polynomial $f \in \mathbb{C}[x, y, z, w]$ and a point $p \in Z(f)$, which is the union of all (complex) lines passing through p and fully contained in $Z(f)$, and that of Σ_p , as the set of directions (considered as points in $\mathbb{P}T_p Z(f)$) of these lines.

Fix a line $\ell \in \Xi_p$, and let $v = (v_0, v_1, v_2, v_3) \in \mathbb{P}^3$ represent its direction. Since $\ell \subset Z(f)$, the four terms $F_i(p; v) = \nabla_v^i f(p)$, for $i = 1, 2, 3, 4$, must vanish at p . These terms, which we denote shortly as $F_i(v)$ at the fixed p , are homogeneous polynomials of respective degrees 1, 2, 3, and 4 in $v = (v_0, v_1, v_2, v_3)$. (Note that when $D \leq 3$, some of these polynomials are identically zero.)

In this subsection we provide a (partial) *algebraic* characterization of points $p \in Z(f)$ for which $|\Sigma_p|$ is infinite; that is, points that are incident to infinitely many lines that are fully contained in $Z(f)$. We refer to this situation by saying that $Z(f)$ is *infinitely ruled* at p . To be precise, here we only characterize points that are incident to infinitely many lines that osculate to $Z(f)$ to order three. The passage from this to the full characterization will be done during the analysis in the next section.

u-resultants. The algebraic tool that we use for this purpose are *u-resultants*. Specifically, following and specializing Cox et al. [5, Chapter 3.5, page 116], define, for a vector $u = (u_0, u_1, u_2, u_3) \in \mathbb{P}^3$,

$$U(p; u_0, u_1, u_2, u_3) = \text{Res}_4\left(F_1(p; v), F_2(p; v), F_3(p; v), u_0 v_0 + u_1 v_1 + u_2 v_2 + u_3 v_3\right),$$

where $\text{Res}_4(\cdot)$ denotes, as earlier, the multipolynomial resultant of the four respective (homogeneous) polynomials, with respect to the variables v_0, v_1, v_2, v_3 . For fixed p , this is the so-called *u-resultant* of $F_1(v), F_2(v), F_3(v)$.

Theorem 2.16. *The function $U(p; u_0, u_1, u_2, u_3)$ is a homogeneous polynomial of degree six in the variables u_0, u_1, u_2, u_3 , and is a polynomial of degree $O(D)$ in $p = (x, y, z, w)$. For fixed $p \in Z(f)$, $U(p; u_0, u_1, u_2, u_3)$ is identically zero as a polynomial in u_0, u_1, u_2, u_3 , if and only if there are infinitely many (complex) directions $v = (v_0, v_1, v_2, v_3)$, such that the corresponding lines $\{p + tv \mid t \in \mathbb{C}\}$ osculate to $Z(f)$ to order three at p .*

Proof. By definition, the osculation property in the theorem, for given p and v , is equivalent to $F_1(p; v) = F_2(p; v) = F_3(p; v) = 0$. Regarding F_1, F_2, F_3 as homogeneous polynomials in v , the degree of U in u_0, u_1, u_2, u_3 is $\deg(F_1) \deg(F_2) \deg(F_3) = 3! = 6$ (see Cox et al. [5, Exercise 3.4.6.b]). Put $d = \deg(F_1) + \deg(F_2) + \deg(F_3) + 1 = 7$. Then the total degree of U in the coefficients of F_i , each being a polynomial in p of degree at most D , is at most $\binom{d}{3} = \binom{7}{3} = 35$ (see also the proof of Lemma 2.9 and Cox et al. [5, Exercises 3.4.6.c, 3.4.19]), and thus the degree of U as a polynomial in p is $O(D)$.

Put $H(u, v) = u_0 v_0 + u_1 v_1 + u_2 v_2 + u_3 v_3$, and, for any $v \in \mathbb{C}^4$, denote by H_v the hyperplane $H(u, v) = 0$. Fix $p \in Z(f)$, and regard $F_1, F_2, F_3, H(u, \cdot)$ as polynomials in v . If the osculation property holds at p then $Z(F_1, F_2, F_3)$ is infinite, so it is at least 1-dimensional. Thus, for any $u = (u_0, u_1, u_2, u_3) \in \mathbb{C}^4$, the variety $Z(F_1, F_2, F_3, H(u, v))$ is non-empty, so the multipolynomial

⁵This property holds for \mathbb{C} but not for \mathbb{R} .

resultant of these four polynomials (in v) vanishes at u . Since this holds for all $u \in \mathbb{C}^4$, It follows from Cox et al. [5, Proposition 1.1.5] that $U \equiv 0$.

Suppose then that the osculation property does not hold at p , so $Z(F_1, F_2, F_3)$ is finite. Pick any $u_0 \notin \bigcup_{v \in Z(F_1, F_2, F_3)} H_v$. Then, for every $v \in Z(F_1, F_2, F_3)$, we have $H(u_0, v) \neq 0$, implying that

$$Z(F_1, F_2, F_3, H(u_0, \cdot)) = \{v \in Z(F_1, F_2, F_3) \mid H(u_0, v) = 0\} = \emptyset.$$

Therefore, by the properties of multipolynomial resultants, $U(u_0) \neq 0$, and U is not identically zero. \square

Remark. Theorem 2.16 shows that the subset of $Z(f)$ consisting of the points incident to infinitely many lines that osculate to $Z(f)$ to order three is contained in a *subvariety* of $Z(f)$, which is the intersection of $Z(f)$ with the common zero set of the coefficients of U (considered as polynomials in x, y, z, w).

Corollary 2.17. *Fix $p \in Z(f)$. The polynomial $U(p; u_0, u_1, u_2, u_3)$ is identically zero, as a polynomial in u_0, u_1, u_2, u_3 , if and only if there are more than six (complex) lines osculating to $Z(f)$ to order 3 at p .*

Proof. The polynomial F_i is either 0 or of degree i (in v , for a fixed value of p), for $i = 1, 2, 3$. By Theorem 2.2, the number of their common zeros $v = (v_0, v_1, v_2, v_3)$ is either six (counting complex projective solutions with multiplicity) or infinite. The result then follows from Theorem 2.16. \square

3 Proof of Theorem 1.3

Let P, L, m, n, q , and s be as in the theorem.

The proof proceeds by induction on m , where the base cases of the induction are the ranges $m \leq \sqrt{n}$ and $m \leq M_0$, for a sufficiently large constant M_0 . In both cases we have $I(P, L) \leq A(m + n)$, for a suitable choice of A . Assume then that the bound holds for all $m' < m$, and consider an instance involving sets P, L , with $|P| = m > \sqrt{|L|} = \sqrt{n}$.

As already discussed, the bound in (5) is qualitatively different in the two ranges $m = O(n^{4/3})$ and $m = \Omega(n^{4/3})$, and the analysis will occasionally have to bifurcate accordingly. Nevertheless, the bifurcation is mainly in the choice of various parameters, and in manipulating them. Most of the technical details that deal with the algebraic structure of the problem are identical. We will therefore present the analysis jointly for both cases, and bifurcate only locally, when the induction itself, or tools that prepare for the induction, get into action, and require different treatments in the two cases.

As promised in the overview, we will use two different partitioning schemes, one with a polynomial of “large” degree, and one with a polynomial of “small” degree. We start naturally with the first scheme.

An important issue to bear in mind is that, unlike most of the material in the preceding section, where the underlying field was \mathbb{C} , the analysis in this section is over the reals. Nevertheless, this is essentially needed only for constructing a polynomial partitioning, which is meaningless over \mathbb{C} . Once this is done, the analysis of incidences between points and lines on the zero set of the partitioning polynomial can be carried out over the complex field just as well as over \mathbb{R} , and then the machinery reviewed and developed in the previous section can be brought to bear.

First partitioning scheme. Fix a parameter r , given by

$$r = \begin{cases} cm^{8/5}/n^{4/5} & \text{if } m \leq an^{4/3} \\ cn^4/m^2 & \text{if } m \geq an^{4/3}, \end{cases}$$

where a and c are suitable constants. Note that, in both cases, $1 \leq r \leq m$, for a suitable choice of the constants of proportionality, unless either $m = \Omega(n^2)$ or $n = \Omega(m^2)$, extreme cases that have already been handled. We refer to the cases $n^{1/2} \leq m \leq an^{4/3}$ and $an^{4/3} < m \leq n^2$ as the cases of *small* m and of *large* m , respectively.

We now apply the polynomial partitioning theorem of Guth and Katz (see [14] and [20, Theorem 2.6]), to obtain an r -partitioning 4-variate (real) polynomial f of degree

$$D = O(r^{1/4}) \leq \begin{cases} c_0 m^{2/5}/n^{1/5} & \text{if } m \leq an^{4/3} \\ c_0 n/m^{1/2} & \text{if } m \geq an^{4/3}, \end{cases} \quad (13)$$

for another suitable constant c_0 . That is, every connected component of $\mathbb{R}^4 \setminus Z(f)$ contains at most m/r points of P , where, as above, $Z(f)$ denotes the zero set of f . By Warren's theorem [43] (see also [20]), the number of components of $\mathbb{R}^4 \setminus Z(f)$ is $O(D^4) = O(r)$.

Set $P_0 := P \cap Z(f)$ and $P' := P \setminus P_0$. We recall that, although the points of P' are more or less evenly partitioned among the cells of the partition, no nontrivial bound can be provided for the size of P_0 ; in the worst case, all the points of P could lie in $Z(f)$. Each line $\ell \in L$ is either fully contained in $Z(f)$ or intersects it in at most D points (since the restriction of f to ℓ is a univariate polynomial of degree at most D). Let L_0 denote the subset of lines of L that are fully contained in $Z(f)$ and put $L' = L \setminus L_0$. Put $m' := |P'|$, $n' := |L'|$, $m_0 := |P_0|$, and $n_0 := |L_0|$. We have

$$I(P, L) = I(P_0, L_0) + I(P_0, L') + I(P', L'). \quad (14)$$

As can be expected (and noted earlier), the harder part of the analysis is the estimation of $I(P_0, L_0)$. Indeed, it might happen that $Z(f)$ is a hyperplane, and then the best (and worst-case tight) bound we can offer is the bound specified by Theorem 1.1. It might also happen that $Z(f)$ contains some 2-flat, in which case we are back in the planar scenario, for which the best (and worst-case tight) bound we can offer is the Szemerédi–Trotter bound (1). Of course, the assumptions of the theorem come to the rescue, and we will see below how exactly they are used.

We first bound the second and third terms of (14). We have

$$I(P_0, L') \leq |L'| \cdot D \leq nD, \quad (15)$$

because, as just noted, a line not fully contained in $Z(f)$ can intersect this set in at most D points. To estimate $I(P', L')$, we put, for each cell τ of the partition, $P_\tau = P \cap \tau$, and let L_τ denote the set of the lines of L' that cross τ ; put $m_\tau = |P_\tau| \leq m/r$, and $n_\tau = |L_\tau|$. Since every line $\ell \in L'$ crosses at most $1 + D$ components of $\mathbb{R}^4 \setminus Z(f)$ (because it has to pass through $Z(f)$ in between cells), we have

$$\sum_{\tau} n_\tau \leq n'(1 + D) \leq n(1 + D). \quad (16)$$

Clearly, we have

$$I(P', L') = \sum_{\tau} I(P_\tau, L_\tau).$$

We now bifurcate depending on the value of m .

Estimating $I(P', L')$: The case of small m . Here we use the easy upper bound (which holds for any pair of sets P_τ, L_τ)

$$I(P_\tau, L_\tau) = O(|P_\tau|^2 + |L_\tau|) = O((m/r)^2 + n_\tau).$$

Summing these bounds over the cells, using (16), and recalling the value of r , we get

$$I(P', L') = \sum_{\tau} I(P_\tau, L_\tau) = O(m^2/r + nr^{1/4}) = O(m^{2/5}n^{4/5}).$$

Estimating $I(P', L')$: The case of large m . Here we use the dual (generally applicable) upper bound $I(P_\tau, L_\tau) = O(|L_\tau|^2 + |P_\tau|)$, which, by splitting L_τ into subsets of size at most $|P_\tau|^{1/2}$, becomes

$$I(P_\tau, L_\tau) = O(|P_\tau|^{1/2}|L_\tau| + |P_\tau|) = O((m/r)^{1/2}n_\tau + m_\tau).$$

Summing these bounds over the cells, using (16), and recalling the value of r , we get

$$I(P', L') = \sum_{\tau} I(P_\tau, L_\tau) = O((m/r)^{1/2}nr^{1/4} + m) = O(m^{1/2}n/D + m) = O(m).$$

Combining both bounds, we have:

$$I(P_0, L') + I(P', L') = O\left(m^{2/5}n^{4/5} + m\right). \quad (17)$$

Note that in this part of the analysis we do not need the assumptions involving q and s .

Estimating $I(P_0, L_0)$. We next bound the number of incidences between points and lines that are contained in $Z(f)$. To simplify the notation, write P for P_0 and L for L_0 , and denote their respective cardinalities as m and n . (The reader should keep this convention in mind, as we will “undo” it towards the end of the analysis.) To be precise, we will not be able to account explicitly for all types of these incidences (for the present choices of D). Our strategy is to obtain an explicit bound for a subset of the incidences, which is subsumed by the bound in (5), and then prune away those lines and points that participate in these incidences. We will be left with “problematic” subsets of points and lines, and we will then handle them in a second, new, induction-based partitioning step. A major goal for the first stage is to show that, for the set of surviving lines, the parameters q and s can be replaced by the respective parameters $O(D^2)$ and $O(D)$ that “pass well” through the induction; see below for details.⁶

By the nature of its construction, f is in general reducible (see [14]). However, to apply successfully certain steps of the forthcoming analysis, we will need to assume that f is irreducible, so we will apply the analysis separately to each irreducible factor of f , and then sum up the resulting bounds.

Write the irreducible factors of f , in an arbitrary order, as f_1, \dots, f_k , for some $k \leq D$. The points of P are partitioned among the zero sets of these factors, by assigning each point $p \in P$ to the first factor in this order whose zero set contains p . A line $\ell \in L$ is similarly assigned to the first factor whose zero set fully contains ℓ (there always exists such a factor). Then $I(P, L)$ is the sum, over $i = 1, \dots, k$, of the number of incidences between the points and the lines that are assigned to the (same) i th factor, plus the number of incidences between points and lines assigned to different factors. The latter kind of incidences is easier to handle. Indeed, if (p, ℓ) is an incident

⁶Note that in general the bounds $O(D^2)$ and $O(D)$ are not necessarily smaller than their respective original counterparts q and s . Nevertheless, they uniformly depend on m and n in a way that makes them fit the induction process.

pair in $P \times L$, so that p is assigned to f_i and ℓ is assigned to f_j , for $i \neq j$ (necessarily $i < j$), then the incidence occurs at an intersection of ℓ with $Z(f_i)$. By construction, ℓ is not fully contained in $Z(f_i)$, so it intersects it in at most $\deg(f_i)$ points, so the overall number of incidences on ℓ of this kind is at most $\sum_{i \neq j} \deg f_i < D$, and the overall number of such incidences is therefore at most nD .

For the former kind of incidences, we assume in what follows that we have a single irreducible polynomial f , and denote by P and m , for short, the set of points assigned to f and its cardinality, and by L and n the set of lines assigned to f (and thus fully contained in $Z(f)$) and its cardinality. We continue to denote the degree of f as D . (Again, we will undo these conventions towards the end of the analysis.)

This is not yet the end of the reduction, because, in most of the analysis about to unfold, we need to assume that the points of P are non-singular points of $Z(f)$. To reduce the setup to this situation we proceed as follows. We construct a sequence of partial derivatives of f that are not identically zero on $Z(f)$. Without loss of generality, assume that this sequence is f, f_x, f_{xx} , and so on. Denote the j -th element in this sequence as f_j , for $j = 0, 1, \dots$ (so $f_0 = f, f_1 = f_x$, and so on). Assign each point $p \in P$ to the first polynomial f_j in the sequence for which p is non-singular; more precisely, we assign p to the first f_j for which $f_j(p) = 0$ but $f_{j+1}(p) \neq 0$. Similarly, assign each line ℓ to the first polynomial f_j in the sequence for which ℓ is fully contained in $Z(f_j)$ but not fully contained in $Z(f_{j+1})$. If ℓ is assigned to f_j then it can only contain points p that were assigned to some f_k with $k \geq j$. Indeed, if ℓ contained a point p assigned to f_k with $k < j$ then $f_{k+1}(p) \neq 0$ but ℓ is fully contained in $Z(f_{k+1})$, since $k + 1 \leq j$, a contradiction that establishes the claim.

Fix a line $\ell \in L$, which is assigned to some f_j . An incidence between ℓ and a point $p \in P$, assigned to some f_k , for $k > j$, can be charged to the intersection of ℓ with $Z(f_{j+1})$ at p (by construction, p belongs to $Z(f_{j+1})$). The number of such intersections is at most $D - j - 1$, so the overall number of incidences of this sort, over all lines $\ell \in L$, is $O(nD)$. It therefore suffices to consider only incidences between points and lines that are assigned to the same zero set $Z(f_i)$.

The reductions so far have produced a finite collection of up to $O(D^2)$ polynomials, each of degree at most D , so that the points of P are *partitioned* among the polynomials and so are the lines of L , and we only need to bound the number of incidences between points and lines assigned to the *same* polynomial. This is not the end yet, because the various partial derivatives might be reducible, which we want to avoid. Thus, in a final decomposition step, we split each derivative polynomial f_j into its irreducible factors, and reassign the points and lines that were assigned to $Z(f_j)$ to the various factors, by the same ‘‘first come first served’’ rule used above. The overall number of incidences that are lost in this process is again $O(nD)$. The number of polynomials remains $O(D^2)$, as can easily be checked. Note also that the last decomposition step preserves non-singularity of the points in the special sense defined above; that is, as is easily verified, a point $p \in Z(f_j)$ with $f_{j+1}(p) \neq 0$, continues to be a non-singular point of the irreducible component it is reassigned to.

We now fix one such final polynomial, still call it f , denote its degree by D (which is upper bounded by the original degree D), and denote by P and L the subsets of the original sets of points and lines that are assigned to f , and by m and n their respective cardinalities. (Again, this simplifying convention will be undone towards the end of the analysis.) We now may assume that P consists exclusively of *non-singular* points of the *irreducible* variety $Z(f)$.

If $D \leq 3$, then, by Lemma 2.12, $Z(f)$ is ruled by lines. Hypersurfaces ruled by lines will be handled in the later part of the analysis. (Note that the cases $D = 1$ or $D = 2$ can be controlled by assumption (i’) of the theorem (see below), whereas the case $D = 3$ requires a different treatment.) Suppose then that $D \geq 4$. The flecnode polynomial FL_f^4 of f (see Section 2.4) vanishes identically on every line of L (and thus also on P , assuming that each point of P is incident to at least one

line of L). If FL_f^4 does not vanish identically on $Z(f)$, then $Z(f, \text{FL}_f^4) := Z(f) \cap Z(\text{FL}_f^4)$ is a two-dimensional variety (see, e.g., Hartshorne [16, Exercise I.1.8]). It contains P and all the lines of L (by Lemma 2.10), and is of degree $O(D^2)$ (by Theorem 2.2). The other possibility is that FL_f^4 vanishes identically on $Z(f)$, and then Theorem 2.11 implies that $Z(f)$ is ruled by lines. This latter case, which requires several more refined tools from algebraic geometry, will be analyzed later.

First case: $Z(f, \text{FL}_f^4)$ is two-dimensional

Put $g = \text{FL}_f^4$. In the analysis below, we only use the facts that $\deg(g) = O(D)$, and that $Z(f, g)$ is two-dimensional, so the analysis applies for any such g ; this comment will be useful in later steps of the analysis. Recall that in this part of the analysis f is assumed to be an irreducible polynomial of degree ≥ 4 .

We have a set P of m points and a set L of n lines in \mathbb{C}^4 , so that P is contained in the two-dimensional algebraic variety $Z(f, g) \subset \mathbb{C}^4$. By pruning away all the lines containing at most $\max(D, \deg(g))$ points of P , we lose $O(nD)$ incidences, and all the surviving lines are contained in $Z(f, g)$, as is easily checked. For simplicity of notation, we continue to denote by L the set of surviving lines.

Let $Z(f, g) = \bigcup_{i=1}^s V_i$ be the decomposition of $Z(f, g)$ into its irreducible components, as described in Section 2.2. By Theorem 2.2, we have $\sum_{i=1}^s \deg(V_i) \leq \deg(f) \deg(g) = O(D^2)$.

Our next step is to analyze the number of incidences between points and lines within the components of $Z(f, g)$ that are not 2-flats. For this we first need the following bound on point-line incidences within a two-dimensional surface in three dimensions. This part of the analysis is taken from our companion paper [36]. Refer to Section 2.4 for properties of ruled surfaces.

For a point p on an irreducible singly ruled surface V , which is not the exceptional point of V , we let $\Lambda_V(p)$ denote the number of generator lines passing through p and fully contained in V (so if p is incident to an exceptional line, we do not count that line). We also put $\Lambda_V^*(p) := \max\{0, \Lambda_V(p) - 1\}$. Finally, if V is a cone and p_V is its exceptional point (that is, apex), we put $\Lambda_V(p) = \Lambda_V^*(p_V) := 0$. We also consider a variant of this notation, where we are also given a finite set L of lines (where not all lines of L are necessarily contained in V), which does not contain any of the (at most two) exceptional lines of V . For a point $p \in V$, we let $\lambda_V(p; L)$ denote the number of lines in L that pass through p and are fully contained in V , and put $\lambda_V^*(p; L) := \max\{0, \lambda_V(p; L) - 1\}$. If V is a cone with apex p_V , we put $\lambda_V(p_V; L) = \lambda_V^*(p_V; L) = 0$. We clearly have $\lambda_V(p; L) \leq \Lambda_V(p)$ and $\lambda_V^*(p; L) \leq \Lambda_V^*(p)$, for each point p .

Lemma 3.1. *Let V be an irreducible singly ruled two-dimensional surface of degree $D > 1$ in \mathbb{R}^3 or in \mathbb{C}^3 . Then, for any line ℓ , except for the (at most) two exceptional lines of V , we have*

$$\begin{aligned} \sum_{p \in \ell \cap V} \Lambda_V(p) &\leq D + 2 && \text{if } \ell \text{ is not fully contained in } V, \\ \sum_{p \in \ell \cap V} \Lambda_V^*(p) &\leq D + 2 && \text{if } \ell \text{ is fully contained in } V. \end{aligned}$$

The following lemma provides the needed infrastructure for our analysis.

Lemma 3.2. *Let V be a possibly reducible two-dimensional algebraic surface of degree $D > 1$ in \mathbb{R}^3 or in \mathbb{C}^3 , with no linear components. Let P be a set of m distinct points on V and let L be a set of n distinct lines fully contained in V . Then there exists a subset $L_0 \subseteq L$ of at most $O(D^2)$ lines, such that the number of incidences between P and $L \setminus L_0$ satisfies*

$$I(P, L \setminus L_0) = O\left(m^{1/2} n^{1/2} D^{1/2} + m + n\right). \quad (18)$$

Sketch of Proof. We provide the following sketch of the proof; the full details are given in the companion paper [36]. Consider the irreducible components W_1, \dots, W_u of V . We first argue that the number of lines that are either contained in the union of the non-ruled components, or those contained in more than one ruled component of V is $O(D^2)$, and we place all these lines, as well as the exceptional lines of any singly ruled component, in the exceptional set L_0 . We may thus assume that each surviving line in $L_1 := L \setminus L_0$ is contained in a unique ruled component of V , and is a generator of that component.

The strategy of the proof is to consider each line ℓ of L_1 , and to estimate the number of its incidences with the points of P in an indirect manner, via Lemma 3.1, applied to ℓ and to each of the ruled components W_j of V .

Specifically, we fix some threshold parameter ξ , and dispose of points that are incident to at most ξ lines of L_1 , losing at most $m\xi$ incidences. Let P_1 denote the set of surviving points.

Now if a line $\ell \in L_1$ is incident to a point $p \in P_1$, it meets at least ξ other lines of L_1 at p . It follows from Lemma 3.1 that the overall number of such lines, over all points in $P_1 \cap \ell$, is roughly D , so the number of such points on ℓ is at most roughly D/ξ , for a total of nD/ξ incidences of this kind. Choosing $\xi = (nD/m)^{1/2}$ yields the bound $O(m^{1/2}n^{1/2}D^{1/2})$, and the lemma follows. \square

We can now proceed, by deriving two upper bounds for certain types of incidences between P and L . The first bound is relevant for the range $m = O(n^{4/3})$, and the second bound is relevant for the range $m = \Omega(n^{4/3})$. Nevertheless, both bounds apply to the entire range of m and n .

Proposition 3.3. *The number of incidences involving non-singular points of $Z(f)$ that are contained in components of $Z(f, g)$ that are not 2-flats is*

$$\min \{O(mD^2 + nD), O(m + nD^4)\}.$$

Proof. We first establish the bound $O(mD^2 + nD)$. Let $p \in Z(f)$ be a non-singular point. The irreducible decomposition of $S_p := Z(f, g) \cap T_p Z(f)$ is the union of one- and two-dimensional components. Clearly, S_p contains all the lines that are incident to p and are fully contained in $Z(f, g)$; it is a variety, embedded in 3-space, of degree $O(D^2)$. The union of the one-dimensional components is a curve of degree $O(D^2)$, so, by Lemma 2.3, it can contain at most $O(D^2)$ lines; when summing over all $p \in P$, the total number of such incidences is $O(mD^2)$.

It remains to bound incidences involving the two-dimensional components of S_p . By Sharir and Solomon [35, Lemma 2.3], the number of lines incident to p inside the two-dimensional components of S_p is at most $O(D^2)$, except possibly for lines that lie in a component that is a cone and has p as its apex. Summing over all $p \in P$, we get a total of $O(mD^2)$ incidences for this case too.

Note that each two-dimensional component of S_p is necessarily also a two-dimensional irreducible component of $Z(f, g)$. Hence the analysis performed so far takes care of all incidences except for those that occur on conic two-dimensional components of $Z(f, g)$ (and on 2-flats, which we totally ignore in this proposition). Let V be a conic component of $Z(f, g)$ with apex p_V , which is not a 2-flat. We note that V cannot fully contain a line that is not incident to p_V . Indeed, suppose to the contrary that V contained such a line ℓ . Since V is a cone with apex p_V , for each point $a \in \ell$, the line connecting a to p_V is fully contained in V , and therefore the 2-flat containing p_V and ℓ is fully contained in V . As V is irreducible and is not a 2-flat, we obtain a contradiction, showing that no such line exists. We conclude that any point on V , except for p_V , is incident to at most one line that is fully contained in V (a “generator” through p_V), for a total of $O(m)$ incidences. Since $Z(f, g)$ is of degree $O(D^2)$, the number of conic components of $Z(f, g)$ is $O(D^2)$, so, summing this bound over all components V , we get again the bound $O(mD^2)$ on the number of relevant incidences.

Therefore, it remains to bound the number of incidences between the points of

$$P_c := \{p_V \mid p_V \text{ is an apex of an irreducible conic component } V \text{ of } Z(f, g)\}$$

and the lines of L . Since there are at most $O(D^2)$ irreducible components of $Z(f, g)$, we have $|P_c| \leq cD^2$, for some suitable constant c . We next let L_c denote the set of lines in L containing fewer than cD points of P_c , and claim that any point $p \in P_c$ is incident to fewer than D lines of $L \setminus L_c$. Indeed, otherwise, we would get at least D lines incident to p , each containing at least $cD + 1$ points of P_c , i.e., at least cD points other than p . As these points are all distinct, we would get that $|P_c| \geq 1 + D \cdot cD > cD^2$, a contradiction. On the other hand, by definition of L_c , we have

$$I(P_c, L_c) = O(nD).$$

We have thus shown that the number of incidences involving points of P_c is

$$I(P_c, L) = I(P_c, L_c) + I(P_c, L \setminus L_c) = O(nD) + O(mD) = O(nD + mD),$$

well within the bound that we seek to establish.

The second bound. We next establish the second bound $O(m + nD^4)$. Let V be an irreducible two-dimensional component of $Z(f, g)$. If V is not ruled, then by Proposition 2.13, it contains at most $11 \deg(V)^2 - 24 \deg(V) < 11 \deg(V)^2$ lines. Summing over all irreducible components of $Z(f, g)$ that are not ruled, we get at most $11 \sum_V \deg(V)^2 = O(D^4)$ lines. Let ℓ be one of those lines, and let $p \in \ell \cap P$. For any other line $\lambda \in L$ that passes through p , we charge its incidence with p to its intersection with ℓ . This yields a total of $O(nD^4)$ incidences, to which we add $O(m)$ for incidences with those points that lie on only one line of L , for a total of $O(m + nD^4)$ incidences.

We next analyze the irreducible components of $Z(f, g)$ that are ruled but are not 2-flats. Let V_1, \dots, V_k denote these components, for some $k = O(D^2)$. Project all these components onto some generic hyperplane, and regard them as a single (reducible) ruled surface in 3-space, whose degree is $\sum_{i=1}^k \deg(V_i) = O(D^2)$. Lemma 3.2 then yields a subset L_0 of L of size $O(D^4)$, and shows that

$$I(P, L \setminus L_0) = O\left(m^{1/2}n^{1/2}D + m + n\right).$$

The lines of L_0 are simply added to the set of $O(D^4)$ lines not belonging to ruled components. This does not affect the asymptotic bound $O(nD^4)$ derived above. In total we get

$$O\left(m^{1/2}n^{1/2}D + m + nD^4\right)$$

incidences. Since

$$m^{1/2}n^{1/2}D \leq \frac{1}{2}(m + nD^2),$$

we obtain the second bound asserted in the proposition. \square

Remark. The term $O(nD^4)$ appears to be too weak, and can probably be improved, using ideas similar to those in the proof of Lemma 3.2. Since such an improvement does not have a significant effect on our analysis, we leave it as an interesting problem for further research.

Restrictedness of hyperplanes and quadrics, and lines on 2-flats. The bounds in Proposition 3.3 might be too large, for the current choices of D , because of the respective terms $O(mD^2)$ and $O(nD^4)$. (Technically, the m and n in the definition of D are not necessarily the same as the m and n that denote the size of the current subsets of the original P and L , but let us assume that they are the same for the present discussion.) For example, when $m = O(n^{4/3})$ and $D = \Theta(m^{2/5}/n^{1/5})$

(recall that this is the “large” value of D for this range), we have $mD^2 = \Theta(m^{9/5}/n^{2/5})$, and this is $\gg m^{2/5}n^{4/5}$ when $m \gg n^{6/7}$. Similarly, when $m = \Omega(n^{4/3})$ and $D = \Theta(n/m^{1/2})$ (which is the value chosen for this range), we have $nD^4 = \Theta(n^5/m^2)$, and this is $\gg m$ when $m \ll n^{5/3}$. These bounds will be used in the second partitioning step, where we use a smaller-degree partitioning polynomial, and for m outside the problematic ranges, i.e., for $m \leq n^{6/7}$ or $m \geq n^{5/3}$; see below for details. Otherwise, for the current D , these bounds need to be finessed and replaced by the following alternative analysis.⁷

In the first step of this analysis, we estimate the number of lines contained in a hyperplane or a quadric (when $Z(f, g)$ is two-dimensional), and establish the following properties.

Lemma 3.4. *Each hyperplane or quadric H is $O(D^2)$ -restricted.*

Proof. Fix a hyperplane or quadric H . Let V be an irreducible component of $Z(f, g)$. If $V \cap H$ is a curve, then (recalling Theorem 2.2) its degree is at most $\deg(V)$ (when H is a hyperplane) or $2\deg(V)$ (when H is a quadric), and can therefore contain at most $2\deg(V)$ lines, by Lemma 2.3. Therefore, the union of all the irreducible components V of $Z(f, g)$ which intersect H in a curve, contains at most $2\sum_V \deg(V) = O(D^2)$ lines. Assume then that $V \cap H$ is two-dimensional. Since V is irreducible, we must have $V \cap H = V$, so V is fully contained in H . Moreover, V is an irreducible two-dimensional surface contained in $Z(f) \cap H$, and therefore must be an irreducible component of $Z(f) \cap H$, which is a two-dimensional surface of degree $\leq D$. By Theorem 2.2, $\sum_{V \subset H} \deg(V) \leq \deg(Z(f) \cap H) \leq \deg(f) \leq D$. If V is not ruled by lines, then by Proposition 2.13, it contains at most $11\deg(V)^2$ lines, and summing over all such components V within H , we get a total of at most $\sum_V 11\deg(V)^2 = O(D^2)$ lines.

The remaining irreducible (two-dimensional) components V of $Z(f, g)$ that meet H (if such components exist) are fully contained in H , and are ruled by lines. As already observed, these components are also irreducible components of $Z(f) \cap H$, and so, with the exception of $O(D^2)$ lines (those contained in the components already analyzed), all the lines of L that lie in H are contained in components of $Z(f) \cap H$ that are ruled by lines. Since f restricted to H is a polynomial of degree $\leq D$, we conclude that H is $O(D^2)$ -restricted. \square

We next analyze the number of lines contained in a 2-flat.

Lemma 3.5. *Let π be a 2-flat that is not fully contained in $Z(f, g)$. Then the number of lines fully contained in $Z(f) \cap \pi$ is $O(D)$.*

Proof. The intersection $Z(f) \cap \pi$ is either π itself, or a curve of degree $\leq D$. The latter case implies (using Lemma 2.3) that π contains at most D lines that are fully contained in $Z(f)$. In the former case $\pi \subset Z(f)$. By assumption, π is not contained in $Z(f, g)$, implying that g intersects π in a curve of degree $O(D)$ (since $Z(f, g) = \pi \cap Z(g)$), and can therefore contain at most $O(D)$ lines that are fully contained in $Z(f)$. \square

Recap. Summing up what was done so far, we can classify the incidences in $I(P, L)$ into the following types. Recall that the analysis is confined to a single irreducible factor f of the original polynomial or of some partial derivative of such a factor.

(a) We treat the cases where f is linear or quadratic separately, using a variant of Theorem 1.1, which takes into account the restrictedness of hyperplanes and quadrics; see Proposition 3.6 below.

⁷As the example worked out above indicates, with a similar phenomenon showing up for the other bound $O(nD^4)$, the bounds in Proposition 3.3 will be within the bound (5) when m is not too large. For such values of m we can bypass the induction process, and obtain the desired bounds directly, in a single step.

(b) We treat the case where $Z(f)$ is ruled by lines separately (this is the second case in the analysis, when $Z(f, \text{FL}_f^{(4)})$ is three-dimensional).

If f is not ruled by lines and is of degree ≥ 4 (recall that each surface of degree at most 3 is ruled by lines—see Lemma 2.12), then there are two kinds of incidences that need to be considered.

(c) Incidences between points and lines that are contained in irreducible components of $Z(f, \text{FL}_f^{(4)})$ (or, more generally, of $Z(f, g)$, for other suitable polynomials g) that are not 2-flats. We have bounded the number of these incidences in Proposition 3.3 in two different ways, but we also ignored these incidences, passing them to the induction in the second partitioning step, to be presented later, where we now know that each hyperplane and quadric is $O(D^2)$ -restricted, and each 2-flat⁸ contains at most $O(D)$ lines of L .

(d) Incidences between points and lines that are contained in some irreducible component of $Z(f, g)$ that is a 2-flat. These incidences will be analyzed explicitly below, using the properties of flat points and lines, as presented in Section 2.5.

Incidences within hyperplanes and quadrics. We next derive a bound that we will use several times later on, in cases where we can *partition* P and L (or, more precisely, subsets thereof) among some finite collection of hyperplanes and quadrics, so that all the relevant incidences occur between points and lines that are assigned to the same surface. Recall that we have already applied a similar partitioning among the factors of f and of its derivatives. The prime application of this bound will be to incidences of type (a) above, but it will also be used in the analysis of type (d) incidences, and in the analysis of the second case (b), where $Z(f, \text{FL}_f^{(4)})$ is three-dimensional, i.e., when $Z(f, \text{FL}_f^{(4)}) = Z(f)$. In particular, we emphasize that the following proposition does not require that $Z(f, \text{FL}_f^{(4)})$ be two-dimensional.

Proposition 3.6. *Let H_1, \dots, H_t be a finite collection of hyperplanes and quadrics. Assume that the points of P and the lines of L are partitioned among H_1, \dots, H_t , so that each point $p \in P$ (resp., each line $\ell \in L$) is assigned to a unique hyperplane or quadric that contains p (resp., fully contains ℓ), and assume further that each H_i is q -restricted and that each 2-flat contains at most s lines of L . Then the overall number of incidences between points and lines that are assigned to the same surface is*

$$O\left(m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + m + n\right). \quad (19)$$

Proof. For $i = 1, \dots, t$, let L_i (resp., P_i) denote the set of lines of L (resp., points of P), that are assigned to H_i , and put $n_i := |L_i|$, $m_i := |P_i|$. We have $\sum_i m_i = m$ and $\sum_i n_i = n$. For each i , since H_i is q -restricted, there exists a polynomial $g_i = g_{H_i}$, defined on H_i and of degree at most $O(\sqrt{q})$, such that all the lines of L_i , with the exception of at most q of them, are fully contained in ruled components of $Z(g_i)$. Write $L_i = L_i^{nr} \cup L_i^r$, where L_i^r is the subset of those lines that are fully contained in ruled components of $Z(g_i)$, and L_i^{nr} is the complementary subset, of size at most q . In fact, we also remove from L_i^r the subset L_{i0}^r of $O(q)$ lines, as provided by Lemma 3.2, and put them in L_i^{nr} ; we continue to use the same notations for these modified sets. (To apply Lemma 3.2 to the case where H_i is a quadric, we first project the configuration onto some generic 3-space.)

⁸We need some care here, because the analysis guarantees this property only for 2-flats not fully contained in $Z(f, g)$. Nevertheless, we will get rid of all the lines of L that are contained in any such 2-flat, and that will make the property hold for every 2-flat; see below for elaboration of this issue.

Since the size of L_i^{nr} is still $O(q)$, we have, by Theorem 1.1,

$$\begin{aligned} I(P_i, L_i^{nr}) &= O\left(m_i^{1/2}|L_i^{nr}|^{3/4} + m_i^{2/3}|L_i^{nr}|^{1/3}s^{1/3} + m_i + |L_i^{nr}|\right) \\ &= O\left(m_i^{1/2}n_i^{1/2}q^{1/4} + m_i^{2/3}n_i^{1/3}s^{1/3} + m_i + n_i\right). \end{aligned}$$

(Note that Theorem 1.1 is directly applicable when H_i is a hyperplane, and that it can also be applied when H_i is a quadric, by projecting the configuration onto some generic hyperplane, similar to what we have just noted for the application of Lemma 3.2.)

We next bound $I(P_i, L_i^r)$, using Lemma 3.2. Since $\deg(g_i) = O(\sqrt{q})$ and we have already removed from L_i^r the subset L_{i0}^r provided by the lemma, the lemma yields the bound

$$I(P_i, L_i^r) = O\left(m_i^{1/2}n_i^{1/2}q^{1/4} + m_i + n_i\right).$$

That is, we have:

$$I(P_i, L_i) = O\left(m_i^{1/2}n_i^{1/2}q^{1/4} + m_i^{2/3}n_i^{1/3}s^{1/3} + m_i + n_i\right).$$

Summing these bounds for $i = 1, \dots, t$, and using Hölder's inequality (twice), we get the bound asserted in (19). \square

The case where f is linear or quadratic. (These are the cases $D = 1, 2$.) Let us apply Proposition 3.6 right away to bound the number of incidences when our (irreducible) f is linear or quadratic, that is, when $Z(f)$ is a hyperplane or a quadric. The proposition then implies the following bound.

$$I(P, L) = O\left(m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + m + n\right), \quad (20)$$

which is subsumed by the main bound (5).

Incidences within 2-flats fully contained in $Z(f, g)$. Assuming generic directions of the coordinate axes, we may assume that, for every non-singular point $p \in P$, $T_p Z(f)$ is not orthogonal to any of the axes. This allows us to use the flatness criterion developed in Section 2.5 to each point of P .

As in previous steps of the analysis, we simplify the notation by denoting the subsets of the points and lines that lie in the 2-flat components of $Z(f, g)$ as P and L , and their respective sizes as m and n . Each point $p \in P$ (resp., each line $\ell \in L$) under consideration is contained (resp., fully contained) in at least one 2-flat that is fully contained in $Z(f, g)$. Let π_1, \dots, π_k denote the 2-flats that are fully contained in $Z(f, g)$ (these are the linear irreducible components of $Z(f, g)$, and we have $k = O(D^2)$). Let $P^{(2)}$ (resp., $P^{(3)}$) denote the set of points $p \in P$ that lie in at most two (resp., at least three) of these 2-flats. Assign each point $p \in P^{(2)}$ to the (at most) two 2-flats containing it. Note that if $p \in P^{(2)}$, then every line ℓ that is incident to p can be contained in at most two of the 2-flats π_i , and we assign ℓ to those 2-flats. Let $L^{(2)}$ denote the set of lines $\ell \in L$ such that ℓ is incident to at least one point in $P^{(2)}$ (and is thus contained in at most two 2-flats π_i), and put $L^{(3)} = L \setminus L^{(2)}$. For $i = 1, \dots, k$, let $L_i^{(2)}$ (resp., $P_i^{(2)}$) denote the set of lines of $L^{(2)}$ (resp., points of $P^{(2)}$), that are contained in π_i , and put $n_i := |L_i^{(2)}|$, $m_i = |P_i^{(2)}|$. (Note that we ignore here lines that are not fully contained in one of these 2-flats; these lines are fully contained in other components of $Z(f, g)$ and their contribution to the incidence count has already been taken care of.) By construction,

$$\sum_{i=1}^k m_i \leq 2m, \quad \text{and} \quad \sum_{i=1}^k n_i \leq 2n.$$

Moreover, a point $p \in P^{(2)}$ can be incident only to lines of L that are contained in one of the (at most) two 2-flats that contain p , so we have $I(P^{(2)}, L^{(2)}) \leq \sum_{i=1}^k I(P_i^{(2)}, L_i^{(2)})$. The Szemerédi–Trotter bound (1) yields

$$I(P_i^{(2)}, L_i^{(2)}) = O\left(m_i^{2/3} n_i^{2/3} + m_i + n_i\right), \quad i = 1, \dots, k. \quad (21)$$

By assumption (ii) of the theorem, $n_i \leq s$ for each $i = 1, \dots, k$, so, summing over $i = 1, \dots, k$ and using Hölder’s inequality, we obtain

$$\begin{aligned} I(P^{(2)}, L^{(2)}) &\leq \sum_{i=1}^k I(P_i^{(2)}, L_i^{(2)}) = O\left(\sum_{i=1}^k \left(m_i^{2/3} n_i^{2/3} + m_i + n_i\right)\right) \\ &= O\left(\left(\sum_{i=1}^k m_i^{2/3} n_i^{1/3} s^{1/3}\right) + m + n\right) \\ &= O\left(\left(\sum_{i=1}^k m_i\right)^{2/3} \left(\sum_{i=1}^k n_i\right)^{1/3} s^{1/3} + m + n\right) \\ &= O\left(m^{2/3} n^{1/3} s^{1/3} + m + n\right). \end{aligned} \quad (22)$$

Consider next the points of $P^{(3)}$, each contained in at least three 2-flats that are fully contained in $Z(f)$. All the points of $P^{(3)}$ are linearly flat (see Section 2.5 for details), and are therefore flat. Notice that each such point can be incident to lines of $L^{(2)}$ and to lines of $L^{(3)}$. We prune away each line $\ell \in L$ that contains fewer than $3D$ points of $P^{(3)}$, losing at most $3nD$ incidences in the process.

Each of the surviving lines contains at least $3D - 3$ flat points, and is therefore flat, because the degrees of the nine polynomials whose vanishing at p captures the flatness of p , are all at most $3D - 4$. In other words, we are left with the task of bounding the number of incidences between flat points and flat lines. To simplify this part of the presentation, we again rename the sets of these points and lines as P and L , and denote their sizes by m and n , respectively.

Incidences between flat points and lines. By Lemma 2.15, all the (non-singular) points of a flat line have the same tangent hyperplane. We assign each flat point $p \in P$ (resp., flat line $\ell \in L$) to $T_p Z(f)$ (resp., to $T_\ell Z(f)$ for some (any) non-singular point $p \in P \cap \ell$; again we only consider lines incident to at least one such point). We have therefore partitioned P and L among *distinct* hyperplanes H_1, \dots, H_t , and we only need to count incidences between points and lines assigned to the same hyperplane. Using Proposition 3.6, the number of these incidences is

$$O\left(m^{1/2} n^{1/2} q^{1/4} + m^{2/3} n^{1/3} s^{1/3} + m + n\right). \quad (23)$$

In summary, combining the bounds in (22) and (23), Proposition 3.3, and Lemmas 3.4 and 3.5, the overall outcome of the analysis for the first case is summarized in the following proposition. (In the proposition, f is one of the irreducible factors of the original polynomial or of one of its derivatives, and P and L refer to the subsets assigned to that factor.)

Proposition 3.7. *Let g be any polynomial of degree $O(D)$ such that $Z(f, g)$ is two-dimensional, let P be a set of m points contained in $Z(f, g)$, and let L be a set of n lines contained in $Z(f, g)$. Then*

$$I(P, L) = I(P^*, L^*) + O\left(m^{1/2} n^{1/2} q^{1/4} + m^{2/3} n^{1/3} s^{1/3} + m + nD\right), \quad (24)$$

where P^* and L^* are subsets of P and L , respectively, so that each hyperplane or quadric is $O(D^2)$ -restricted with respect to L^* , and each 2-flat contains at most $O(D)$ lines of L^* . We also have the explicit estimates

$$I(P^*, L^*) = \begin{cases} O(mD^2 + nD) & \text{for } m = O(n^{4/3}) \\ O(m + nD^4) & \text{for } m = \Omega(n^{4/3}) . \end{cases}$$

Remarks. (1) As already noted, lines that are contained in 2-flats that are fully contained in $Z(f, g)$ have already been taken care of, and thus do not belong to L^* , so the application of Lemma 3.5 shows that every 2-flat has the property that it contains only $O(D)$ lines of L^* .

(2) When m and n are such that the bounds on $I(P^*, L^*)$ are dominated by the bound in (5), we use these bounds explicitly, and get the induction-free refined bound in (6). This remark will be expanded and highlighted later, within the part that presents the induction process.

Second case: $Z(f)$ is ruled by lines

We next consider the case where the four-dimensional flecnode polynomial FL_f^4 vanishes identically on $Z(f)$. By Theorem 2.11, this implies that $Z(f)$ is ruled by lines.

In what follows we assume that $D \geq 3$ (the cases $D = 1, 2$ have already been treated earlier, using Proposition 3.6). We prune away points $p \in P$, with $|\Sigma_p| \leq 6$ (the number of incidences involving these points is at most $6m = O(m)$). For simplicity of notation, we still denote the set of surviving points by P . Thus we now have $|\Sigma_p| > 6$, for every $p \in P$.

Recalling the properties of the u -resultant of f (that is, the u -resultant associated with $F_1(p; v)$, $F_2(p; v)$, $F_3(p; v)$), as reviewed in Section 2.6, we conclude, by Corollary 2.17, that $U(p; u_0, u_1, u_2, u_3) \equiv 0$ (as a polynomial in u_0, \dots, u_3) for every $p \in P$.

We will use the following theorem of Landsberg, which generalizes Theorem 2.11. It is stated here in a specialized and slightly revised form; but still for an arbitrary hypersurface in any dimension, and for any choice of the parameter k . Recall that Σ^k is the union of Σ_p^k (where these sets are now interpreted as sets of lines rather than of directions) over all $p \in X$, namely, it is the set of all lines that osculate to $Z(f)$ to order three at some point on $Z(f)$.

The actual application of the theorem will be for $X = Z(f)$ (and $d = 4, k = 3$).

We refer the reader to Section 2.1 for notations and further details.

Theorem 3.8 (Landsberg [17, Theorem 3.8.7]). *Let $X \subset \mathbb{P}^d(\mathbb{C})$ be a hypersurface, and let $2 \leq k$ be an integer, such that there is an irreducible component $\Sigma_0^k \subset \Sigma^k$ satisfying, for every point p in a Zariski open set $\mathcal{O} \subset Z(f)$, $\dim \Sigma_{0,p}^k > d - k - 1$, where $\Sigma_{0,p}^k$ is the set of lines in Σ_0^k incident to p . Then, for each point $p \in \mathcal{O}$, all lines in $\Sigma_{0,p}^k$ are contained in X .*

To appreciate the theorem, we note that, informally, lines through p have $d - 1$ degrees of freedom, and the constraint that such a line osculates to X to order k removes k degrees of freedom, leaving $d - k - 1$ degrees. The theorem asserts that if the dimension of this set of lines is larger, for most points on X , then these lines are fully contained in X . Note also that this is a “local-to-global” theorem—the large dimensionality condition has to hold at every point of some Zariski open subset of $Z(f)$, for the conclusion to hold.

If $U(p; u_0, u_1, u_2, u_3)$ does not vanish identically (as a polynomial in u_0, u_1, u_2, u_3) at every point $p \in Z(f)$, then at least one of its coefficients, call it c_U , does not vanish identically on $Z(f)$. In this case, as U vanishes identically at every point of P (as a polynomial in u_0, u_1, u_2, u_3), it follows that P is contained in the two-dimensional variety $Z(f, c_U)$. Since c_U has degree $O(D)$ in x, y, z, w (by

Theorem 2.16), we can proceed exactly as we did in the case where $Z(f, \text{FL}_f^4)$ was 2-dimensional. That is, we obtain the bound (24) in Proposition 3.7, namely,

$$I(P, L) = I(P^*, L^*) + O\left(m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + m + nD\right), \quad (25)$$

where P^* and L^* are subsets of P and L , respectively, so that each hyperplane or quadric is $O(D^2)$ -restricted with respect to L^* , and each 2-flat contains at most $O(D)$ lines of L^* . We also have the explicit estimates

$$I(P^*, L^*) = \begin{cases} O(mD^2 + nD) & \text{for } m = O(n^{4/3}) \\ O(m + nD^4) & \text{for } m = \Omega(n^{4/3}). \end{cases}$$

Therefore, since this case does not require the following analysis, it suffices to consider the complementary situation, where we assume that $U(p; u_0, u_1, u_2, u_3) \equiv 0$ at every point $p \in Z(f)$ (as a polynomial in u_0, u_1, u_2, u_3). By Theorem 2.16, Σ_p^3 is infinite, so its dimension is positive, for each such p .

Informally, the analysis proceeds as follows. Since Σ_p^3 is (at least) one-dimensional for each non-singular point $p \in Z(f)$, the set Σ^3 , which is the union of Σ_p^3 , over all $p \in Z(f)$, has (at least) three degrees of freedom—three for specifying p , at least one for specifying the line in Σ_p^3 , and one removed because the same line may arise at each of its points (if it is fully contained in $Z(f)$). In what follows we show that we can find a single irreducible component Σ_0^3 of Σ^3 , which is three-dimensional, and such that for any point $p \in Z(f)$, the variety $\Sigma_{0,p}^3$ is at least one-dimensional. This will facilitate the application of Theorem 3.8 in our context.

Theorem 3.9. *There exists an irreducible component Σ_0^3 of Σ^3 of dimension at least three, such that for each non-singular $p \in Z(f)$, the variety $\Sigma_{0,p}^3$ is at least one-dimensional.*

Proof. The proof makes use of the Theorem of the Fibers and related results, as reviewed in Section 2.3. Put

$$W := \{(p, \ell) \mid p \in \ell, \ell \in \Sigma_p^3\} \subset Z(f) \times \Sigma^3.$$

W can formally be defined as the zero set of homogeneous polynomials; the condition $p \in \ell$ is expressed as the vanishing of several bilinear forms, and the other polynomials are those defining the projective variety Σ_p^3 (when f is homogeneous, then osculation to order 3 can be expressed via homogeneous polynomials. See Section 2.1 for details). Therefore, W is a projective variety.

Let

$$\Psi_1 : W \rightarrow Z(f), \quad \Psi_2 : W \rightarrow \Sigma^3$$

be the (restrictions to W of the) projections to the first and second factors of the product.

For an irreducible component Σ_0^3 of Σ^3 (which is also a projective variety), put

$$W_0 := \Psi_2^{-1}(\Sigma_0^3) = \{(p, \ell) \in W \mid \ell \in \Sigma_{0,p}^3\}.$$

Since W and Σ_0^3 are projective varieties, so is W_0 . (Indeed, if $W = Z(\{f_i(p, \ell)\})$, and $\Sigma_0^3 = Z(\{g_j(\ell)\})$, for suitable sets of homogeneous polynomials $\{f_i\}, \{g_j\}$, then $W_0 = Z(\{f_i(p, \ell), g_j(\ell)\})$.)

Let \tilde{W}_0 denote some irreducible component of W_0 , and put $Y := \Psi_1(\tilde{W}_0) \subset Z(f)$. By the projective extension theorem (see, e.g., [4, Theorem 8.6]), Y is also a projective variety.

For a point $p \in Y$, the fiber of the map $\Psi_1|_{\tilde{W}_0} : \tilde{W}_0 \rightarrow Y$ over p is contained in $\{p\} \times \Sigma_{0,p}^3 = \{(p, \ell) \mid \ell \in \Sigma_{0,p}^3\}$ (this is the fiber of $\Psi_1|_{W_0}$ over p , which clearly contains the fiber of $\Psi_1|_{\tilde{W}_0}$ over p , as $\tilde{W}_0 \subseteq W_0$).

We will show that there exists some component Σ_0^3 , and some irreducible component \tilde{W}_0 of $W_0 = \Psi_2^{-1}(\Sigma_0^3)$, such that (i) $Y = \Psi_1(\tilde{W}_0)$ is equal to $Z(f)$, and (ii) for every point $p \in Z(f)$, the fiber of $\Psi_1|_{\tilde{W}_0} : \tilde{W}_0 \rightarrow Y$ over p is (at least) one-dimensional; in this case we say that Σ_0^3 and \tilde{W}_0 form a *one-dimensional line cover* of $Z(f)$. Suppose that we have found such a pair Σ_0^3, \tilde{W}_0 . As noted above, the fiber of $\Psi_1|_{\tilde{W}_0}$ over p is contained in (or equal to) $\{p\} \times \Sigma_{0,p}^3$, and $\dim(\{p\} \times \Sigma_{0,p}^3) = \dim(\Sigma_{0,p}^3)$. Therefore, since $Y = Z(f)$, this would imply that, for every $p \in Z(f)$, we have $\dim(\Sigma_{0,p}^3) \geq 1$, which is what we want to prove.

We pick some component Σ_0^3 , and some irreducible component \tilde{W}_0 of $W_0 = \Psi_2^{-1}(\Sigma_0^3)$, and analyze when do Σ_0^3 and \tilde{W}_0 form a one-dimensional line cover of $Z(f)$. Put, as above, $Y = \Psi_1(\tilde{W}_0)$. For a point $p \in Y$, put $\lambda(p) = \dim(\Psi_1|_{\tilde{W}_0}^{-1}(\{p\}))$, and let $\lambda = \min_{p \in Y} \lambda(p)$. As noted above, $\lambda(p) \leq \dim(\Sigma_{0,p}^3)$.

By the Theorem of the Fibers (Theorem 2.6), applied to the map $\Psi_1|_{\tilde{W}_0} : \tilde{W}_0 \rightarrow Y \subseteq Z(f)$, we have

$$\dim(\tilde{W}_0) = \dim(Y) + \lambda. \quad (26)$$

Observe that $\lambda \leq 1$. Indeed, if $\lambda = 2$, then there exists some non-singular point $p \in Y$, such that $\Sigma_{0,p}^3$ is (at least) two-dimensional, implying that $Z(f)$ is a three-dimensional cone; since p is non-singular, $Z(f)$ is thus a hyperplane, contrary to our assumptions.

Assume first that $Y = \Psi_1(\tilde{W}_0)$ is equal to $Z(f)$ (this is part (i) of the definition of a one-dimensional line cover). If $\lambda = 1$, part (ii) of this property also holds, and we are done. Assume then that $\lambda = 0$. By the first part of the Theorem of the Fibers (Theorem 2.6), the subset $Y_1 = \{p \in Y \mid \lambda(p) \geq 1\}$ is Zariski closed in Y , so it is a subvariety of $Z(f)$, of dimension at most 2. Hence, for each p in the Zariski open complement $Y \setminus Y_1$, the fiber $\Psi_1|_{\tilde{W}_0}^{-1}(\{p\})$ is finite.

The remaining case is when $Y = \Psi_1(\tilde{W}_0)$ is properly contained in $Z(f)$. Since $Z(f)$ is irreducible, Y is of dimension at most two.

To recap, we have proved that for each component Σ_0^3 of Σ^3 , and each component \tilde{W}_0 of W_0 , if the associated Y is properly contained in $Z(f)$, then the image of \tilde{W}_0 under Ψ_1 (that is, Y) is at most two-dimensional; we refer to this situation as being of the first kind. If $Y = Z(f)$ but $\lambda = 0$ (these are referred to as situations of the second kind), then, except for a two-dimensional subvariety Y_1 of $Z(f)$, the fibers of the map $\Psi_1|_{\tilde{W}_0}$ are finite.

However, in the case under consideration, we have argued that, for any non-singular point $p \in Z(f)$, the fiber $\Psi_1^{-1}(p) = \{p\} \times \Sigma_p^3$ is (at least) one-dimensional.

We apply this analysis to all the irreducible components Σ_0^3 of Σ^3 , and to all the irreducible components of the corresponding $W_0 = \Psi_2^{-1}(\Sigma_0^3)$. Let Y^* denote the union of all the images Y of the first kind, and of all the excluded subvarieties Y_1 of the second kind. Being a finite union of two-dimensional varieties, Y^* is two-dimensional.

The union, over the irreducible components Σ_0^3 of Σ^3 , of all the corresponding components \tilde{W}_0 , covers W , and therefore, for any non-singular point $p \in Z(f)$, the union over all the components \tilde{W}_0 of the fibers of $\Psi_1|_{\tilde{W}_0}$ over p is equal to the fiber of Ψ_1 over p , which is one-dimensional (and thus infinite).

We claim that there must exist some irreducible component Σ_0^3 of Σ^3 , and a corresponding irreducible component \tilde{W}_0 of W_0 , such that $Y = \Psi_1(\tilde{W}_0)$ is equal to $Z(f)$, and the corresponding λ is equal to 1. Indeed, if this were not the case, take any non-singular point p in $Z(f) \setminus Y^*$. Since p is not in the image $\Psi_1(\tilde{W}_0)$, for any \tilde{W}_0 of the first kind, the fiber of $\Psi_1|_{\tilde{W}_0}$ at p is empty. Similarly, since p is not in the excluded set Y_1 for any \tilde{W}_0 of the second kind, the fiber of $\Psi_1|_{\tilde{W}_0}$

at p is finite. But then the fiber of Ψ_1 at p , being a finite union of (empty or) finite sets, must be finite, a contradiction that establishes the claim.

Since for any $p \in Y = Z(f)$, $\lambda \leq \lambda(p) \leq \dim(\Sigma_{0,p}^3)$, it follows that all the fibers $\Sigma_{0,p}^3$ are (at least) one-dimensional, completing the proof. \square

Remark. One interesting corollary of the Theorem of the Fibers is that if we know that for any point p in a Zariski open subset \mathcal{O} of $Z(f)$, the fiber of Ψ_1 over p (which is equal to $\{p\} \times \Sigma_p^3$) is one-dimensional, then this is true for the entire $Z(f)$. Indeed, by the Theorem of the Fibers (Theorem 3.9), the set $\{p \in Z(f) \mid \dim(\Psi_1^{-1}(\{p\})) \geq 1\}$ is Zariski closed, and, since it contains the Zariski open set \mathcal{O} , it must be equal to $Z(f)$.

Theorem 3.8 (with $d = 4, k = 3, \mathcal{O} = Z(f) \setminus Z(f)_{\text{sing}}$, and Σ_0^3 as specified by Theorem 3.9) then implies that $Z(f)$ is *infinitely ruled* by lines, in the sense defined in Section 2.6; that is, each point $p \in Z(f)$ is incident to infinitely many lines that are fully contained in $Z(f)$, and, moreover, $\Sigma_{0,p}^3 = \Sigma_{0,p}$ (which is the set of lines in Σ_0 incident to p); in other words, $\Sigma_{0,p}$ is of dimension at least 1, or, equivalently, the cone $\Xi_{0,p}$ (which is the union of the lines in $\Sigma_{0,p}$) is at least two-dimensional. If, for some non-singular $p \in Z(f)$, the cone $\Xi_{0,p}$ were three-dimensional, then, as already noted, $Z(f)$ would be a hyperplane, contrary to assumption. Thus, for each non-singular point $p \in Z(f)$, the cone $\Xi_{0,p}$ is two-dimensional, and $\Sigma_{0,p}$ is one-dimensional. We also have $\dim(\Sigma_0) = \dim(\Sigma_0^3) \geq 3$. We thus have

Corollary 3.10. *The union of lines in $\Sigma_0^3 = \Sigma_0$, namely $\bigcup_{\ell \in \Sigma_0^3} \ell$, is equal to $Z(f)$, and $\dim(\Sigma_0) = \dim(\Sigma_0^3) \geq 3$.*

Severi's theorem. The following theorem is a major ingredient in the present part of our analysis. It has been obtained by Severi [32] in 1901, and a variant of it is also attributed to Segre [31]; it is mentioned in a recent work of Rogora [29], in another work of Mezzetti and Portelli [24], and also appears in the unpublished thesis of Richelson [28]. Severi's paper is not easily accessible (and is written in Italian). In a future version of this paper we will sketch, for the sake of completeness, a proof of this theorem (or rather of a special case of the theorem that arises in our context).

Theorem 3.11 (Severi's Theorem [32]). *Let $X \subset \mathbb{P}^d(\mathbb{C})$ be a k -dimensional irreducible variety, and let Σ_0 be an irreducible component of maximal dimension of $F(X)$, such that the lines of Σ_0 cover X . Then the following holds.*

1. *If $\dim(\Sigma_0) = 2k - 2$, then X is a copy of $\mathbb{P}^k(\mathbb{C})$ (that is, a complex projective k -flat).*
2. *If $\dim(\Sigma_0) = 2k - 3$, then either X is a quadric, or X is ruled by copies of $\mathbb{P}^{k-1}(\mathbb{C})$, i.e., every point⁹ $p \in X$ is incident to a copy of $\mathbb{P}^{k-1}(\mathbb{C})$ that is fully contained in X .*

As is easily checked, the maximum dimension of Σ_0 is $2k - 2$. Note also that the cases where $\dim \Sigma_0 < 2k - 3$ are not treated by the theorem (although they might occur); see Rogora [29] for a (partial) treatment of these cases.

We apply Severi's theorem to $Z(f)$, with $k = 3$ and with $\dim(\Sigma_0) = 3 = 2k - 3$ (by Corollary 3.10). We thus conclude that either $Z(f)$ is a quadric or it is ruled by 2-flats. If $Z(f)$ is a quadric, then again we can use Proposition 3.6, and obtain the bound

$$I(P, L) = O\left(m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + m + n\right). \quad (27)$$

⁹One can also define this property as done in Section 2) for the case of lines, that is, requiring that the ruling holds at every point in some Zariski-open subset of X . Here too one can show that the two definitions are equivalent.

The case where $Z(f)$ is ruled by 2-flats. It remains to handle the case where $Z(f)$ is ruled by 2-flats. That is, every (non-singular) point $p \in Z(f)$ is incident to at least one 2-flat $\tau_p \subset Z(f)$. Let D_p denote the set of 2-flats that pass through p and are contained in $Z(f)$.

For a non-singular point $p \in Z(f)$, if $|D_p| > 2$, then p is a (linearly flat and thus) flat point of $Z(f)$. Recall that we have bounded the number of incidences involving flat points (and lines) by partitioning them among a finite number of hyperplanes, and by bounding the incidences within each hyperplane. (Recall that lines incident to fewer than $3D - 3$ points of P have been pruned away, losing only $O(nD)$ incidences, and that the remaining lines are all flat.) Repeating this argument here, we obtain the bound

$$O\left(m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + m + n\right).$$

In what follows we therefore assume that all points of P are non-singular and non-flat (call these points *ordinary* for short), and therefore $|D_p| = 1$ or 2 , for each such p . Put $H_1(p)$ (resp., $H_1(p), H_2(p)$) for the 2-flat (resp., two 2-flats) in D_p , when $|D_p| = 1$ (resp., $|D_p| = 2$).

Clearly, each line in L , containing at least one ordinary point $p \in Z(f)$ is fully contained in at most two 2-flats (namely, the 2-flats of D_p).

Assign each point $p \in P$ to *each* of the at most two 2-flats in D_p , and assign each line $\ell \in L$ that is incident to at least one ordinary point to the at most two 2-flats that fully contain ℓ and are fully contained in $Z(f)$ (it is possible that ℓ is not assigned to any 2-flat—see below). Changing the notation, enumerate these 2-flats as U_1, \dots, U_k , and, for each $i = 1, \dots, k$, let P_i and L_i denote the respective subsets of points and lines assigned to U_i , and let m_i and n_i denote their cardinalities. We then have $\sum_i m_i \leq 2m$ and $\sum_i n_i \leq 2n$, and the total number of incidences within the 2-flats U_i (excluding lines not assigned to any 2-flat) is at most $\sum_{i=1}^k I(P_i, L_i)$. This can be established exactly as in the first case, using the bound in (22). That is, we have

$$\sum_{i=1}^k I(P_i, L_i) = O\left(m^{2/3}n^{1/3}s^{1/3} + m + n\right).$$

As noted, this bound does not take into account incidences involving lines which are not contained in any of the 2-flats U_i (and are therefore not assigned to any 2-flat). If ℓ is a non-singular line, and is not fully contained in any of the U_i , we call it a *piercing line* of $Z(f)$.

Lemma 3.12. *If ℓ is a piercing line of $Z(f)$, then the union of lines fully contained in $Z(f)$ and intersecting ℓ is equal to $Z(f)$.*

Proof. Let V denote this union. By a suitable extension to four dimensions of a similar result of Sharir and Solomon [35, Lemma 2.5], V is a variety. Clearly $V \subseteq Z(f)$. If V is strictly contained in $Z(f)$, then, since $Z(f)$ is irreducible, V must be a finite union of irreducible components V_1, \dots, V_k , each of dimension at most two. Let $p \in \ell$ be a non-singular point of $Z(f)$, and let $H_1(p) \in D_p$. Note that $H_1(p)$ is contained in V (because it is a union of lines fully contained in $Z(f)$ and intersecting ℓ at p). We claim that there exists some V_j such that $H_1(p) \subseteq V_j$. Indeed, otherwise, the intersection $H_1(p) \cap V_j$ would be one-dimensional for each $j = 1, \dots, k$ (a variety strictly contained in a 2-flat is of dimension at most one), and therefore

$$V \cap H_1(p) = \left(\bigcup_{j=1}^k V_j \right) \cap H_1(p) = \bigcup_{j=1}^k (V_j \cap H_1(p))$$

is a finite union of one-dimensional varieties, contradicting the fact that $H_1(p)$ is contained in V . This contradiction establishes the claim. Since $H_1(p)$ and V_j are two-dimensional irreducible varieties and $H_1(p) \subseteq V_j$, it follows that $H_1(p) = V_j$.

In other words, for each non-singular point $p \in \ell$ there exists a 2-flat $H_1(p) \in D_p$ which is equal to some component V_j . Consider only the components V_j that coincide with such a 2-flat. Since there are only finitely many components V_j of this kind, one of them, call it V_{j_0} has to intersect ℓ in infinitely many points, and therefore $\ell \subseteq V_{j_0}$. That is, ℓ is contained in the 2-flat V_{j_0} that is fully contained in $Z(f)$.

Now pick any point $p \in P \cap \ell$. By definition, since $p \in V_{j_0}$, V_{j_0} must be one of the (at most) two 2-flats in D_p . But then ℓ is fully contained in that 2-flat, which is one of the U_i 's, and therefore is not a piercing line. This contradiction completes the proof. \square

Remark. The last step of the proof shows that if a line ℓ contains a point of P then it is piercing (in and) only if it is not contained in any 2-flat fully contained in $Z(f)$.

Lemma 3.13. *Let $p \in Z(f)$ be a non-singular point. Then p is incident to at most one piercing line.*

Proof. Assume to the contrary that p is incident to two piercing lines $\ell_1, \ell_2 \in L$. We claim that the 2-flat π_{12} that is spanned by ℓ_1 and ℓ_2 is fully contained in $Z(f)$ (and thus, by the preceding remark, ℓ_1 and ℓ_2 are not piercing lines). Indeed, for any point $q \in \ell_1$, Lemma 3.12 implies that there exists some line $\ell_q \neq \ell_1$, incident to q , that intersect ℓ_2 and is fully contained in $Z(f)$. When q varies along the non-singular points of ℓ_1 , we get an infinite collection of lines, fully contained in both $Z(f)$ and π_{12} , i.e., in their intersection $Z(f) \cap \pi_{12}$. If π_{12} is not contained in $Z(f)$ then $Z(f) \cap \pi_{12} \neq \pi_{12}$, and Lemma 2.3 implies that it contains at most D lines, and therefore cannot contain the infinite union of lines $\bigcup_p \ell_p$. \square

Therefore, each non-singular point $p \in P$ is incident to at most one piercing line, and the total contribution of incidences involving piercing lines is at most m .

In summary, combining the bounds that we have obtained for the various subcases of the second case, we get the following proposition. As in the first case, here f refers to a single irreducible factor, D to its degree, and P and L refer to the subsets, of the original respective sets of points and lines, that are assigned to f .

Proposition 3.14. *Let P be a set of m points contained in $Z(f)$, and let L be a set of n lines contained in $Z(f)$, and assume that $Z(f)$ is ruled by lines and that f is of degree ≥ 3 . Then*

$$I(P, L) = I(P^*, L^*) + O\left(m^{1/2}n^{1/2}q^{1/4} + m^{2/3}n^{1/3}s^{1/3} + m + nD\right), \quad (28)$$

where P^* and L^* are subsets of P and L , respectively, so that each hyperplane or quadric is $O(D^2)$ -restricted with respect to L^* , and each 2-flat contains at most $O(D)$ lines of L^* . We also have the explicit estimates

$$I(P^*, L^*) = \begin{cases} O(mD^2 + nD) & \text{for } m = O(n^{4/3}) \\ O(m + nD^4) & \text{for } m = \Omega(n^{4/3}) \end{cases}. \quad (29)$$

The induction. In summary, after having exhausted all possible cases, we are in the following situation; we finally undo the shorthand notations that we have used, and re-express the various bounds in terms of the original parameters.

The first partitioning step has resulted in a collection of irreducible polynomials, which we write as f_1, \dots, f_k , with respective degrees D_1, \dots, D_k , all upper bounded by D . The points of P have been *partitioned* among the zero sets $Z(f_1), \dots, Z(f_k)$, into respective pairwise disjoint subsets P_1, \dots, P_k , including a leftover subset P' of points outside the zero set, and the lines of L have

been *partitioned* among the zero sets, into respective pairwise disjoint subsets L_1, \dots, L_k , so that the zero set to which a line is assigned fully contains it, and including a leftover subset L' of lines not fully contained in any zero set. Put $m_i = |P_i|$, $n_i = |L_i|$, for $i = 1, \dots, k$, and $m' = |P'|$, $n' = |L'|$. Then $m' + \sum_{i=1}^k m_i = m$, and $n' + \sum_{i=1}^k n_i = n$.

Then $I(P, L)$ is $I(P', L') + \sum_{i=1}^k I(P_i, L_i)$ plus the number of incidences between points in some $Z(f_i)$ and lines not fully contained in $Z(f_i)$. (Note that $I(P \setminus P', L')$ also counts incidences of this kind.) As we have argued, the total number of these additional incidences is $O(nD)$. That is, we have, for any choice of the degree D ,

$$I(P, L) \leq I(P', L') + nD + \sum_{i=1}^k I(P_i, L_i). \quad (30)$$

For each i , the preceding analysis culminates at the following bound.

$$I(P_i, L_i) = I(P_i^*, L_i^*) + O\left(m_i^{1/2} n_i^{1/2} q^{1/4} + m_i^{2/3} n_i^{1/3} s^{1/3} + m_i + n_i D\right), \quad (31)$$

where, for each i , P_i^* and L_i^* are subsets of P_i and L_i , respectively, so that each hyperplane or quadric is $O(D_i^2)$ -restricted with respect to L_i^* , and each 2-flat contains at most $O(D_i)$ lines of L_i^* . We also have the explicit estimates, for each i ,

$$I(P_i^*, L_i^*) = \begin{cases} O(m_i D_i^2 + n_i D_i) & \text{for } m_i = O(n_i^{4/3}) \\ O(m_i + n_i D_i^4) & \text{for } m_i = \Omega(n_i^{4/3}). \end{cases} \quad (32)$$

In addition, for the large values of D in (13), we have

$$I(P', L') = O\left(m^{2/5} n^{4/5} + m\right). \quad (33)$$

Induction-free derivation of the bound. To proceed with the analysis, for general values of m and n , we bound the various quantities $I(P_i^*, L_i^*)$ using induction. However, as asserted in the theorems, the cases where $m \leq n^{6/7}$ or $m \geq n^{5/3}$ admit an induction-free argument that yields the improved bound in (6), and we first dispose of these cases. (Recall that these are the original values of m and n , the respective sizes of the entire input sets P and L .)

Assume first that $m \leq n^{6/7}$. Substituting (31), (32), and (33) into (30), and using Hölder's inequality twice, we get, for the "large" choices of D ,

$$\begin{aligned} I(P, L) &= O\left(m^{2/5} n^{4/5} + m + m^{1/2} n^{1/2} q^{1/4} + m^{2/3} n^{2/3} s^{1/3} + mD^2 + nD\right) \\ &= O\left(m^{2/5} n^{4/5} + m + m^{1/2} n^{1/2} q^{1/4} + m^{2/3} n^{2/3} s^{1/3} + n\right), \end{aligned}$$

where we have used the fact that $mD^2 + nD = O(m^{2/5} n^{4/5} + n)$. This establishes (6). The case $m \geq n^{5/3}$ is handled in the same manner, using the bound $O(m + nD^4)$ instead, and the fact that this bound is $O(m)$ when $m \geq n^{5/3}$.

The induction via a new partitioning. We now proceed with the general case, where induction is needed. To simplify the notation, we (again, but only temporarily) drop the indices, and consider one of many (possibly a nonconstant number of) subproblems, involving a set $P (= P_i^*)$ of m ($\leq m_i$) points and a set $L (= L_i^*)$ of n ($\leq n_i$) lines, so that each hyperplane or quadric is $O(D^2)$ -restricted for L , and each 2-flat contains at most $O(D)$ lines of L ; here $D (= D_i)$ is the degree of the corresponding factor $f (= f_i)$, which is upper bounded by the value in (13).

To make the induction work, we choose a degree E , typically much smaller than D (see below for the actual value), and construct a new partitioning polynomial h of degree E for P . (Although $P \subset Z(f)$ and each line of L is fully contained in $Z(f)$, we ignore here f completely, possibly losing some structural properties of P and L , and consider only the partitioning induced by h .) With an appropriate value of $r = \Theta(E^4)$, we obtain $O(r)$ cells, each containing at most m/r points of P , and each line of L either crosses at most $E + 1$ cells, or is fully contained in $Z(h)$.

Set $P_0 := P \cap Z(h)$ and $P' := P \setminus P_0$. Similarly, denote by L_0 the set of lines of L that are fully contained in $Z(h)$, and put $L' := L \setminus L_0$. We repeat the whole analysis done so far, but with h and its degree E instead of f and D , for the points of P and the lines of L . That is, we apply, to our P and L (which themselves are, as we recall, subsets of the form P_i^* , L_i^* that arise in the partitions of the original P and L , as effected by the first partitioning step), the bounds given in (30), (31), and (32) (but not the one in (33)), with E instead of D . Moreover, in this application we exploit the property that each hyperplane or quadric is $O(D^2)$ -restricted with respect to L , and each 2-flat contains at most $O(D)$ lines of L . We thus get (where the parameters k , P_i , L_i , etc. are new and depend on h , but we recycle the notation in the interest of simplicity)

$$\begin{aligned} I(P, L) &\leq I(P', L') + nE + \sum_{i=1}^k I(P_i, L_i) \\ &= I(P', L') + nE + \sum_{i=1}^k I(P_i^*, L_i^*) + \sum_{i=1}^k O\left(m_i^{1/2} n_i^{1/2} D^{1/2} + m_i^{2/3} n_i^{1/3} D^{1/3} + m_i + n_i E\right). \end{aligned}$$

Using Hölder's inequality (twice) in the second sum, we get, for a suitable absolute constant a ,

$$I(P, L) \leq I(P', L') + a \left(m^{1/2} n^{1/2} D^{1/2} + m^{2/3} n^{1/3} D^{1/3} + m + nE \right) + \sum_{i=1}^k I(P_i^*, L_i^*).$$

The analysis bifurcates depending on whether $m = O(n^{4/3})$ or $m = \Omega(n^{4/3})$ (recall that these are not the original values of m and n).

For $m = O(n^{4/3})$ we have

$$\sum_{i=1}^k I(P_i^*, L_i^*) \leq a' \left(\sum_{i=1}^k (m_i E^2 + n_i E) \right) \leq a' (m E^2 + n E),$$

and for $m = \Omega(n^{4/3})$ we have

$$\sum_{i=1}^k I(P_i^*, L_i^*) \leq a' \left(\sum_{i=1}^k (m_i + n_i E^4) \right) \leq a' (m + n E^4),$$

for a suitable absolute constant a' .

That is, slightly increasing the coefficient a , we have

$$\begin{aligned} I(P, L) &\leq I(P', L') + a \left(m^{1/2} n^{1/2} D^{1/2} + m^{2/3} n^{1/3} D^{1/3} + m + nE \right) \\ &\quad + \begin{cases} a m E^2 & \text{for } m = O(n^{4/3}) \\ a n E^4 & \text{for } m = \Omega(n^{4/3}). \end{cases} \end{aligned} \tag{34}$$

We next turn to bound $I(P', L')$. For each cell τ of $\mathbb{R}^4 \setminus Z(h)$, put $P_\tau := P' \cap \tau$, and let L_τ denote the set of the lines of L' that cross τ ; put $m_\tau = |P_\tau| \leq m/r$ (where $r = \Theta(E^4)$), and $n_\tau = |L_\tau|$.

Since every line $\ell \in L'$ crosses at most $E + 1$ components of $\mathbb{R}^4 \setminus Z(h)$, we have $\sum_{\tau} n_{\tau} \leq n(1 + E)$. (By the same token, we also have $I(P_1, L') \leq nE$, which has already been subsumed by the bound in (34).)

To simplify the application of the induction hypothesis within the cells of the partition, we want to make the subproblems of uniform size, so that $m_{\tau} = m/E^4$ and $n_{\tau} = n/E^3$ for each τ (the latter quantity, up to some constant, is the average number of lines crossing a cell). This is easy to enforce: To achieve $m_{\tau} = m/E^4$, we simply partition P_{τ} into $\lceil m_{\tau}/(m/E^4) \rceil = O(1)$ subsets, each consisting of at most m/E^4 points, and analyze each subset separately. Similarly, if τ is crossed by $\xi n/E^3$ lines, for $\xi > 1$, we treat τ as if it occurs $\lceil \xi \rceil$ times, where each incarnation involves all the points of (each of the constantly many corresponding subsets of) P_{τ} , and at most n/E^3 lines of L_{τ} . The number of subproblems remains $O(E^4)$, with a larger constant of proportionality.

We apply the induction hypothesis for each cell τ , to obtain

$$\begin{aligned} I(P_{\tau}, L_{\tau}) &\leq 2^{c\sqrt{\log m_{\tau}}} \left(m_{\tau}^{2/5} n_{\tau}^{4/5} + m_{\tau} \right) + \beta A \left(m_{\tau}^{1/2} n_{\tau}^{1/2} D^{1/2} + m_{\tau}^{2/3} n_{\tau}^{1/3} D^{1/3} + n_{\tau} \right) \\ &= 2^{c\sqrt{\log(m/E^4)}} \left((m/E^4)^{2/5} (n/E^3)^{4/5} + m/E^4 \right) \\ &\quad + \beta A \left((m/E^4)^{1/2} (n/E^3)^{1/2} D^{1/2} + (m/E^4)^{2/3} (n/E^3)^{1/3} D^{1/3} + n/E^3 \right), \end{aligned}$$

for a suitable absolute constant β . Summing this bound over all cells, that is, multiplying it by $O(E^4)$, we get, for a suitable absolute constant b ,

$$\begin{aligned} \sum_{\tau} I(P_{\tau}, L_{\tau}) &\leq b \cdot 2^{c\sqrt{\log(m/E^4)}} \left(m^{2/5} n^{4/5} + m \right) \\ &\quad + bA \left(m^{1/2} n^{1/2} D^{1/2} E^{1/2} + m^{2/3} n^{1/3} D^{1/3} E^{1/3} + nE \right). \end{aligned} \tag{35}$$

We have

$$\begin{aligned} 2^{c\sqrt{\log(m/E^4)}} &= 2^{c\sqrt{\log m - 4 \log E}} = 2^{c\sqrt{\log m}} \left(1 - \frac{4 \log E}{\log m} \right)^{1/2} \\ &< 2^{c\sqrt{\log m}} \left(1 - \frac{2 \log E}{\log m} \right) = \frac{2^{c\sqrt{\log m}}}{2^{2c \log E / \sqrt{\log m}}}. \end{aligned}$$

We choose E to ensure that, for a suitable sufficiently large constant κ ,

$$2^{2c \log E / \sqrt{\log m}} > \kappa b, \quad \text{or} \quad \frac{2c \log E}{\sqrt{\log m}} > \log(\kappa b), \quad \text{or} \quad \log E > \frac{\log(\kappa b)}{2c} \sqrt{\log m}.$$

That is, we choose

$$E > 2^{c^* \sqrt{\log m}}, \quad \text{for} \quad c^* = \frac{\log(\kappa b)}{2c} < c/3, \tag{36}$$

where the last constraint can be enforced if c is chosen sufficiently large. With this constraint on the choice of E , (35) becomes

$$\begin{aligned} \sum_{\tau} I(P_{\tau}, L_{\tau}) &\leq \frac{1}{\kappa} 2^{c\sqrt{\log m}} \left(m^{2/5} n^{4/5} + m \right) \\ &\quad + bA \left(m^{1/2} n^{1/2} D^{1/2} E^{1/2} + m^{2/3} n^{1/3} D^{1/3} E^{1/3} + nE \right). \end{aligned} \tag{37}$$

Adding this bound to the one in (34), we get

$$\begin{aligned} I(P, L) &\leq \frac{1}{\kappa} 2^{c\sqrt{\log m}} \left(m^{2/5} n^{4/5} + m \right) \\ &\quad + (bA + a) \left(m^{1/2} n^{1/2} D^{1/2} E^{1/2} + m^{2/3} n^{1/3} D^{1/3} E^{1/3} + nE \right) + am \\ &\quad + \begin{cases} amE^2 & \text{for } m = O(n^{4/3}) \\ anE^4 & \text{for } m = \Omega(n^{4/3}). \end{cases} \end{aligned} \tag{38}$$

We now bifurcate depending on the relation between m and n (again, recall that these are not the original values of these parameters).

The case $m = O(n^{4/3})$. Recall that here we take $D = O(m^{2/5}/n^{1/5})$ (for the current values of m and n). It is easily checked that, for this choice of D , each of the terms $m^{1/2}n^{1/2}D^{1/2}$, $m^{2/3}n^{1/3}D^{1/3}$, m , and n , is $O(m^{2/5}n^{4/5})$, because $n^{1/2} \leq m = O(n^{4/3})$.

We choose $E = 2^{c^* \sqrt{\log m}}$. This turns (38) into the bound

$$I(P, L) \leq \left(\frac{1}{\kappa} 2^{c \sqrt{\log m}} + \mu 2^{2c^* \sqrt{\log m}} \right) \left(m^{2/5} n^{4/5} + m \right),$$

for a suitable absolute constant μ . The choice of c^* , and the assumption that $m \geq M_0$ and that M_0 is sufficiently large, ensure that

$$\mu 2^{2c^* \sqrt{\log m}} < \frac{1}{\kappa} 2^{c \sqrt{\log m}},$$

and thus we get

$$I(P, L) \leq \frac{2}{\kappa} 2^{c \sqrt{\log m}} \left(m^{2/5} n^{4/5} + m \right).$$

The case $m = \Omega(n^{4/3})$. Here we take $D = O(n/m^{1/2})$. It is easily checked that, for this choice of D , each of the terms $m^{1/2}n^{1/2}D^{1/2}$, $m^{2/3}n^{1/3}D^{1/3}$, $m^{2/5}n^{4/5}$, and n , is $O(m)$, because $m = \Omega(n^{4/3})$.

We choose, as before, $E = 2^{c^* \sqrt{\log m}}$, and note that, for $m \geq M_0$ sufficiently large, the term nE^4 is also $O(m)$. This turns (38) into the bound

$$I(P, L) \leq \left(\frac{1}{\kappa} 2^{c \sqrt{\log m}} + \mu 2^{2c^* \sqrt{\log m}} \right) \left(m^{2/5} n^{4/5} + m \right),$$

for a suitable absolute constant μ . As above, the choice of c^* , and the assumption that $m \geq M_0$ and that M_0 is sufficiently large, ensure that

$$\mu 2^{2c^* \sqrt{\log m}} < \frac{1}{\kappa} 2^{c \sqrt{\log m}},$$

and thus we get

$$I(P, L) \leq \frac{2}{\kappa} 2^{c \sqrt{\log m}} \left(m^{2/5} n^{4/5} + m \right).$$

Returning to the original notations, we have just bounded $I(P_i^*, L_i^*)$, for any i . Concretely, we have shown that

$$I(P_i^*, L_i^*) \leq \frac{2}{\kappa} 2^{c \sqrt{\log m_i}} \left(m_i^{2/5} n_i^{4/5} + m_i \right).$$

We next plug these bounds into (31), taking $\kappa > 4$, and get, for each i ,

$$I(P_i, L_i) \leq b \left(m_i^{1/2} n_i^{1/2} q^{1/4} + m_i^{2/3} n_i^{1/3} s^{1/3} + m_i + n_i D \right) + \frac{1}{2} 2^{c \sqrt{\log m_i}} \left(m_i^{2/5} n_i^{4/5} + m_i \right).$$

Since $m_i \leq m$ for each i , and since $m_i + n_i D_i = O \left(m_i^{2/5} n_i^{4/5} + m_i \right)$, we get

$$I(P_i, L_i) \leq b \left(m_i^{1/2} n_i^{1/2} q^{1/4} + m_i^{2/3} n_i^{1/3} s^{1/3} \right) + \left(O(b) + \frac{1}{2} 2^{c \sqrt{\log m}} \right) \left(m_i^{2/5} n_i^{4/5} + m_i \right).$$

Summing these bounds over all i , and applying Hölder's inequality twice, using the fact the $m > M_0$, and taking A to be larger than b , we finally get

$$I(P, L) \leq 2^{c\sqrt{\log m}} \left(m^{2/5} n^{4/5} + m \right) + A \left(m^{1/2} n^{1/2} q^{1/4} + m^{2/3} n^{1/3} s^{1/3} + n \right),$$

thereby completing, at last, the induction step and thus establishing the general upper bound (5) in the theorem. We have already shown how to get the improved bound in (6) for $m \leq n^{6/7}$ or for $m \geq n^{5/3}$, and the lower bound construction is given in the following section. Altogether, these complete the proof of the theorem. \square

4 The lower bound

In this section we present a construction that shows that the bound asserted in the theorem is worst-case tight (except for the factor $2^{c\sqrt{\log m}}$), for each m and n , and for q and s in suitable corresponding ranges. The construction is a generalization to four dimensions of a construction due to Elekes; see [8].

We have already remarked that the “lower order” terms $m^{1/2} n^{1/2} q^{1/4}$ and $m^{2/3} n^{1/3} s^{1/3}$ are both worst-case tight, as they can be attained by a suitable packing of points and lines into hyperplanes (for the first term) or planes (for the second term). Specifically, create n/q parallel hyperplanes, and place on each of them q lines and mq/n points in a configuration that attains the three-dimensional lower bound as in Guth and Katz [14]. Overall, we get

$$(n/q) \cdot \Theta((mq/n)^{1/2} q^{3/4}) = \Theta(m^{1/2} n^{1/2} q^{1/4})$$

incidences. A similar construction can be carried out for the second term $m^{2/3} n^{1/3} s^{1/3}$.

We therefore focus on the term $m^{2/5} n^{4/5}$ (the remaining terms m and n are trivial to attain).

We fix two integer parameters k and ℓ , with concrete values that will be set later, and take P to be the set of vertices of the integer grid

$$\{(x, y, z, w) \mid 1 \leq x \leq k, 1 \leq y, z, w \leq 2k\ell\}.$$

We have $|P| = 8k^4\ell^3$.

We then take L to be the set of all lines of the form

$$y = ax + b, \quad z = cx + d, \quad w = ex + f, \tag{39}$$

where $1 \leq a, c, e \leq \ell$ and $1 \leq b, d, f \leq k\ell$. We have $|L| = k^3\ell^6$. Note that each line in L has k incidences with the points of P , one for each $x = 1, 2, \dots, k$, so

$$I(P, L) = k^4\ell^6 = \Theta(|P|^{2/5}|L|^{4/5}),$$

as is easily checked. Note that $|L|^{1/2} \leq |P| \leq 8|L|^{4/3}$, which is (asymptotically) the range of interest for this bound to be significant: when $|P| < |L|^{1/2}$ we have the trivial bound $I(P, L) = O(|L|)$, and when $|P| > |L|^{4/3}$, the leading term in the bound changes qualitatively to $O(m)$, which is trivial for a lower bound. Moreover, for any pair of integers m, n , with $n^{1/2} \leq m \leq n^{4/3}$, we can find k and ℓ for which $|P| = \Theta(m)$ and $|L| = \Theta(n)$. Specifically, choose $k = \Theta(m^{2/5}/n^{1/5})$ and $\ell = \Theta(n^{4/5}/m^{3/5})$; both are ≥ 1 for the range of m and n under consideration.

To complete the construction, we show that no hyperplane or quadric can contain more than $q_0 := O(|L|^{6/5}/|P|^{2/5}) = O(k^2\ell^6)$ lines of L , and no plane can contain more than $s_0 := O(|L|^{7/5}/|P|^{4/5}) =$

$O(k\ell^6)$ lines of L . As an easy calculation shows, these threshold values of q and s are such that, for $q > q_0$ or $s > s_0$, the corresponding “lower-dimensional” term $m^{1/2}n^{1/2}q^{1/4}$ or $m^{2/3}n^{1/3}s^{1/3}$ dominates the “leading” term $m^{2/5}n^{4/5}$, making the above construction pointless (see below for more details). The actual values of q and s that we will now derive are actually much smaller.

To estimate our q and s , let h be an arbitrary hyperplane. If h is orthogonal to the x -axis then it does not contain any line of L , as is easily checked, so we may assume that h intersects any hyperplane of the form $x = i$ in a 2-plane π_i . The intersection of P with $x = i$ is a $2k\ell \times 2k\ell \times 2k\ell$ lattice, that we denote as Q_i . Every line $\lambda \in L$ in h meets π_i at a single point (as noted, it cannot be fully contained in π_i), which is necessarily a point in Q_i (every line of L contains a point of every Q_i). The size of $\pi_i \cap Q_i$ is easily seen to be $O((k\ell)^2)$, and each point is incident to at most ℓ^2 lines that lie in h . To see this latter property, substitute the equations (39) of a line of L into the linear equation defining h , say $Ax + By + Cz + Dw - 1 = 0$ (where B, C and D are not all 0). This yields a linear equation in x , whose x -coefficient has to vanish. This in turn yields a linear equation in a, c , and e , which can have at most ℓ^2 solutions over $[1, \dots, \ell]^3$ (it is easily checked that the x -coefficient cannot be identically zero for all choices of a, c, e). The number of lines of point-line incidences of P and L within h is thus $O(\ell^2(k\ell)^2) = O(k^2\ell^4)$. Since each line is incident to k points, necessarily all lying in h , it follows that the number of lines of L in h is $O(k^2\ell^4/k) = O(k\ell^4)$, which is always smaller than q_0 .

This analysis easily extends to show that no quadric contains more than $O(k\ell^4)$ lines of L ; we omit the routine details.

Finally, let π be a 2-plane, where again we may assume that π is not orthogonal to the x -axis. Then π meets a hyperplane $x = i$ in a line μ , and $\mu \cap Q_i$ contains at most $k\ell$ points. Every line λ in π meets μ at one of these points and, arguing as above, each such point can be incident to at most ℓ lines that lie in π (now instead of one linear equation in a, c, e , we get two). Hence, π contains at most $k\ell^2/k = \ell^2$ lines of L , which is always smaller than s_0 .

We have thus shown that the bound in Theorem 1.3 is (almost) tight in the worst case. The bound will be tight when $|P| \leq |L|^{6/7}$, which occurs when $k \leq \ell^{3/2}$, as an easy calculation shows.

5 Conclusion

The results of this paper (almost) settle the problem of point-line incidences in four dimensions, but they raise several interesting and challenging open problems. Among them are:

(a) Get rid of the factor $2^{c\sqrt{\log m}}$ in the bound. We have achieved this improvement when m is not too close to $n^{4/3}$, so to speak, allowing us to use the weak but non-inductive bounds and complete the analysis in one step. We believe that the ranges of m where this can be done can be enlarged, e.g. by improving the weak bounds. A concrete step in this direction would be to improve the term $O(nD^4)$ in the second bound in Proposition 3.3, which, as already remarked, appears to be too weak. It would also be interesting to improve the bound using the strategy in [33, 35], which generates a sequence of ranges of m , converging to $m = \Theta(n^{4/3})$, where in each range the improved bound (6) holds, with a different constant of proportionality A .

(b) Extend (and sharpen) the bound of Corollary 1.4 for any value of k . In particular, is it true that the number of intersection points of the lines (this is the case $k = 2$; the intersection points are also known as *2-rich points*) is $O(n^{4/3} + nq^{1/2} + ns)$? We conjecture that this is indeed the case. A deeper question, extending a similar open problem in three dimensions that has been posed by Guth and others (see, e.g., Katz’s expository note [22]), is whether the above conjectured bound can be improved when $q = o(n^{2/3})$ and $s = o(n^{1/3})$, that is, when the second and third terms in

the conjectured bound become much smaller than the term $n^{4/3}$. We also note that if we could establish such a bound for the number of k -rich points, for any constant k (when q and s are not too large), then the case of large m (that is, $m = \Omega(n^{4/3})$) would become vacuous, as only $O(n^{4/3})$ points could be incident to more than k lines.

(c) Extend the study to five and higher dimensions. In a preliminary ongoing study, joint with Adam Sheffer, we can do it using a constant-degree partitioning polynomial, with the disadvantages discussed above (slightly weaker bounds, significantly more restrictive assumptions, and inferior “lower-dimensional” terms). The leading terms in the resulting bounds, for points and curves in \mathbb{R}^d , are $O(m^{2/(d+1)+\varepsilon}n^{d/(d+1)} + m^{1+\varepsilon})$. Obtaining sharper results, like the ones obtained in this paper, is quite challenging algebraically, although some of the tools developed in this work seem promising for higher dimensions too.

(d) If we are given in advance that the points and lines lie in some algebraic surface of a given degree $D > 2$, can we improve the bound and/or simplify the analysis? In our companion work [36] we achieve these goals for the three-dimensional case, improving the bound of Guth and Katz [14].

(e) Elaborating on item (a) above, we note that the “culprit” Proposition 3.3, which produces the weak bounds that force us to go into the induction, is only used in the case where $Z(f, g)$ is two-dimensional, and the difficulty there lies in bounding the number of incidences within a two-dimensional ruled surface (be it either one irreducible ruled surface of large degree, or the union of many irreducible ruled surfaces of small degree). The analysis of the three-dimensional analogous situation (addressed in Guth and Katz [14]), cannot be applied here, since the degree of the underlying surface is $O(D^2)$ instead of D in [14]. In a recent study of Szemerédi-Trotter type theorems in three dimensions [18], Kollár uses the *arithmetic genus* of curves to prove effective bounds on the number of point-line incidences in three dimensions. In four dimensions, the situation is more involved, but we hope that the arithmetic genus of the surface $Z(f, g)$ may yield effective bounds for the number of incidences within this surface.

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