Problem 1

Let $T$ be a set of $n$ triangles in the plane.

(a) Describe in the dual plane the structure of the set $K$ of points dual to all the lines that intersect all the triangles.

(b) The following is known: The complexity (number of edges and vertices) of the lower (or upper) envelope of $n$ line segments and rays in the plane is $O(n\alpha(n))$, where $\alpha(n)$ is the inverse Ackermann function, which grows extremely slowly with $n$. Since $\alpha(n) \ll \log n$, use instead the weaker (but simpler) bound of $O(n \log n)$ for the complexity of the envelope.

Assuming this, give a near-linear algorithm for computing $K$. (Hint: Use a divide-and-conquer approach to construct the relevant envelopes; if you solved (a) correctly, the envelopes are there...)

(c) Assume now that all the triangles of $T$ contain the origin in their interior. Describe the structure of the set $M$ of points dual to the lines that miss all the triangles in $T$. Show that this set has a near-linear complexity and compute it in near-linear time. I prefer to see two solutions: One that does all the analysis in the primal plane, and the other that maps each triangle $t_i \in T$ to the set $M_i$ of points dual to the lines that intersect $t_i$, and then reasons directly about the sets $M_i$. 
Problem 2

Given two sets $A = \{p_1, \ldots, p_n\}, B = \{q_1, \ldots, q_m\}$ of points in the plane. Determine in linear time whether the two sets can be separated from one another by a line, and, if so, produce such a separating line. How can the problem be extended into higher dimensions and how efficiently can it be solved?

Problem 3

Given $n$ half-planes of the form $a_ix + b_iy + c_i \geq 0$, for $i = 1, \ldots, n$. Find, in $O(n)$ time, the largest circle that is fully contained in their intersection.

Problem 4

Use duality to solve efficiently the following problem: Given a set $S$ of $n$ points in the plane, find the triangle of smallest area whose vertices belong to $S$. (As a special case, determine whether $S$ has three collinear points—they define a triangle of zero area.) (Hint: Prove that if $abc$ is the smallest such triangle, then, up to some permutation of $a, b, c$, the line $c^*$ lies immediately above or below the point of intersection of the lines $a^*, b^*$. Use this observation to get an $O(n^2 \log n)$ solution, using, e.g., line-sweeping.)