

Advanced Topics in Computational and Combinatorial Geometry

Assignment 1

Due: March 27, 2012

Problem 1

Davenport-Schinzel sequences of order 2 and triangulations: Let P be any convex polygon with n vertices. A triangulation of P is a collection of $n - 3$ non-intersecting chords connecting pairs of vertices of P and partitioning P into $n - 2$ triangles. Set up a correspondence between such triangulations and $DS(n - 1, 2)$ sequences, as follows. Number the vertices $1, 2, \dots, n$ in their order along ∂P . Let T be a given triangulation. Include in T the edges of P too. For each vertex i , let $T(i)$ be the sequence of vertices $j < i$ connected to i in T and arranged in *decreasing* order, and let U_T be the concatenation of $T(2), T(3), \dots, T(n)$.

(a) Show that U_T is a $DS(n - 1, 2)$ sequence of maximum length.

(b) Show that any $DS(n - 1, 2)$ -sequence of maximum length can be realized in this manner, perhaps with an appropriate renumbering of its symbols.

(c) Use (a) and (b) to show that the number of different $DS(n, 2)$ sequences of maximum length is $\frac{1}{n-1} \binom{2n-4}{n-2}$ (where two sequences are different if one cannot obtain one sequence from the other by renumbering its symbols).

(**Note:** Obviously, you have to show that this is the number $F(n)$ of triangulations of P . Show it, e.g., by deriving a recurrence relation on $F(n)$ and solving it inductively.)

Problem 2

(a) Show that $\lambda_3(n) \geq 5n - 8$.

(b) Show that the lower bound in (a) can be realized as the lower envelope sequence of n segments.

Problem 3

Let F be a collection of n partially-defined and continuous functions over the reals. Suppose that F is the disjoint union of c subcollections $F_1, \dots, F_c$, such that (i) Any pair of functions in F_i intersect in at most s_i points, for $i = 1, \dots, c$. (ii) Any pair of functions of F intersect in at most s points. Here c , s , and the s_i 's are all constants.

(a) Show that the complexity of the lower envelope of F is $O(\lambda_{q+2}(n))$, where $q = \max_{i=1}^c s_i$.

(b) Show that the envelope can be computed in time $O(\lambda_{q+1}(n) \log n)$.

(c) As a corollary, what is the complexity of the lower envelope of a collection of n line segments and unit-radius circles? (For the purpose of the lower envelope, each circle can be replaced by its lower semi-circle.)

Problem 4

Let $P = \{p_1(t), \dots, p_n(t)\}$ be a set of n moving points in the plane, so that each point p_i moves at some constant speed v_i along some line ℓ_i . At time t , let $q = q(t)$ be the point of P that satisfies: (i) q lies in the right halfplane $x \geq 0$, and (ii) the segment $\vec{oq}$ (o is the origin) has the largest slope (among all such segments for points in the right halfplane). If no point of P lies in $x \geq 0$ at time t then q is undefined. There are discrete times when more than one point satisfies (i) and (ii).

(a) Obtain an upper bound on the number of discrete changes in $q(t)$ (including transitions from being defined to being undefined and vice versa).

(b) Using (a), and its symmetric version for points in the left halfplane $x \leq 0$, to obtain an upper bound on the number of discrete changes in the left or right neighbors of o on the upper boundary of the convex hull of $\{o\} \cup P$ (including transitions where q starts or stops being a vertex of the upper hull).

Problem 5

Given n ellipses in the plane, all containing the origin. What is the combinatorial complexity of their union? What is the complexity of their intersection?