Problem 1: Incidences

(a) Prove the Kővári–Sós–Turán theorem: Let $G = (V, E)$ be a bipartite graph with vertex sets $P, Q$ (so $E \subseteq P \times Q$), such that $G$ does not contain $K_{r,s}$ as a subgraph, where $r$ and $s$ are constants. Show that $|E| = O(mn^{1-1/r} + n)$ (and, symmetrically, $|E| = O(nm^{1-1/s} + m)$), where $m = |P|$ and $n = |Q|$.

(Hints: (i) Double count the number of tuples $(p_1, p_2, \ldots, p_r, q)$, such that $p_1, p_2, \ldots, p_r$ are distinct vertices in $P$, $q \in Q$, and all the edges $(p_1, q), (p_2, q), \ldots, (p_r, q)$ are in $E$. (ii) As a further hint, one of the counts should be $\sum_{q \in Q} \deg(q)^r$. (iii) Note that $|E| = \sum_{q \in Q} \deg(q)$, and use Hölder’s inequality to estimate $|E|$.)

(b) For a set $P$ of $m$ points in the plane, and a set $C$ of $n$ curves in the plane, denote by $G(P, C)$ the incidence graph of $P$ and $C$; its edges are all the pairs $(p, c) \in P \times C$ such that $p$ is incident to $c$. Apply (a) to the incidence graphs for the cases where $C$ is (i) a set of lines; (ii) a set of unit circles; (iii) a set of circles with arbitrary radii. Get three respective weak incidence bounds for $I(P, C)$ in these three cases.

(c) Consider case (iii) from (b) (where $C$ is a set of arbitrary circles). Combine the bound from (b) with the cutting method, to show that $I(P, C) = O(m^{3/5} n^{4/5} + m + n)$. (No need to give full details, but try to discuss issues where the analysis here is somewhat different from the one shown in class for lines.)

Problem 2

Extend the proof technique of the Crossing Lemma to show the following: Let $P$ be a set of $n$ points in the plane in general position, and let $D$ be a set of $M$ disks, each having a pair of points of $P$ as a diameter. If $M \geq 4n$ then there exists a point of $P$ that lies in the interior of $\Omega(M^2/n^2)$ disks of $D$.

(Hints: (a) Show that if $M \geq 3n$ then there exists a disk $d \in D$ and a point $p \in P$ such that $p$ lies in the interior of $d$. (To show this, use the graph $G$ drawn on the set $P$ as vertices, where for each disk $d \in D$ we draw in $G$ the straight edge which is the diameter of $d$ that connects its two defining points.) (b) Apply the random sampling technique used in the proof of the Lemma to get a good lower bound on the number
of such pairs \((p,d)\). (c) Conclude from (b) the existence of a point \(p \in P\) that has the desired property.)

**Problem 3**

For each of the following range spaces \((X, \mathcal{R})\), obtain an upper bound on the number of possible ranges for any subset of \(m\) points of \(X\). Whenever possible, compute the VC-dimension too.

(a) \(X\) is a set of points in \(\mathbb{R}^3\) and each range of \(\mathcal{R}\) is the set of points of \(X\) inside some axis-parallel box.

(b) \(X\) is a set of points in \(\mathbb{R}^3\) and each range of \(\mathcal{R}\) is the set of points of \(X\) inside some ball.

(c) \(X\) is a set of lines in \(\mathbb{R}^3\) and each range of \(\mathcal{R}\) is the set of lines of \(X\) that intersect some unit ball. (Hint: Move to a dual space where each ball is represented by its center, and each line of \(X\) is represented by . . .)

**Problem 4**

Using Problem 3(c), give a simple-minded solution for the following problem. Given a set \(X\) of \(n\) lines in \(\mathbb{R}^3\), and a parameter \(\varepsilon > 0\), preprocess them into a data structure that supports efficiently queries of the form: For a query point \(q \in \mathbb{R}^3\), estimate the number of lines at distance at most 1 from \(q\), up to an error of \(\varepsilon\). (The problem is somewhat vaguely defined, but you should know what to do...)