Polygonal Approximation of a Jordan Curve

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May 19, 2002

1 Preliminaries

Let γ be a closed Jordan curve in the plane. Let $P = \{p_1, \ldots, p_n\}$ be a set of n points on γ, which appear in this counterclockwise order along the curve. For each $i = 1, \ldots, n$, let γ_i denote the portion of γ between p_i and p_{i+1} (where we put $p_{n+1} = p_1$). let C_i denote the convex hull of γ_i, and let R_i be a circumscribed rectangle of C_i (or of γ_i), one of whose sides is parallel to the straight segment $p_i p_{i+1}$. Put $U = \bigcup_{i=1}^n C_i$, and $W = \bigcup_{i=1}^n R_i$. Clearly, $U \subseteq W$.

The union of the rectangles may have quadratic complexity. We first observe that the union $W = \bigcup_{i=1}^n R_i$ may have quadratic complexity. A construction that illustrates the lower bound is shown in Figure 1.

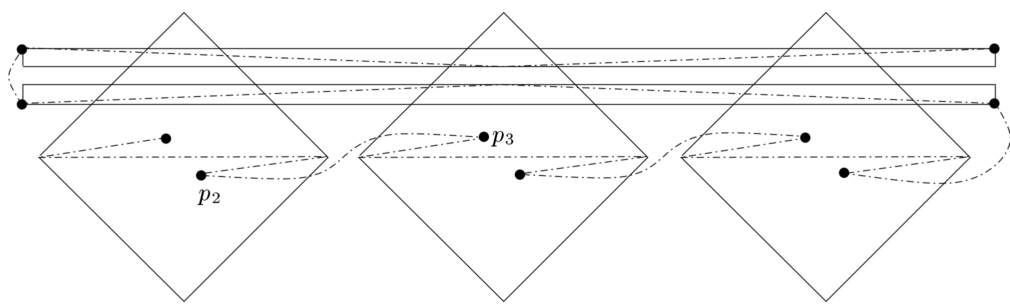


Figure 1: The union W of the rectangles R_i may have quadratic complexity. Not all rectangles are shown (e.g., the one constructed for p_2, p_3 is missing), but their presence would not have affected the quadratic complexity of W.

Hence, our goal is to find an intermediate polygonal region U^* that contains U, is contained in W, and has only linear complexity. In what follows we show how to construct such a region.

Lemma 1.1 *U has linear complexity. That is, the number of intersection points of the boundaries of the sets C_i that lie on ∂U is at most $6n - 12$, for $n \geq 3$.*

Proof: We claim that $\{C_i\}$ is a collection of *pseudo-disks*, i.e., simply connected planar regions, each pair of whose boundaries intersect at most twice. This is a well known property (see, e.g., [1]). We include its simple proof for the sake of completeness. Let C_i, C_j be a fixed pair of these sets, and suppose to the contrary that ∂C_i and ∂C_j intersect each other in at least four points. Since these sets are convex, $C_i \cup C_j \setminus (C_i \cap C_j)$ consists of at least four nonempty connected components, at least two of which, denoted C_i', C_i'', are contained in $C_i \setminus C_j$, and at least two others, denoted C_j', C_j'', are contained in $C_j \setminus C_i$; see Figure 2.

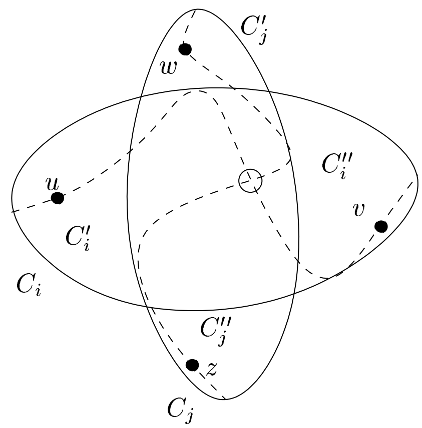


Figure 2: Two hulls C_i, C_j cannot intersect at four points.

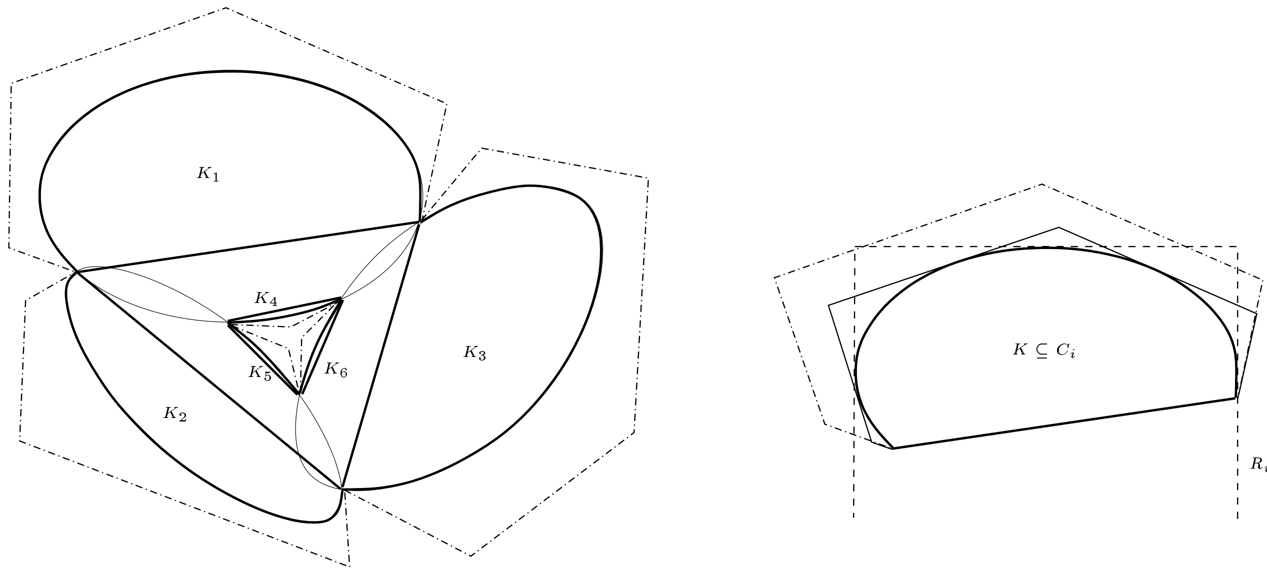
Note that each of the components C_i', C_i'' (resp., C_j', C_j'') must intersect γ_i (resp., γ_j), for otherwise, if, say, $\gamma_i \cap C_i' = \emptyset$, then we can replace C_i by $C_i \setminus C_i'$, which is a convex set that contains γ_i, contradicting the fact that C_i is the convex hull of γ_i. Choose four points $u \in C_i' \cap \gamma_i$, $v \in C_i'' \cap \gamma_i$, $w \in C_j' \cap \gamma_j$, and $z \in C_j'' \cap \gamma_j$, and observe that the portion of γ_i between u and v must cross the portion of γ_j between w and z; see Figure 2. This contradiction implies that $\{C_i\}$ is a family of pseudo-disks. The claim is now an immediate consequence of the linear bound on the complexity of the union of pseudo-disks, given in [3]. $\Box$

For each pair of consecutive intersection points u, v along (some connected component of) ∂U, connect u and v by a straight segment. This chord and the portion of ∂U between u and v bound a convex subregion of U. Let $\mathcal{K}$ denote the set of resulting subregions. The regions in $\mathcal{K}$ are pairwise openly-disjoint, as is easily verified. See Figure 3(a) for an illustration. We now use the following result, due to Edelsbrunner et al. [2]:

Lemma 1.2 *Let $\mathcal{K}$ be a collection of m pairwise openly disjoint convex regions in the plane. One can cover each region in $\mathcal{K}$ by a convex polygon, so that the resulting polygons are also pairwise openly disjoint, and the total number of their edges is at most $3m - 6$.*

Let K be a region in $\mathcal{K}$, and let P be the convex polygon that covers K. We shrink P by translating each of its edges so that it becomes tangent to K. The resulting polygon P' is clearly contained in P and contains K. Finally, let C_i be the (unique) convex hull that contains K, and let R_i be the rectangle containing C_i. We replace P' by $P' \cap R_i$. This increases the number of edges of P' by at most four. See Figure 3(b).

In summary, we have obtained a collection $\mathcal{P}$ of at most $6n - 12$ pairwise openly disjoint convex polygons with a total of at most $3(6n - 12) - 6 = 18n - 42$ edges. Let U^* denote the



(a) (b)

Figure 3: (a) The subregions in $\mathcal{K}$ and their containing polygons. (b) Shrinking a covering polygon.

union of U with the union of $\mathcal{P}$. Then $U \subseteq U^* \subseteq W$. Moreover, ∂U^* consists exclusively of edges of the polygons in $\mathcal{P}$, so U^* is a polygonal region with $18n - 42$ edges. We have thus shown:

Theorem 1.3 *Let γ, P, and W be as above. There exists a polygonal region U^* with at most $18|P| - 42$ edges that contains γ and is contained in W.*

References

[1] B. Chazelle and L.J. Guibas, Fractional cascading: II. Applications, *Algorithmica* 1 (1986), 163–191.

[2] H. Edelsbrunner, A. D. Robison and X. Shen, Covering convex sets with non-overlapping polygons, *Discrete Math.* 81 (1990), 153–164.

[3] K. Kedem, R. Livne, J. Pach and M. Sharir, On the union of Jordan regions and collision-free translational motion amidst polygonal obstacles, *Discrete Comput. Geom.* 1 (1986), 59–71.