Assignment 5 - Computational Geometry (0368-4211)

Due: June 29, 2015 (in Omer's mailbox or by email to him)

Problem 1

Let S be a given set of n points in the plane, and let R denote the x-axis. Define $Vor(S, R)$, the Voronoi diagram of S on R, to be the partitioning of R into regions such that for each of these regions J there exists a point a in S so that the distance of each point x in J from a is less than or equal to the distance of x from any other point of S.

(a) How many regions does $Vor(S, R)$ contain, and what is their shape?

(b) Give a direct method, having time complexity $O(n \log n)$, for the calculation of $Vor(S, R)$. (Do not compute the 2-dimensional Voronoi diagram of S!)

(c) Show that when a new point is added to S, the diagram can be updated in time $O(\log n)$.

Problem 2

Let S be a set of n points in the plane, and suppose that its Voronoi diagram $Vor(S)$ is given. Describe an algorithm which, given a new point x, calculates $Vor(S \cup \{x\})$ in $O(n)$ time. Show that the number of changes in $Vor(S)$ needed to produce $Vor(S \cup \{x\})$ can actually be $\Omega(n)$ in the worst case.

Problem 3

Prove the following local property of the Delaunay triangulation of a set P of n points in the plane. Let T be a triangulation of (the convex hull of) P that has the following property. For each edge ab of T, which is not an edge of the hull, let Δabc and Δabd be the two triangles of T that are adjacent to ab. We say that ab is *locally Delaunay* if d lies outside the circle circumscribing Δabc (which is equivalent to c lying outside the circle circumscribing Δabd). Show that if every (internal) edge of T is locally Delaunay then T is the Delaunay triangulation of P. (**Hint:** Argue using the three-dimensional representation of $DT(P)$.)

Problem 4

Let $\mathcal{D} = \{D_1, \ldots, D_n\}$ be a set of n pairwise disjoint disks in the plane (of different radii). Let $\text{Vor}(\mathcal{D})$ denote the Voronoi diagram of $\mathcal{D}$. As usual, it is the partitioning of the plane into Voronoi cells $V(D_1), \ldots, V(D_n)$, where

$$V(D_i) = \{x \in \mathbb{R}^2 \mid dist(x, D_i) \leq dist(x, D_j) \text{ for each } j\},$$

where $dist(x, D)$ is the distance from x to D (it is 0 if $x \in D$ and otherwise it is the smallest distance from x to a point of D).

(a) How does a bisector between two disks D, D' look like?

(b) Show that, for each i, $V(D_i)$ is *star-shaped* with respect to the center of D_i: If $x \in V(D_i)$ and y lies on the segment connecting x to the center, then y is also in $V(D_i)$. (**Hint:** Assume to the contrary that y is in another cell, and use the triangle inequality to get a contradiction.)

(c) Conclude that each Voronoi region is connected, and derive an upper bound for the complexity of the diagram.