Distinct distances between points and lines

Micha Sharir[1]   Shakhar Smorodinsky[2]   Claudiu Valculescu[3]   Frank de Zeeuw[3]

June 28, 2015

Abstract

We show that for m points and n lines in $\mathbb{R}^2$, the number of distinct distances between the points and the lines is $\Omega(m^{1/5}n^{3/5})$, as long as $m^{1/2} \le n \le m^2$. We also show that for any m points in the plane, not all on a line, the number of distances between these points and the lines that they span is $\Omega(m^{4/3})$. We reduce the problem of bounding the number of distinct point-line distances to the problem of bounding the number of tangent pairs among a finite set of lines and a finite set of circles in the plane. We believe that this latter question is of independent interest. We also show that n circles in the plane determine at most $O(n^{3/2})$ points where two or more circles are tangent. This improves the best previously known bound of $O(n^{3/2} \log n)$. Finally, we study the higher-dimensional version of the distinct point-line distances, namely, the number of distinct point-hyperplane distances in $\mathbb{R}^d$.

1   Introduction

In 1946 Paul Erdős [4] posed the following two problems, which later became extremely influential and central questions in combinatorial geometry: the so-called *repeated distances* and *distinct distances* problems. The first problem deals with the number of repeated distances, namely the number of pairs of points at some fixed distance, in the plane. The best known upper bound is $O(n^{4/3})$ [12], but the best known lower bound, attained by the $\sqrt{n} \times \sqrt{n}$ grid, is only $\Omega(n^{1+\frac{c}{\log \log n}})$ [4]. The second problem asks at least how many distinct distances are determined by a set of n points in the plane? In a recent breakthrough, Guth and Katz [7] proved that the number of distinct distances is $\Omega(n/\log n)$ for any set of n points in the plane. This bound is nearly tight in the worst case, since an upper bound $O(n/\sqrt{\log n})$ is attained by the $\sqrt{n} \times \sqrt{n}$ grid [4].

In this paper we consider questions similar to those above, but for distances *between points and lines*.

Let P be a set of m distinct points and L a set of n distinct lines in the plane. Let $I(P, L)$ denote the number of incidences between P and L. Namely, the number of pairs $(p, \ell) \in P \times L$

such that the point p lies on the line ℓ. The classical result of Szemerédi and Trotter [14] asserts that

$$I(P, L) = O\left(m^{2/3}n^{2/3} + m + n\right). \tag{1}$$

This bound is best possible in general, by constructions due to Erdős and to Elekes. See the survey of Pach and Sharir [9] for more details on geometric incidences.

The point-line incidence setup can be viewed as a special instance of a repeated distance problem between points and lines. Specifically, the Szemerédi-Trotter result provides a sharp bound on the number of point-line pairs such that the point is at distance 0 from the line. As a matter of fact, the same bound holds if we consider pairs $(p, \ell) \in P \times L$ that have any fixed positive distance, say 1. Indeed, replace each line $\ell \in L$ by a pair ℓ^+, ℓ^- of lines parallel to ℓ and at distance 1 from it. Then any point $p \in P$ at distance 1 from ℓ must lie on one of these lines. Hence the number of point-line pairs at distance 1 is at most the number of incidences between the m points of P and the $2n$ lines ℓ^+, ℓ^-, for $\ell \in L$. (Actually, a line in the shifted set might arise twice, but this does not affect the asymptotic bound.) This proves that the number of times a single distance can occur between m points and n lines is $O\left(m^{2/3}n^{2/3} + m + n\right)$.

Distinct point-line distances. Our first main result concerns distinct distances between m points and n lines in the plane. In contrast with the repeated distances question, as discussed above, the distinct distances variant seems harder than for point-point distances, and the lower bound that we are able to derive is inferior to that of [7]. Nevertheless, deriving this bound is not an easy task, and follows by a combination of several advanced tools from incidence geometry. We hope that our work will trigger further research into this problem.

We write $D(m, n)$ for the minimum number of point-line distances determined by a set of m points and a set of n lines in $\mathbb{R}^2$. Our first main theorem is the following lower bound for $D(m, n)$.

Theorem 1.1. *For $m^{1/2} \leq n \leq m^2$, the minimum number $D(m, n)$ of point-line distances between m points and n lines in $\mathbb{R}^2$ satisfies*

$$D(m, n) = \Omega\left(m^{1/5}n^{3/5}\right).$$

Our proof also yields a stronger statement: For any set P of m points, and any set L of n lines in the plane, there always exists a point $p \in P$ such that the number of distinct distances from p to L satisfies the bound in Theorem 1.1.

In order to derive the above bound, we study the problem of bounding the number of tangencies between n lines and k circles; see Theorem 1.3 below.

We note that the upper bound $D(m, n) \leq \lceil n/2 \rceil$ is easy to achieve by the following construction. Place n parallel lines, say the horizontal lines $y = j$ for integers $j = 1, \ldots, n$. Place all points on the line $y = n/2 + 1/2$. Since all points on the median line have the same distance from any given horizontal line, the number of possible distinct point-lines distances is $\lceil n/2 \rceil$. Note that this upper bound does indeed dominate the bound in Theorem 1.1, because $m^{1/2} \leq n$.

Distinct distances between points and their spanned lines. We also study the number of distinct point-line distances between a finite set of non-collinear points (that is, not all points lie on a common line) and the set of lines that they span. This question has a different flavor, because the number of lines spanned by m non-collinear points varies from m to $\binom{m}{2}$. When the points span many lines, Theorem 1.1 provides a reasonable bound, which can get as high as $\Omega(m^{7/5})$, but when the points span few lines, we use a different approach to obtain a better overall bound.

We write $H(m)$ for the minimum number of distances between m points in $\mathbb{R}^2$, not all collinear, and the lines spanned by these points. Note that the distance between a point p and the line spanned by two points q, r equals the *height* to qr in the triangle pqr. Thus $H(m)$ is also the minimum number of heights occuring in the triangles determined by a set of m non-collinear points.

Theorem 1.2. *The minimum number $H(m)$ of point-line distances between m non-collinear points and their spanned lines satisfies*

$$H(m) = \Omega\left(m^{4/3}\right).$$

An upper bound $H(m) \leq m^2$ follows from a simple construction. Place $m-1$ points on a line, and one point off the line. This configuration spans only m lines, and therefore it determines at most m^2 point-line distances. The same bound also holds for other constructions, like the vertex set of a regular polygon, or an integer grid.

The lower bound in Theorem 1.2 is most likely not tight, but it currently seems hard to improve, because in the extreme configuration with $m - 1$ points on a line, say the x-axis, and one point off the line, say at $(0, 1)$, it corresponds to a lower bound on the number of distinct values of the rational function $f(x, y) = (x - y)^2/(1 + y^2)$, with x, y from a set $S \subset \mathbb{R}$ of size m. Even for simpler functions, such as bivariate polynomials in x, y, no better bound than $\Omega(m^{4/3})$ is known (see, e.g., [10]).

Tangencies involving lines and circles. As mentioned, our proof of Theorem 1.1 is based on an analysis of the number of tangencies between lines and circles. We believe this question to be of independent interest, and we now consider it in more detail.

Given a finite set L of lines and a finite set C of circles, we write $T(L, C)$ for the number of *tangencies* between lines from L and circles from C, i.e. the number of pairs $(\ell, c) \in L \times C$ such that the line ℓ is tangent to the circle c. We prove the following upper bound.

Theorem 1.3. *Let L be a set of n distinct lines and C a set of k distinct circles in the plane. Then*

$$T(L, C) = O\left(n^{2/3}k^{2/3} + n^{6/11}k^{9/11}\log^{2/11} k + k + n\right).$$

From this theorem we can deduce a lower bound on $D(m, n)$, but the resulting bound (see Corollary 2.2 below) is weaker than that in Theorem 1.1. In the proof of Theorem 1.1 we improve this bound by exploiting the specific structure of the problem.

It is natural to consider the corresponding question for tangencies between circles: Given n circles in the plane, how many pairs of circles can be tangent? This is related to the problem of bounding the number of pairwise non-overlapping *lenses* in an arrangement of n circles in the plane, which is of central significance in the derivation of the best bound on the number of point-circle incidences; see [1]. The best known upper bound for lenses is $O(n^{3/2} \log n)$ (see [8]), and the best known lower bound is $\Omega(n^{4/3})$, for both versions. The analogy between tangencies and lenses is not tight, since one can easily construct n circles, all pairs of which are tangent to each other at a common point.[1] Nevertheless, under the restructions imposed in the following two theorems, we can slightly improve the upper bound to $O(n^{3/2})$ for the tangency problem.

Theorem 1.4. *Let C be a family of n circles in $\mathbb{R}^2$ with arbitrary radii. Assume that no three circles of C are mutually tangent at a common point. Then C has at most $O(n^{3/2})$ pairs of tangent circles.*

---

[1] In such a case, any slight perturbation of the circles generates only $O(n)$ pairwise non-overlapping lenses near the previous common tangency point.

We also consider the following variant of this question, which bounds the number of *tangency points* without any conditions on the circles.

Theorem 1.5. *Given a family C of n circles in $\mathbb{R}^2$ with arbitrary radii, there are $O(n^{3/2})$ points where (at least) two circles of C are tangent.*

Distinct point-hyperplane distances. Finally, we consider higher-dimensional variants of the question about distinct point-line distances. Theorem 1.7 gives a lower bound on the number of distances between points and hyperplanes in any dimension, where certain configurations have to be excluded. Theorem 1.6 gives an improvement on this bound in $\mathbb{R}^3$.

Due to lack of space, we have moved the proofs of these theorems, and further introductory remarks, to Appendix A. The definitions of the excluded configurations can also be found in Appendix A.

Theorem 1.6. *Let P be a set of m distinct points and let Π be a set of n distinct planes in $\mathbb{R}^3$. Assume that there is no cone or cylinder configuration of three points and three planes. Then the number of distinct point-plane distances determined by P and Π is*

$$\Omega(m^{1/3}n^{1/3}),$$

unless $m = O(1)$ or $m = \Omega(n^2)$.

Theorem 1.7. *Let P be a set of m distinct points and let Π be a set of n distinct hyperplanes in $\mathbb{R}^d$. Assume that these points and hyperplanes determine no cone or cylinder configuration of k points and k hyperplanes, for some constant k. Then the number of distinct point-hyperplane distances determined by P and Π is*

$$\Omega\left(m^{\frac{1}{d}-\frac{1}{d^2}}n^{\frac{1}{d}-\varepsilon}\right),$$

for any $\varepsilon > 0$, unless $m = O(1)$ or $m = \Omega(n^2)$, where the implied constant of proportionality depends on d and ε.

2 Distinct distances between points and lines

In this section we provide a lower bound on the minimum number $D(m,n)$ of distinct point-line distances determined by m points and n lines in the plane. In Subsection 2.1, we prove an upper bound on the number of tangencies between a set of lines and a set of circles, and as a corollary we obtain an initial weaker lower bound on $D(m,n)$. To do this we dualize this problem to an incidence problem between points and algebraic curves, to which we can apply a known bound. In Subsection 2.2, we use the special structure of the problem to derive a better bound on $D(m,n)$, as given in Theorem 1.1.

2.1 A bound on line-circle tangencies

Recall that, given a set L of lines and a set C of circles, $T(L,C)$ denotes the number of pairs $(\ell, c) \in T \times C$ such that the line ℓ is tangent to the circle c. We restate Theorem 1.3 for the convenienve of the reader.

Theorem 2.1. *Let L be a set of n lines and C a set of k circles in the plane. Then*

$$T(L,C) = O\left(n^{2/3}k^{2/3} + n^{6/11}k^{9/11}\log^{2/11}k + k + n\right).$$

Proof. We dualize the lines and circles as follows. We rotate the plane so that no line of L is vertical, and then we map a line $y = ax + b$ to the dual point (a, b). Under this map, an algebraic curve is mapped to the set of points that are dual to the non-vertical tangent lines of the curve; these dual points form an algebraic curve, called the *dual curve*. We refer to the original xy-plane as the *primal plane*, and to the ab-plane as the *dual plane*. Applying this to our setting, the set L of n lines in the primal plane is mapped to a set L^* of n points in the dual plane, and, as we next argue, the set C of k circles in the primal plane is mapped to a set C^* of k hyperbolas in the dual plane.

Indeed, the dual curve c^* of a circle c is the locus of all points (a, b) dual to lines that are tangent to c. If c is centered at a point $p = (p_1, p_2)$ and has radius r, then the equation in a, b that defines c^* is

$$\frac{|p_2 - p_1 a - b|}{\sqrt{1 + a^2}} = r, \qquad \text{or} \qquad (p_2 - p_1 a - b)^2 - r^2(1 + a^2) = 0,$$

which is indeed the equation of a hyperbola.

We treat each branch of c^* as a separate curve, and obtain a collection C^* of $2k$ such curves. Standard considerations show that one of the two branches of c^* is the locus of points dual to the lines tangent to c from above, and the other branch is the locus of points dual to the lines tangent to c from below.

Each tangency between a line ℓ and a circle c corresponds to an incidence between the dual hyperbola c^* and the point ℓ^*. It is easily checked that any two hyperbola branches in C^* intersect at most twice, so they are *pseudo-parabolas*. Moreover, the curves in C^* admit a 3-parameter algebraic representation, because the circles in C have such a representation. Then the desired bound follows from Agarwal et al. [1, Theorem 6.6]. $\qquad \square$

We now deduce a lower bound on distinct point-line distance from the line-circle tangency bound above. In Subsection 2.2, we will significantly improve on this corollary.

Corollary 2.2. *The minimum number $D(m, n)$ of point-line distances between m points and n lines in $\mathbb{R}^2$ satisfies*

$$D(m, n) = \Omega\left(m^{2/9} n^{5/9} \log^{-2/9} m\right),$$

provided that $m^{1/2}/\log^{1/2} m \le n \le m^5 \log^4 m$, and that m is at least some sufficiently large constant.

Proof. Let P be a set of m points and let L be a set of n lines in the plane. Let t be the number of distinct distances between points of P and lines of L. For each point $p \in P$, draw at most t circles centered at p with radii equal to the (at most t) distances from p to the lines in L. We obtain a family C of at most mt (distinct) circles.

We double count $T(L, C)$. On the one hand, we trivially have $T(L, C) = mn$, since for each of the mn point-line pairs $(p, \ell) \in P \times L$, there is exactly one tangency between the line ℓ and the circle centered at p whose radius is the distance from p to ℓ. On the other hand, we can apply Theorem 2.1 with $|C| \le mt$ and $|L| = n$ to obtain

$$mn = T(L, C) = O\left(n^{2/3}(mt)^{2/3} + n^{6/11}(mt)^{9/11} \log^{2/11}(mt) + mt + n\right).$$

Eliminating t from this inequality yields

$$t = \Omega\left(\min\left\{m^{1/2} n^{1/2}, \ n^{5/9} m^{2/9} \log^{-2/9} m, \ n\right\}\right),$$

assuming that m is at least some sufficiently large constant. The minimum is attained by the second term, unless either $n > m^5 \log^4 m$ or $n < m^{1/2}/\log^{1/2} m$. $\qquad \square$

2.2 Proof of Theorem 1.1

In this section we prove Theorem 1.1, which we restate below as Theorem 2.3. We start, as in the proof of Corollary 2.2, by reducing the problem to counting line-circle tangencies, and then dualize. Instead of directly using an incidence bound for the dual curves, we derive a better bound by taking a closer look at the structure of the problem. In particular, we make use of the fact that in the set of circles there are many concentric circles. Our approach is similar to that used by Székely [13] to prove the bound $\Omega(m^{4/5})$ on the number of distinct point-point distances determined by m points.

Theorem 2.3. *For $m^{1/2} \leq n \leq m^2$, the minimum number $D(m,n)$ of point-line distances between m points and n lines in $\mathbb{R}^2$ satisfies*

$$D(m,n) = \Omega\left(m^{1/5}n^{3/5}\right).$$

Proof. Let P be a set of m distinct points and L be a set of n distinct lines in the plane. Again, let t denote the number of distinct point-line distances, draw at most t circles around every point of P, and denote the resulting set of circles by C. As before, we have $T(L,C) = mn$.

In the dual plane we have a set L^* of n points. Recall that the dual curve c^* of a circle c is a hyperbola. As before, we treat each branch of the hyperbola as a separate curve, and we let C^* be the set of these $2mt$ hyperbola branches.

To bound the number $I(L^*, C^*)$ of incidences between L^* and C^*, we draw a topological (multi-)graph $G = (L^*, E)$ in the dual plane. We assume without loss of generality that each hyperbola branch in C^* contains at least two points of L^*. Indeed, we can discard all curves of C^* containing at most one point of L^*, thereby discarding at most $2mt$ incidences.

For every curve in C^*, we connect each pair of consecutive points of L^* on that curve by an edge drawn along the portion of the curve between the two points. Write E for the set of edges in this graph. The number of edges on each curve of C^* is exactly one less than the number of points on it, so overall the number of edges in G satisfies

$$|E| \geq I(L^*, C^*) - 2mt.$$

Note that an edge can have high multiplicity, when many curves of C^* pass through its two endpoints, and the endpoints are consecutive on each of these curves. This situation corresponds to the case in the primal plane where we have many circles touching a pair of lines, and the corresponding tangencies are consecutive on each of the circles.

We define a parameter s, to be chosen later. Let E_1 denote the set of edges with multiplicity at most s and let E_2 denote the set of edges with multiplicity larger than s. In order to bound $|E_1|$ we use the crossing lemma (see [13]), which states that a graph G with n vertices, e edges, and maximum edge multiplicity s, has at least $\Omega(e^3/sn^2)$ edge crossings in any drawing, unless $e < 4ns$. We apply it to the graph with vertex set L^* and edge set E_1. Since any pair of these hyperbola branches intersect at most twice, the total number of crossings between curves in C^* is at most $2 \cdot \binom{mt}{2} = O(m^2t^2)$. Combining the two bounds on the number of crossings, we get

$$m^2t^2 = \Omega\left(\frac{|E_1|^3}{sn^2}\right),$$

so, taking also into account the alternative case $|E_1| < 4ns$,

$$|E_1| = O(m^{2/3}n^{2/3}t^{2/3}s^{1/3} + ns). \tag{2}$$

6

Next, we consider the edges of E_2. If an edge with endpoints ℓ_1^*, ℓ_2^* has multiplicity larger than x, then the lines ℓ_1 and ℓ_2 in the primal plane have x common tangent circles. The centers of these circles lie on the two angular bisectors defined by ℓ_1, ℓ_2, so there must be at least $x/2$ incidences between the m points and one of the bisectors of ℓ_1, ℓ_2.

We charge each edge of E_2 to the incidence between the angular bisector and the center of the circle c dual to the curve that the edge lies on. We claim that each such incidence can be charged at most $2t$ times. Indeed, in the primal plane, consider such an incidence between a point p and and an angular bisector ℓ. There are at most t distinct circles with the same center p, and each of these circles can have at most two pairs of tangent lines such that the angular bisector of those lines is ℓ, *and* such that the tangencies are consecutive. (In this argument, we acknowledge the possibility that ℓ might be the angular bisector of many pairs of lines, all of which are tangent to the same circle.)

It follows from the Szemerédi-Trotter theorem (see (1)) that the number of lines containing at least $s/2$ of the m points is $O(m^2/s^3 + m/s)$ (as long as $s/2 > 1$), and that the number of incidences between these m points and $O(m^2/s^3 + m/s)$ lines is $O(m^2/s^2 + m)$. Thus we have

$$|E_2| = O\left(\frac{m^2 t}{s^2} + mt\right). \tag{3}$$

To balance (2) and (3), we choose $s = O(m^{4/7} t^{1/7}/n^{2/7})$, noting that, with a proper choice of the constant of proportionality, we have $s > 2$. Indeed, this amounts to requiring that $n = O(m^2 t^{1/2})$, which holds since $m \geq n^{1/2}$. Adding together (2) and (3) gives

$$|E| = O\left(m^{6/7} n^{4/7} t^{5/7} + m^{4/7} n^{5/7} t^{1/7} + mt\right).$$

Thus the same bound holds for $T(L, C)$. Combining this with $T(L, C) = mn$, we get $t = \Omega(m^{1/5} n^{3/5})$ from the first term, $t = \Omega(m^3 n^2)$ from the second term, and $t = \Omega(n)$ from the third term. Thus

$$t = \Omega\left(m^{1/5} n^{3/5}\right),$$

using the assumption $m \leq n^2$. This completes the proof of Theorem 1.1. $\qquad\square$

Note that in the proof above we could set t to be the maximum number of distances from one of the m points to the n lines. We can therefore conclude that there is a single point from which the number of distances satisfies the bound. We note here that the proof of Theorem 1.2 that is provided in Section 3 below (restated as Theorem 3.1) does not lead to such a stronger conclusion.

3   Distances between points and spanned lines

We now consider the problem of bounding from below the number of distinct distances between a point set P and the set of lines spanned by P. Equivalently, we want to bound from below the number of distinct heights of triangles spanned by P. Write ℓ_{bc} for the line spanned by points b and c, write

$$H(P) = \left|\{d(a, \ell_{bc}) \mid a, b, c \in P\}\right|$$

for the number of distances between points of P and lines spanned by P, and write $H(m)$ for the minimum value of $H(P)$ over all non-collinear point sets P of size m.

For point sets with not too many points on a line, a better bound than that below follows from our Theorem 1.1. Our goal here is to provide a reasonably good bound that holds also when many points are collinear. To prove this, we reduce it to showing that the rational function

$$f(x, y) = \frac{(x - y)^2}{1 + y^2}$$

is "expanding", in the sense that $f(x, y)$ takes $\Omega(m^{4/3})$ distinct values for x, y in any set of m real numbers. If f were a polynomial, this would follow directly from a result of Raz et al. [10]. However, for rational functions in general the only known result is that of Elekes and Rónyai [5], which says that the number of values is superlinear (both these statements have certain exceptions). To extend the bound $\Omega(m^{4/3})$ to the rational function f, we use the same approach as in [10], which originated in Sharir, Sheffer, and Solymosi [11].

Theorem 3.1. *The minimum number $H(m)$ of distances between m points and their spanned lines satisfies*

$$H(m) = \Omega(m^{4/3}).$$

Proof. By a theorem of Beck [2, Theorem 3.1] (sometimes referred to as "Beck's two extremities theorem"), there is a constant c such that either the points of P span at least cm^2 distinct lines, or at least cm points of P are collinear.

In the first case, Theorem 1.1 gives

$$H(P) \geq D(m, cm^2) = \Omega\left(m^{7/5}\right).$$

Consider the second case, when $k = cm$ of the points are collinear; we can assume that k is an integer. Since not all the points are collinear, at least one other point $q \in P$ does not belong to this line. By translating, rotating, and scaling, we can assume that $q = (0, 1)$ and that the k collinear points lie on the x-axis, and by removing at most half the points we can assume that they all lie in the positive x-axis. We denote them by $p_i = (x_i, 0)$ for $i = 1, \ldots, k$, with all x_i positive, and we set $W = \{x_1, \ldots, x_k\}$.

The distance $d(p_i, \ell_{p_j q})$ from a point p_i to the line $\ell_{p_j q}$ spanned by p_j and q is easily seen to be $\dfrac{|x_i - x_j|}{\sqrt{1 + x_j^2}}$. Thus, putting

$$f(x, y) = \frac{(x - y)^2}{1 + y^2},$$

we get that $f(x_i, x_j) = d^2(p_i, \ell_{p_j q})$. Hence, in order to obtain a lower bound for the number of point-line distances, it suffices to find a lower bound for the cardinality of the set $f(W) = \{f(x, y) \mid x, y \in W\}$.

Following the setup in [11], we define the set of quadruples

$$Q = \{(x, y, x', y') \in W^4 \mid f(x, y) = f(x', y')\}.$$

Writing $f^{-1}(a) = \{(x, y) \in W^2 \mid f(x, y) = a\}$ and using the Cauchy-Schwarz inequality, we have

$$|Q| = \sum_{a \in f(W)} |f^{-1}(a)|^2 \geq \frac{k^4}{|f(W)|}.$$

Thus a lower bound for $|f(W)|$ will follow from an upper bound on $|Q|$.

We define algebraic curves C_{ij} in $\mathbb{R}^2$, for $i, j = 1, \ldots, k$, by

$$C_{ij} = \{(z, z') \in \mathbb{R}^2 \mid f(z, x_i) = f(z', x_j)\}.$$

We have $(x_k, x_l) \in C_{ij}$ if and only if $(x_k, x_i, x_l, x_j) \in Q$. Thus, denoting by Γ the set of curves C_{ij}, and by $S = W^2$ the set of pairs (x_k, x_l), we have

$$|Q| = |I(S, \Gamma)|.$$

It is not hard to show that the curves C_{ij} with $i = j$ contribute at most $O(k^2)$ quadruples, which is a negligible number, so in the rest of the proof we will assume that $i \neq j$. The equation $f(z, x_i) = f(z', x_j)$ is equivalent to $(z - x_i)^2/(1 + x_i^2) = (z' - x_j)^2/(1 + x_j^2)$, or

$$z' - x_j = \pm A_{ij} \cdot (z - x_i),$$

where

$$A_{ij} = \sqrt{(1 + x_j^2)/(1 + x_i^2)}.$$

Every curve C_{ij} is thus the union of two lines in the zz'-plane, given by

$$L_{ij}^+ : \ z' = A_{ij} z + (x_j - A_{ij} x_i), \qquad L_{ij}^- : \ z' = -A_{ij} z + (x_j + A_{ij} x_i).$$

Therefore, we need only consider the two families $\Gamma^+ = \{L_{ij}^+ \mid i \neq j\}$ and $\Gamma^- = \{L_{ij}^- \mid i \neq j\}$ and bound $I(S, \Gamma^+ \cup \Gamma^-)$. Some of the lines in Γ^+ and Γ^- may coincide, but the maximum multiplicity of a line is at most 2. Indeed, given the equation $z' = \alpha z + \beta$ of a line, say with $\alpha > 0$, we have $A_{ij} = \alpha$ and $x_j - \alpha x_i = \beta$. This leads to a quadratic equation in x_i (or x_j), which has at most two solutions.

We thus have an incidence problem for points and lines, with k^2 points and $O(k^2)$ distinct lines, each with multiplicity at most six. The Szemerédi-Trotter theorem (see (1)) gives

$$|I(S, \Gamma^+ \cup \Gamma^-)| = O((k^2)^{2/3}(k^2)^{2/3} + k^2 + k^2) = O(k^{8/3}).$$

Taking into account the discarded quadruples, we have

$$|Q| = |I(S, \Gamma^+ \cup \Gamma^-)| + O(k^2) = O(k^{8/3}),$$

so

$$|f(W)| \geq \frac{k^4}{|Q|} = \Omega(k^{4/3}) = \Omega(m^{4/3}),$$

completing the proof of Theorem 1.2.

$\square$

4   A bound on tangencies between circles

In this section we study tangencies between circles. We need a lemma about the "problem of Apollonius", which asks for a circle tangent to three given circles. All solution sets to this problem have been classified; see for instance [3]. The lemma below follows from this classification.

Lemma 4.1. *Given three circles in $\mathbb{R}^2$ that are not mutually tangent at a common point, there are at most eight other circles tangent to all three.*

We use the following incidence bound due to Zahl [15]. We state it in a slightly different form that is convenient for us, and that follows from the proof given in [15]. Given a set P of points and a set $\mathcal{S}$ of geometric objects, the *incidence graph* of $P \times \mathcal{S}$ is the bipartite graph with vertex set $P \cup \mathcal{S}$, and an edge between $p \in P$ and $S \in \mathcal{S}$ if $p \in S$.

Theorem 4.2 (Zahl). *Let $P \subset \mathbb{R}^3$ be a set of m points and let $\mathcal{S}$ be a set of n irreducible algebraic surfaces in $\mathbb{R}^3$ whose degree is bounded by a constant. Let I be a subgraph of the incidence graph of $P \times \mathcal{S}$ that contains neither $K_{M,3}$ nor $K_{3,M}$, for some constant parameter M. Then*

$$|E(I)| = O\left(m^{3/4}n^{3/4} + m + n\right),$$

where $E(I)$ denotes the set of edges of the graph I, and where the constant of proportionality depends on the maximum degree of the surfaces and on M.

Theorem 4.3. *Given a family C of n circles in $\mathbb{R}^2$ with arbitrary radii, there are at most $O(n^{3/2})$ points where (at least) two circles of C are tangent.*

Proof. Let the *tangency graph* of C be the bipartite graph whose vertex classes are two copies of C, with an edge between a circle from one copy and a (different) circle of the other copy if the two circles are tangent.

First we modify the tangency graph by removing certain edges. For any point where three or more circles are mutually tangent, we remove all but (an arbitrary) one of the corresponding edges from the tangency graph, and we call the resulting subgraph T. Then an upper bound on the number of edges of T will still be an upper bound on the number of tangency points. Moreover, the graph T contains no $K_{3,9}$ or $K_{9,3}$: If three circles are mutually tangent, at some point q, then there exist two of them, C, C', such that C' contains C in its interior, except for q (or vice versa). Any common neighbor of C and C' must be tangent to both of them at q, but then, by construction, at most one of the corresponding edges belongs to T. Otherwise, the three circles have at most eight common neighbors by Lemma 4.1.

For each circle $c \in C$, with center (a, b) and radius r, we define a point $p_c = (a, b, r) \in \mathbb{R}^3$ and two surfaces in $\mathbb{R}^3$ given by

$$\sigma_c^+ = \{(x, y, z) \mid (x - a)^2 + (y - b)^2 = (z + r)^2\},$$
$$\sigma_c^- = \{(x, y, z) \mid (x - a)^2 + (y - b)^2 = (z - r)^2\}.$$

Note that these surfaces are cones. Put $P = \{p_c \mid c \in C\}$ and $\Sigma = \{\sigma_c^+ \mid c \in C\} \cup \{\sigma_c^- \mid c \in C\}$.

Two circles $c_1, c_2 \in C$ are tangent if and only if the point p_{c_1} is incident to one of the surfaces $\sigma_{c_2}^+, \sigma_{c_2}^-$. Thus the tangency graph of C has the same number of edges as the incidence graph of P and Σ. Let I be the subgraph of the incidence graph corresponding to the subgraph T, i.e., an incidence is in I if the corresponding tangency is in T. As noted, I does not contain a copy of $K_{3,9}$ or $K_{9,3}$. It follows from Theorem 4.2 that the number of tangency points is

$$|E(T)| = |E(I)| = O\left(n^{3/4}n^{3/4}\right) = O\left(n^{3/2}\right).$$

This completes the proof. $\qquad\square$

Theorem 4.4. *Let C be a family of n circles in $\mathbb{R}^2$ with arbitrary radii. Assume that no three circles of C are mutually tangent at a common point. Then C has at most $O(n^{3/2})$ pairs of tangent circles.*

Proof. We proceed exactly as in the proof of Theorem 4.3, but now we do not need to remove any edges. By Lemma 4.1 and the condition on the circles, the tangency graph contains no $K_{3,9}$ or $K_{9,3}$. Then Theorem 4.2 gives the upper bound $O(n^{3/2})$ for the number of tangent pairs. $\quad\square$

Acknowledgements

References

[1] P. K. Agarwal, E. Nevo, J. Pach, R. Pinchasi, M. Sharir, and S. Smorodinsky, Lenses in arrangements of pseudo-circles and their applications, *J. ACM* **51** (2004), 139–186.

[2] J. Beck, On the lattice property of the plane and some problems of Dirac, Motzkin, and Erdős in combinatorial geometry, *Combinatorica* **3** (1983), 281–297.

[3] A. Bruen, J. C. Fisher, and J. B. Wilker, Apollonius by inversion, *Mathematics Magazine* **56** (1983), 97–103.

[4] P. Erdős, On sets of distances of n points, *Amer. Math. Monthly* **53** (1946), 248–250.

[5] G. Elekes and L. Rónyai, A combinatorial problem on polynomials and rational functions, *J. Combinat. Theory, Ser. A* **89** (2000), 1–20.

[6] J. Fox, J. Pach, A. Sheffer, A. Suk, and J. Zahl, A semi-algebraic version of Zarankiewicz's problem, `arXiv:1407.5705`, 2014.

[7] L. Guth and N. H. Katz, On the Erdős distinct distances problem in the plane, *Annals Math.* **181** (2015), 155–190.

[8] A. Marcus and G. Tardos, Intersection reverse sequences and geometric applications, *J. Combinat. Theory, Ser. A* **113** (2006), 675–691.

[9] J. Pach and M. Sharir, Geometric incidences, in *Towards a Theory of Geometric Graphs* (J. Pach, ed.), Contemporary Mathematics, Vol. 342, Amer. Math. Soc., Providence, RI, 2004, 185–223.

[10] O. E. Raz, M. Sharir, and J. Solymosi, Polynomials vanishing on grids: The Elekes-Rónyai problem revisited, *Amer. J. Math.*, to appear. Also in *Proc. 30th Annu. Sympos. Comput. Geom.* (2014), 251–260.

[11] M. Sharir, A. Sheffer, and J. Solymosi, Distinct distances on two lines, *J. Combinat. Theory, Ser. A* **120** (2013), 1732–1736.

[12] J. Spencer, E. Szemerédi, and W.T. Trotter, Unit distances in the Euclidean plane, in: *Graph Theory and Combinatorics* (B. Bollobás, ed.), Academic Press, 1984, 293–303.

[13] L. Székely, Crossing numbers and hard Erdős problems in discrete geometry, *Combinat. Probab. Comput.* **6** (1997), 353–358.

[14] E. Szemerédi and W.T. Trotter, Extremal problems in discrete geometry, *Combinatorica* **3** (1983), 381–392.

[15] J. Zahl, An improved bound on the number of point-surface incidences in three dimensions, *Contrib. Discrete Math.* **8** (2013), 100–121.

A Distinct distances between points and hyperplanes in $\mathbb{R}^d$

In this section we provide a lower bound on the number of distinct distances between m points and n hyperplanes in $\mathbb{R}^d$, under appropriate non-degeneracy assumptions.

We first note that, without further assumptions, there is no non-trivial lower bound on the number of distinct distances determined by m points and n hyperplanes, in any dimension $d \geq 3$. Indeed, let Π be a set of n distinct hyperplanes all containing a line ℓ, and let P be a set of m points on that common line. Obviously, every point is at distance 0 from any hyperplane, so this configuration determines only one distance. The construction can be modified to yield a set P of m points and a set Π of n hyperplanes, with only one *positive* point-hyperplane distance, by placing all the points of P on the axis of a cylinder C, and by making all the hyperplanes of Π tangent to C. Note also that the same example demonstrates that there is no non-trivial upper bound on the number of repeated point-hyperplane distances (without further assumptions).

Given a finite set Π of (hyper-)planes and a finite set S of (hyper-)spheres, we write $T(\Pi, S)$ for the number of pairs $(\pi, s) \in \Pi \times S$ such that π is tangent to s, and we define the *tangency graph* to be the bipartite graph whose vertex classes are Π and S, with an edge between π and s if π is tangent to s.

A.1 Distinct point-plane distances

We first consider the problem in three dimensions, where a better lower bound can be obtained. Before proceeding to the lower bound analysis, we need an analogue of Theorem 2.1 that bounds plane-sphere tangencies.

Lemma A.1. *Let Π be a set of n planes and let S be a set of k spheres in $\mathbb{R}^3$. Assume that the tangency graph of $\Pi \times S$ contains no $K_{M,3}$ or $K_{3,M}$, for some constant parameter M. Then*

$$T(\Pi, S) = O\left(n^{3/4}k^{3/4} + n + k\right),$$

where the constant of proportionality depends on M.

Proof. The proof is similar to the proof of Theorem 2.1. We apply a rotation so that all planes in Π are non-vertical. We map a plane $\pi \in \Pi$ defined by $z = ax + by + c$ to the point $\pi^* = (a, b, c)$. We map each sphere $s \in S$ to the surface s^* which is the locus of all points π^* dual to planes π tangent to s. The surface s^* dual to the sphere s with center (x, y, z) and radius r is the set of points (a, b, c) satisfying

$$\frac{|z - ax - by - c|}{\sqrt{1 + a^2 + b^2}} = r, \qquad \text{or} \qquad (z - ax - by - c)^2 - r^2(1 + a^2 + b^2) = 0,$$

which is a smooth quadric (specifically, a hyperboloid of two sheets) in $\mathbb{R}^3$.

Denote the set of resulting points by $\Pi^* = \{\pi^* \mid \pi \in \Pi\}$, and the family of resulting hyperboloids by $S^* = \{s^* \mid s \in S\}$. Notice that every pair (π, s) that is counted in $T(\Pi, S)$ corresponds to an incidence between the point π^* and the surface s^*, and the tangency graph of $\Pi \times S$ is the same as the incidence graph of $\Pi^* \times S^*$. Thus applying Theorem 4.2 to $I(\Pi^*, S^*) = T(\Pi, S)$ gives the stated bound. $\square$

We call a configuration of points and planes a *cone configuration* (resp., a *cylinder configuration*) if all the points lie on a line ℓ, and the planes are all tangent to the same cone (resp.,

cylinder) with axis ℓ. A cylinder configuration is exactly the kind of configuration, mentioned above, that gives only one point-plane distance. A cone configuration determines as many distances as it has points on the axis, no matter how many planes there are. In both types of configurations, there is a complete bipartite graph in the tangency graphs between the planes and the spheres around the points that are tangent to the cone or cylinder.

Corollary A.2. *Let Π be a set of n planes and S a set of k spheres in $\mathbb{R}^3$, such that there is no cone or cylinder configuration of three planes and three centers of the spheres. Then*

$$T(\Pi, S) = O\left(n^{3/4}k^{3/4} + n + k\right).$$

Proof. Any sphere tangent to three given planes must have a center that lies on one of four lines (this is easily seen to hold in both cases, where the planes either intersect at a point, or are all parallel to some line), and a given sphere center (on one of these lines) uniquely determines the tangent sphere. Hence, the tangency graph does not contain $K_{3,9}$. (MICHA SAYS: And what about the other direction?!) $\qquad\square$

Theorem A.3. *Let P be a set of m points and let Π be a set of n planes in $\mathbb{R}^3$. Assume that there is no cone or cylinder configuration of three points and three planes. Then the number of distinct point-plane distances determined by P and Π is*

$$\Omega(m^{1/3}n^{1/3}),$$

unless $m = O(1)$ or $m = \Omega(n^2)$.

Proof. The proof is analogous to the proof of Corollary 2.2. Let t denote the total number of distinct distances between points in P and planes in Π. We place at most t spheres centered at each $p \in P$ according to the occurring distances from p to the planes in Π. Let S denote the resulting family of at most mt distinct spheres.

As in the proof of Corollary A.2, the tangency graph of the planes of Π and the spheres of S contains no $K_{3,9}$, because this would lead to a 3×3 cone or cylinder configuration. (MICHA SAYS: And the other direction?)

On the one hand, the number of tangencies satisfies $T(\Pi, S) = mn$. On the other hand, applying Lemma A.1 to S and Π leads to

$$mn = T(\Pi, S) = O\left(n^{3/4}(mt)^{3/4} + n + mt\right).$$

The second term gives $m = O(1)$ and the third term gives $t = \Omega(n)$. Thus we have

$$t = \Omega\left(m^{1/3}n^{1/3}\right),$$

unless $m = O(1)$ or $m = \Omega(n^2)$. $\qquad\square$

As in the case of point-line distances, our proof yields a stronger statement: Under the same assumptions, there is a point $p \in P$ with at least $\Omega(m^{1/3}n^{1/3})$ distinct distances from p to the planes in Π.

A.2 Distinct point-hyperplane distances

To obtain higher-dimensional analogues, we will use the following incidence bound from Fix et al. [6]. We refer to [6] for the relevant definitions.

Theorem A.4 (Fox-Pach-Sheffer-Suk-Zahl). *Let G be a semialgebraic bipartite graph that has bounded complexity, with parts $A \subset \mathbb{R}^{d_1}$ and $B \subset \mathbb{R}^{d_2}$. If G is $K_{k,k}$-free, then*

$$|E(G)| = O\left(|A|^{\frac{d_2(d_1-1)}{d_1 d_2 - 1} + \varepsilon}|B|^{\frac{d_1(d_2-1)}{d_1 d_2 - 1}} + |A| + |B|\right),$$

for any $\varepsilon > 0$, where the constant of proportionality depends on d_1, d_2, k, ε, and the description complexity of the graph.

We define cylinder and cone configurations analogously to the definitions in Subsection A.1.

Theorem A.5. *Let P be a set of m points and let Π be a set of n hyperplanes in $\mathbb{R}^d$. Assume that these points and hyperplanes determine no cone or cylinder configuration of k points and k hyperplanes, for some constant k. Then the number of distinct point-hyperplane distances determined by P and Π is*

$$\Omega\left(m^{\frac{1}{d} - \frac{1}{d^2}} n^{\frac{1}{d} - \varepsilon}\right),$$

for any $\varepsilon > 0$, where the constant of proportionality depends on ε, unless $m = O(1)$ or $m = \Omega(n^2)$.

Proof. Let $A = \Pi^* \subset \mathbb{R}^d$ be the set of dual points corresponding to the hyperplanes in Π. Again let t be the total number of distinct point-hyperplane distances and draw t (hyper-)spheres around each point with radii equal to these distances. Let $B \subset \mathbb{R}^{d+1}$ be the set of at most mt points (p, r) corresponding to the spheres with center p and radius r.

Let G be the bipartite graph with parts A and B, where a hyperplane and a sphere are connected by an edge if they are tangent. We claim that this is a (semi-)algebraic graph of bounded description complexity, a property that follows from the same formula for point-hyperplane distance that we used above.

Thus we get

$$mn = |E(G)| = O\left(n^{\frac{(d+1)(d-1)}{d(d+1)-1} + \varepsilon}(mt)^{\frac{d^2}{d(d+1)-1}} + n + mt\right)$$

$$= O\left(m^{\frac{d^2}{d^2+d-1}} n^{\frac{d^2-1}{d^2+d-1} + \varepsilon} t^{\frac{d^2}{d^2+d-1}} + n + mt\right),$$

so

$$t = \Omega\left(m^{\frac{d-1}{d^2}} n^{\frac{1}{d} - \varepsilon}\right),$$

for any $\varepsilon > 0$, unless $m = O(1)$ or $m = \Omega(n^2)$. $\qquad\square$

Note that Zahl's theorem corresponds to the case in Theorem A.4 with $d_1 = d_2 = 3$ (except for the ε). The reason for the weaker bound in Theorem A.5 (for $d = 3$) is that, in order to apply Theorem A.4, we had to "lift" the spheres to points in $\mathbb{R}^{d+1}$, rather than work directly with the spheres as surfaces in $\mathbb{R}^d$, as we were able to do in the three-dimensional case.