Assignment 4 - Geometric Optimization (0368-4144)

Due: January 4, 2010

Problem 1

k-Medians. (1) Let $P = \{p_1, \ldots, p_n\}$ be a set of n points on the real line. The *1-median* of P is a point x that minimizes the sum $\sum_{i=1}^{n} |x - p_i|$. Show that the real median of P is a 1-median. (You may assume that n is odd.)

(2) The *2-median* problem for P is to find two numbers x, y that minimize the sum

$$\sum_{i=1}^{n} \min\{|x - p_i|,\ |y - p_i|\}.$$

Give an $O(n \log n)$-time algorithm to compute a 2-median of P.

(3) Solve the *1-line median* problem for a set $P = \{p_1, \ldots, p_n\}$ of n points in the plane: We want to find a line ℓ that minimizes the sum $\sum_{i=1}^{n} d(\ell, p_i)$. Give an algorithm that runs in near-quadratic time (i.e., in $O(n^2\text{polylog}(n))$ time). (**Hint:** Use (1) to solve the problem if the slope of ℓ is fixed, and then ...)

Problem 2

Clustering by partitioning. Let P be a set of n points in the plane. We want to partition P into two sets P_1, P_2, so as to minimize some objective function. In this exercise, the function is any function of the radii of the smallest enclosing disks of P_1 and P_2 (e.g., their maximum, their sum, etc.).

Show that there is always an optimal solution in which P_1 and P_2 are separated from each other by a line. Using this, and duality, derive an efficient algorithm for this problem. (Try to get a near-quadratic algorithm, but I will also accept, with almost the full grade, solutions that are slower.)

Problem 3

The discrete 2-center problem. Let P be a set of n points in the plane. We want to find two congruent disks of smallest radius, each *centered at some point of* P, whose union covers P. Give a near-quadratic algorithm for this problem. (**Hints:** (a) Show that the discrete 1-center problem can be solved in near-linear time. (b) Solve the decision problem in near-quadratic time. (c) What are the critical radii? Show that there is no need for real parametric searching.)

Problem 4

Let R be a finite set of axis-parallel rectangles. Show that R is 3-pierceable if and only if every subset of 16 or fewer rectangles is 3-pierceable. (That is, show that if R is not 3-pierceable then there exists a subset of 16 or fewer rectangles that is not 3-pierceable.)

Problem 5

(1) Let P be a set of n points in d dimensions, and let μ be the center of mass of P. Let c be any point in $\mathbb{R}^d$. Show that

$$\sum_{p \in P} d(p, \mu) \leq 2 \sum_{p \in P} d(p, c).$$

(2) Use (1) to give a factor-2 approximation for the 1-median problem for a set P of n points in any dimension. Can you suggest a way of using this idea to approximate the k-median problem? For example, for a set of points on a line? (This latter question is open-ended; no concrete answer is expected...)