# Advanced Topics in Computational and Combinatorial Geometry

# Assignment 4 (short answers and hints)

4c is missing

# Problem 1

1. Hint: Prove only for the direction of the x-axis; follows more or less by definition.
2. First prove that the Minkowski sum of two convex polygons is a convex polygon; it follows from (a) that each vertex is the sum of a vertex of $A\_{i}$ and a vertex of $B$. Then use the normal diagram to prove that there are only $n\_{i}+k$ vertices.
3. Hint: Prove that each common outer tangent to *Ki* and *Kj* has a corresponding outer tangent to original polygons *Ai* and *Aj* (by "subtracting" the corresponding extreme point of $B$). Since the original polygons *Ai* and *Aj* do not intersect – they have exactly two common outer tangents.
4. We’ve shown in class how to compute the union of *n* pseudo-disks in *O(n log2 n)* time. As shown in (c), the Minkowski sums are pseudo-disks. In this case, though, we have *m* pseudo-disks with a total complexity of *n+mk* (denote *N = n+mk)*. It’s easy to see that the “conquer” step will run now in *O(N log N)* time. And the whole algorithm in *O(N log N log m)* time.

# Problem 2

**a+b+c:** Pick a hyperplane *h* and look at *Zone(H – {h}; h)*. From the Zone Theorem its complexity is *O(nd-1)*. This complexity counts all the features of the cells (in the original arrangement *A(H)*) that *h* is part of them except for the features located on *h* itself. (Some pairs of features of these cells will be counted only once in the zone.)

Repeat for all *h* in *H*; the total count is *O(nd)* and: each *(d-1)-*face of cell *c* will be counted *|c|d-1 – 1* times (it will not be counted for the hyperplane it lies on), each *(d-2)-*face of cell *c* will be counted *|c|d-1 – 2* times and so on. Since these are constants, we can assume that each feature of the cell is counted *|c|d-1* times. Thus we get $ \sum\_{c}^{}\left|c\right||c|\_{d-1}=O(n^{d})$.

**d:** Pick *m* arbitrary faces: *f1, … , fm*. Denote $k=\sum\_{i=1}^{m}|f\_{i}|$, so we want to find a bound for *k*. It’s easy to see that: $\sum\_{i=1}^{m}|f\_{i}|^{2}\geq \sum\_{i=1}^{m}\left(\frac{k}{m}\right)^{2}=\frac{k^{2}}{m}$ (Cauchy-Schwarz)

On the other side: $\sum\_{i=1}^{m}|f\_{i}|^{2}\leq Cn^{2}$ (by (a))

We get: $\frac{k^{2}}{m}\leq Cn^{2}⟹k\leq Cn\sqrt{m}$

# Problem 3

Let’s define a *balanced* angle: angle that is bounded by one ray in down-left direction and other ray in down-right direction.

There is one-to-one correspondence between balanced faces and balanced angles in the arrangement of lines on the plane. Now use the Clarkson-Shor technique and note that there is (at most) only one balanced angle on the lower envelope.

# Problem 4

1. A vertex of weight k is generated in this process with probability$\frac{3!k!}{\left(k+3\right)!}$, because the three function graphs defining such a vertex need to be inserted before the k functions that pass below it are inserted. The number $N\_{\leq k}$ of vertices of weight at most k is $O(k^{1-ε}n^{2+ε})$, using Clarkson-Shor technique and the fact that the complexity of the lower envelope of the family is $O\left(n^{2+ε}\right)$. The expected number of vertices that are generated by the algorithm is then:

$$\sum\_{k=0}^{n}N\_{k}⋅\frac{3!k!}{\left(k+3\right)!}=\sum\_{k=0}^{n}\left(N\_{\leq k}-N\_{\leq k-1}\right)⋅\frac{3!k!}{\left(k+3\right)!}≈\sum\_{k=0}^{n}N\_{\leq k}⋅\frac{1}{k^{4}}=O(n^{2+ε}\sum\_{k=0}^{n}k^{1-ε}⋅\frac{1}{k^{4}})=O(n^{2+ε})$$

1. Similarly, the expected sum of the weights of the vertices generated is

$$\sum\_{k=0}^{n}k⋅N\_{k}⋅\frac{3!k!}{\left(k+3\right)!}=\sum\_{k=0}^{n}k⋅\left(N\_{\leq k}-N\_{\leq k-1}\right)⋅\frac{3!k!}{\left(k+3\right)!}≈\sum\_{k=0}^{n}N\_{\leq k}⋅\frac{1}{k^{3}}=O(n^{2+ε}\sum\_{k=0}^{n}k^{1-ε}⋅\frac{1}{k^{3}})=O\left(n^{2+ε}\right).$$

Bad example: $F$consists of $n/3$ steep wedges whose bottom edges are in the $xy$-plane and are parallel to the $x$-axis, $n/3$ similar wedges with bottom edges parallel to the $y$-axis, and $n/3$ horizontal planes at height $jε$, $ε>0$, $j=1,2,…,n/3$. Insert first the $2n/3$ wedges; the lower envelope has quadratic complexity. Now insert the remaining $n/3$ planes from top to bottom.

# Problem 5

The proof here is similar to the proof that the complexity of lower envelope of 3d triangles is $O(n^{2+ϵ})$.

The vertices of the lower envelope can be divided into two groups:

1. Vertices created by the border of a disk.
2. Vertices created by intersection of 3 disks.

Step 1: for each disk create a curtain and count the number of vertices created on this curtain by other disks. Other disks create at most $n$ curves on the curtain of a fixed disk and each two intersect at most twice. Thus the total number of vertices of the lower envelope created by the border of a disk is $O(nλ\_{4}\left(n\right))$ (over all disks).

Step 2: The intersection of 3 disks looks locally like the intersection of 3 hyperplanes. If you draw the intersection near the intersection point it looks like this:

Either two edges go the left and one to the right or two to the right and one to the left. Now use the counting technique (start collecting $k$ vertices and use Clarkson-Shor technique) we used in class to prove that the number of vertices that are the result of 3 disks intersecting is $O(n^{2+ϵ})$. (Here the intersection curves are the straight segments. Either we manage to collect k vertices (at levels $\leq k$), or we reach a curtain, at a vertex in the 2D arrangement within the curtain, at level $\leq k$.)

The construction: use $\frac{n}{2}$ huge near vertical disks parallel to each other. Use another $\frac{n}{2}$ huge near vertical disks parallel to each other above and perpendicular to the first disks. From below these disks look like a grid.