

Assignment 5

Advanced Topics in Computational Geometry

Due: June 19, 2012

Problem 1

Extend the proof technique of the Crossing Lemma to show the following: Let P be a set of n points in the plane in general position, and let D be a set of M disks, each having a pair of points of P as a diameter. If $M \geq 4n$ then there exists a point of P that lies in the interior of $\Omega(M^2/n^2)$ disks of D .

(**Hints:** (a) Show that if $M \geq 3n$ then there exists a disk $d \in D$ and a point $p \in P$ such that p lies in the interior of d . (b) Apply the random sampling technique used in the proof of the Lemma to get a good lower bound on the number of such pairs (p, d) . (c) Conclude from (b) the existence of a point $p \in P$ that has the desired property.)

Problem 2

(a) Let L be a set of n lines in the plane. Preprocess L into a data structure that supports efficiently queries of the form: Given a query segment s , count the number of lines of L that s crosses. Give two solutions: One that uses near-quadratic storage (and preprocessing) and answers a query in (poly)logarithmic time, and one that uses near-linear storage (and preprocessing) and answers a query in time close to $O(n^{1/2})$.

Problem 3

Let S be a set of n line segments in the plane in general position. Design an algorithm that counts the number of intersections between the segments of S in time $O^*(n^{4/3})$. (Here O^* ignores additional polylogarithmic factors.)

(a) First show that, given a triangle Δ , a set L of m lines, and a set S of n segments, (i) The number of intersections between the lines of L inside Δ can be counted in time $O(m \log m)$. (ii) The number of intersections between the lines of L and the segments of S inside Δ can be counted in time $O^*(n^2 + m)$. (iii) The number of intersections between the segments of S inside Δ can be counted in time $O^*(n^2)$.

(b) Returning to the original problem, construct a $(1/r)$ -cutting with $r = n^{1/3}$, and count the number of intersections within each trapezoid of the cutting separately,

using the results of (a).

Problem 4

Let Γ be a collection of n x -monotone arcs, each pair of which intersect in at most s points. Let κ denote the total number of intersections between the arcs of Γ . Let $r \leq n$ be a parameter. Using random sampling, show that there exists a $(1/r)$ -cutting of $\mathcal{A}(\Gamma)$ that consists of $O^*\left(r + \kappa \frac{r^2}{n^2}\right)$ pseudo-trapezoids.

Use this result to refine the algorithm in Problem 3, so that its running time depends on the output size, that is, on the number of intersections between the segments of S .

Problem 5

Let H be a set of n planes and P a set of m points in three dimensions, and let $I(P, H)$ denote the number of incidences between P and H .

(a) Show that, without any further assumptions, $I(P, H)$ can be mn .

(b) Assume that no line is contained in more than two planes of H . Show that $I(P, H) = O(m^{4/5}n^{3/5} + m + n)$.

Hint: (a) Show that the incidence graph of P and H does not contain $K_{2,3}$, and derive from this an initial weak upper bound on $I(P, H)$.

(b) Use a $(1/r)$ -cutting of $\mathcal{A}(H)$ and apply the weak bound in each cell of the cutting. Pay special attention to what happens on the cell boundaries.