# Advanced Topics in Computational and Combinatorial Geometry

# Assignment 4 (short answers and hints)

4c is missing

# Problem 1

1. Hint: Prove only for the direction of the x-axis; follows more or less by definition.
2. First prove that the Minkowski sum of two convex polygons is a convex polygon; it follows from (a) that each vertex is the sum of a vertex of and a vertex of . Then use the normal diagram to prove that there are only vertices.
3. Hint: Prove that each common outer tangent to *Ki* and *Kj* has a corresponding outer tangent to original polygons *Ai* and *Aj* (by "subtracting" the corresponding extreme point of ). Since the original polygons *Ai* and *Aj* do not intersect – they have exactly two common outer tangents.
4. We’ve shown in class how to compute the union of *n* pseudo-disks in *O(n log2 n)* time. As shown in (c), the Minkowski sums are pseudo-disks. In this case, though, we have *m* pseudo-disks with a total complexity of *n+mk* (denote *N = n+mk)*. It’s easy to see that the “conquer” step will run now in *O(N log N)* time. And the whole algorithm in *O(N log N log m)* time.

# Problem 2

**a+b+c:** Pick a hyperplane *h* and look at *Zone(H – {h}; h)*. From the Zone Theorem its complexity is *O(nd-1)*. This complexity counts all the features of the cells (in the original arrangement *A(H)*) that *h* is part of them except for the features located on *h* itself. (Some pairs of features of these cells will be counted only once in the zone.)

Repeat for all *h* in *H*; the total count is *O(nd)* and: each *(d-1)-*face of cell *c* will be counted *|c|d-1 – 1* times (it will not be counted for the hyperplane it lies on), each *(d-2)-*face of cell *c* will be counted *|c|d-1 – 2* times and so on. Since these are constants, we can assume that each feature of the cell is counted *|c|d-1* times. Thus we get .

**d:** Pick *m* arbitrary faces: *f1, … , fm*. Denote , so we want to find a bound for *k*. It’s easy to see that: (Cauchy-Schwarz)

On the other side: (by (a))

We get:

# Problem 3

Let’s define a *balanced* angle: angle that is bounded by one ray in down-left direction and other ray in down-right direction.

There is one-to-one correspondence between balanced faces and balanced angles in the arrangement of lines on the plane. Now use the Clarkson-Shor technique and note that there is (at most) only one balanced angle on the lower envelope.

# Problem 4

1. A vertex of weight k is generated in this process with probability, because the three function graphs defining such a vertex need to be inserted before the k functions that pass below it are inserted. The number of vertices of weight at most k is , using Clarkson-Shor technique and the fact that the complexity of the lower envelope of the family is . The expected number of vertices that are generated by the algorithm is then:
2. Similarly, the expected sum of the weights of the vertices generated is

Bad example: consists of steep wedges whose bottom edges are in the -plane and are parallel to the -axis, similar wedges with bottom edges parallel to the -axis, and horizontal planes at height , , . Insert first the wedges; the lower envelope has quadratic complexity. Now insert the remaining planes from top to bottom.

# Problem 5

The proof here is similar to the proof that the complexity of lower envelope of 3d triangles is .

The vertices of the lower envelope can be divided into two groups:

1. Vertices created by the border of a disk.
2. Vertices created by intersection of 3 disks.

Step 1: for each disk create a curtain and count the number of vertices created on this curtain by other disks. Other disks create at most curves on the curtain of a fixed disk and each two intersect at most twice. Thus the total number of vertices of the lower envelope created by the border of a disk is (over all disks).

Step 2: The intersection of 3 disks looks locally like the intersection of 3 hyperplanes. If you draw the intersection near the intersection point it looks like this:

Either two edges go the left and one to the right or two to the right and one to the left. Now use the counting technique (start collecting vertices and use Clarkson-Shor technique) we used in class to prove that the number of vertices that are the result of 3 disks intersecting is . (Here the intersection curves are the straight segments. Either we manage to collect k vertices (at levels ), or we reach a curtain, at a vertex in the 2D arrangement within the curtain, at level .)

The construction: use huge near vertical disks parallel to each other. Use another huge near vertical disks parallel to each other above and perpendicular to the first disks. From below these disks look like a grid.