Advanced Topics in Computational and Combinatorial Geometry
Assignment 2

Due: April 17, 2012

Problem 1

Show that a single cell in an arrangement of n rays in the plane has complexity $O(n)$. Use this to show that the complexity of the *outer zone* of any closed convex curve γ in an arrangement of n lines in the plane is $O(n)$. (The outer zone is the portion of the zone that lies outside γ.) What can you say about the complexity of the inner zone (the portion of the zone inside γ)?

Problem 2

Show that the zone of a line in an arrangement of n unit circles in the plane has complexity $O(n\alpha(n))$. What happens when the circles have arbitrary radii? (**Hint:** For the first part, consider separately the top and bottom portions of each circle.)

Problem 3

Let L be a set of n lines in the plane. Define the *separation distance* $d_L(p, q)$ between a pair of points $p, q \in \mathbb{R}^2$ to be the number of lines of L crossed by the segment pq. Show that d_L satisfies the triangle inequality.

(a) Show that for any point x and integer $k \leq n$ there are $\Omega(k^2)$ vertices of $\mathcal{A}(L)$ at separation distance $\leq k$ from x.

(b) Conclude that, for any set P of m points in the plane there are always two points $p, q \in P$ at separation distance at most $O(n/\sqrt{m})$ apart. (**Hint:** Use a packing argument involving the vertices of $\mathcal{A}(L)$.)

(c) Use (b) to show that, for a given set P of m points, the minimum spanning tree T of P under the separation distance has total weight $O(n\sqrt{m})$. Note that this means that, on average, each line of L crosses only $O(\sqrt{m})$ edges of T. (**Hint:** Construct tree edges iteratively, using (b) and deleting points of P.)

Problem 4

Minkowski sums: Let $A_1, \ldots, A_m$ be a collection of m pairwise-disjoint convex polygons in the plane, so that A_i has n_i edges, for a total of $n = \sum_{i=1}^{m} n_i$ edges. Let B be a convex polygon with k edges. Assume general position of the A_i's and B. The *Minkowski sum* of A_i and B, for $i = 1, \ldots, m$, is defined as

$$K_i = A_i \oplus B = \{x + y \mid x \in A_i, y \in B\}.$$

(a) Show that, for any (generic) direction θ, the extreme point of K_i in direction θ is $x_\theta + y_\theta$, where x_θ (resp., y_θ) is the extreme point of A_i (resp., B_i) in direction θ.

(b) Show that each K_i is a polygon with $n_i + k$ edges.

(c) Show that the boundaries of any pair of the polygons K_i, K_j intersect in at most 2 points (give a rigorous proof). (**Hint:** Use (a) to show that K_i and K_j have at most two common outer tangents.)

(d) Give an efficient algorithm for computing the union $K = \bigcup_{i=1}^{m} K_i$ (use a divide-and-conquer approach).