# Assignment 2 - Geometric Optimization (0368-4144)

Due: November 23, 2009

## Problem 1

Let $A$ and $B$ be two sets of $n$ points each, where the points of $A$ lie on the line $x = 0$ and the points of $B$ lie on the line $x = 1$. Let $k$ be an integer between 1 and $n^2$. The goal is to find the $k$-th smallest distance between a point in $A$ and a point in $B$.

Solve the problem in near linear time using three approaches: (1) parametric searching (plus Cole's improvement); (2) monotone matrix searching; (3) a randomized approach. Explain each solution.

## Problem 2

Apply the algorithm for searching in totally monotone matrices to solve the following problem in $O(n \log n)$ time: We are given a set $P$ of $n$ points lying on a closed convex curve $C$, and the goal is to find the triangle of largest area whose vertices are three of the points of $P$. To simplify the solution, consider only triangles $abu$, where $a$ and $b$ lie on the lower portion $C^-$ of $C$ and $u$ lies on the upper portion $C^+$.

The solution is not simple. Here are some guiding steps:

(a) Show that, for a given base $ab$, the third vertex $u$ which maximizes the area of $\triangle abu$ can be found in $O(\log n)$ time (possibly after some (cheap) preprocessing).

(b) For each base $ab$, with $a, b \in P \cap C^-$, let $A_{ab}$ denote the maximum area of a triangle $abu$, with $u \in P \cap C^+$. Show that $A$ has the inverse Monge property.

(c) For (b), fix a vertex $u$ in $P \cap C^+$, and four vertices $a, b, c, d$ in $P \cap C^-$, in this left-to-right order, and prove that

$$\text{Area}(adu) + \text{Area}(bcu) \leq \text{Area}(acu) + \text{Area}(bdu).$$

Complete the details of implementing the algorithm.

## Problem 3

Apply Chan's technique to solve efficiently (in close to quadratic time) the following problem. Let $A$ be a set of $n$ red points in the plane and let $B$ be a set of $n$ blue points in the plane. Find the triangle $uvw$ of smallest perimeter, such that $u, v \in A$ and $w \in B$.

## Problem 4 (bonus)

Let $K$ be a set of $n$ lines in $\mathbb{R}^3$ that are parallel to the $xz$-plane; that is, each of them is of the form $y = a$, $z = bx + c$. Let $L$ be another set of $n$ lines in $\mathbb{R}^3$ that are parallel to the $yz$-plane; that is, each of them is of the form $x = a'$, $z = b'y + c'$. (When projected on the $xy$-plane, they look like an axis-parallel grid.)

We are given that each line of $K$ lies below all the lines of $L$. Find the smallest vertical distance between a line in $K$ and a line in $L$; the vertical distance between two lines $\ell, \ell'$ is the difference of the $z$-coordinates of the unique pair of points that lie vertically above each other, one on each line. (In our case, these points are indeed unique for any pair of lines in $K \times L$.)

Show that this can be done in $O(n \log n)$ time.

**Hint:** Write down the explicit expressions, and try to find a simpler interpretation of the minimization problem. If you do it right, you will not need any of the new methods (but you are welcome to use them if...)