Geometrically Aware Communication in Random Wireless Networks Reconsidered*

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Abstract

Some of the first routing algorithms for geographically aware wireless networks used the Delaunay triangulation among the network's nodes as the underlying connectivity graph [4]. These solutions were considered impractical, however, because in general the Delaunay triangulation may contain arbitrarily long edges, and because calculating the Delaunay triangulation generally requires a global view of the network. Many other algorithms were then suggested for geometric routing, often assuming random placement of network nodes for analysis or simulation [30, 5, 31, 16]. We show that, when the nodes are uniformly placed in the unit disk, the Delaunay triangulation does not contain long edges, it is easy to compute locally and it is in many ways optimal for geometric routing and flooding.

In particular, we prove that, with high probability, the maximal length of an edge in $Del(P)$, the Delaunay triangulation of a set P of n nodes uniformly placed in the unit disk, is $O(\sqrt[3]{\frac{\log n}{n}})$, and that the expected sum of squares of all the edges in $Del(P)$ is $O(1)$. These geometric results imply that for wireless networks, randomly distributed in a unit disk **(1)** computing the Delaunay triangulation locally is asymptotically easy; **(2)** simple "face routing" through the Delaunay triangulation optimizes, up to poly-logarithmic factors, the energy load on the nodes, and **(3)** flooding the network, an operation quite common in sensor nets, is with high probability optimal up to a constant factor. The last property is particularly important for geocasting [20] because the Delaunay triangulation is known to be a spanner [12].

*Work by M.S. has been supported by a grant from the U.S.-Israeli Binational Science Foundation, by a grant from the Israel Science Fund (for a Center of Excellence in Geometric Computing), by NSF Grants CCR-97-32101, CCR-00-98246, and by the Hermann Minkowski–MINERVA Center for Geometry at Tel Aviv University.

1. Introduction

We consider energetically efficient geometric routing and flooding in position-aware wireless networks and, in particular, in sensor networks. Such networks are typically created among a large number of stationary nodes, randomly placed in some bounded geographic region, that communicate by radio transmission. Usually it is assumed that there is no centralized fixed infrastructure that directly communicates with each node. Instead, nodes that are within each other's radio range may communicate directly, while those that are far apart use intermediate nodes to relay messages.

Wireless sensor nets were suggested for applications ranging from disaster recovery and surveillance [13] to the exploration of Mars [10]. In many cases, thousands of nodes are expected to interconnect while deployed in a hostile inaccessible environment. Thus, while such networks share some of the aspects of classical radio networks (geographical positioning, for example), they have also created some unique challenges. Sensors are usually required to be small, light and cheap, which means that power management is particularly important [1, 13]. While there may be many parameters which affect the power consumption in each node, it is common to assume that message transmission is the dominating factor. Another property unique to sensor networks is that in many cases sensors cannot have a dedicated identity. Instead, the locally sensed properties of each node are used as its identifier, making network floods (the propagation of a message by sending it over all the links in the network) far more common [1] [18, 19, 15].

Several architectures have been proposed for constructing efficient sensor networks [19, 34, 21]. A notable difference between these architectures and existing wireless architectures is that they strive to keep the network as *homogeneous* as possible. That is, global or local master nodes are not allowed. Instead, workload is distributed as evenly as possible among all the nodes. Even architectures which do cluster nodes [19] change the cluster-heads from time to time to amortize the load.

Homogeneity is a crucial property for network robustness. A robust sensor network is expected to work even if a constant fraction of the sensor-nodes fail. Thus, while we consider conventional measures of efficiency, such as the time until the first node fails [30, 8, 7] we also propose a new measure, the *half life* of a network, defined to be the number of communication rounds (routing or flooding) it takes until more than half of the nodes consume all their available energy and fail. As is the case in many homogeneous systems, the half life is equal, up to poly-logarithmic factors, to the time it takes for any constant fraction of nodes to fail. This is opposed to the time it takes for the first node to fail, or for the last node to fail, which may be significantly different. We use this measure to evaluate the cost of flooding and routing over the edges of the Delaunay triangulation, showing them to be asymptotically optimal in randomly placed sensor nets.

Several earlier studies of geometric routing have proposed the use of the Delaunay triangulation as an efficient routing graph among the network's nodes [4, 27]. These results, however, were considered impractical, because in general the Delaunay triangulation may contain arbitrarily long edges and because constructing the triangulation, even locally near a point, might require a global view of the entire point set. An extensive amount of research was then done to overcome thses shortcomings, while keeping the routing efficient [6, 7, 30, 37, 19, 7, 16, 28, 31, 23, 22, 39, 5, 25]. Various algorithms were suggested, including for the case where the nodes are uniformly distributed in some fixed region. We show that, under such an assumption, the original suggestions of [4] do indeed lead to efficient algorithms, at least when the nodes are randomly placed in a fixed disk. Furthermore, while we leave the analysis of random node placement in the unit square, or in other simple regions, as open, it follows from our analysis that, for a uniform distribution, the boundary effects of the Delaunay triangulation, which make edges near the boundary longer than "inner edges", involve, for most simple regions, an asymptotically diminishing number of nodes that lie very close to the boundary. This might suggest that it is more advantangeous to place the nodes according to a distribution that has larger density near the boundary (see [16] for example).

Given an instance, P, of n random points uniformly chosen from the unit disk, denote by $Del(P)$ the Delaunay triangulation of P, and by t_u the distance of node u from the boundary of the unit disk (that is, $t_u = 1 - \|u\|$). We show that

Theorem 1. *Depending on the distances of the endpoints u, v from the boundary, the maximum length of a Delaunay edge uv is, with high probability,*

$$\begin{cases} O\left(\sqrt[3]{\frac{\log n}{n}}\right) & t_u + t_v \le \left(\frac{\log n}{n}\right)^{2/3} \\ O\left(\frac{1}{t_u+t_v} \cdot \frac{\log n}{n}\right) & \left(\frac{\log n}{n}\right)^{2/3} \le t_u + t_v \le \left(\frac{\log n}{n}\right)^{1/2} \\ O\left(\sqrt{\frac{\log n}{n}}\right) & \left(\frac{\log n}{n}\right)^{1/2} \le t_u + t_v. \end{cases}$$

An interesting outcome of this theorem is that, with high probability, the number of nodes in the convex hull of P must be $\Theta(\sqrt[3]{\frac{n}{\log n}})$.

We also show that

Theorem 2. *The expected value of the sum of the squared lengths of the edges of $Del(P)$ is $\Theta(1)$.*

Some implications of these results are straightforward. For example, since there are no long edges, one can expect the computation of the triangulation to be local. We say that the triangulation $Del(P)$ is *r-local* if every edge in $Del(P)$ is contained in some disk of radius smaller than r that does not contain any other point in P. We show that

Theorem 3. *$Del(P)$ is an $O\left(\sqrt[3]{\frac{\log n}{n}}\right)$-local triangulation, with very high probability.*

However, showing that routing and flooding over the Delaunay edges are energetically efficient requires some assumptions on the communication patterns. We consider two homogeneous communication patterns: a *random point-to-point communication process* which models communication among nodes as a random process where

[1] In practice, floods are usually limited to some portion of the network domain, typically restricted to a specific subregion. However, since these constraints are not related to the number of deployed nodes, floods may span a constant fraction of the network.

in each step two random nodes are chosen to communicate, and an *iterated flooding process* which models repeated floods as a process where at each time step one message is propagated to the entire network by being sent over each link in the system. We then show that

THEOREM 4. *The half life of the iterated flooding process over $Del(P)$ is $\Theta(n)$.*

Noting that the upper bound in this case follows from the well known bounds for the Euclidean minimum spanning tree [17, 26], which is contained in the Delaunay triangulation. We also show that

THEOREM 5. *The time until the first node fails and the half life of a system executing a random point-to-point communication process, when routing over Delaunay edges, is, with high probability,* $\Omega((\frac{n}{\log n})^{1.5})$.

Here we assume a routing algorithm that sends a message from node u to node v along a path that lies on the union of the boundaries of all Delaunay faces crossed by the segment uv; see, e.g., [24]. We refer to such a routing strategy as *face routing*.

We also derive *upper bounds* for the network's half life, when it executes a random point-to-point communication process, where the bounds are *independent* of the routing method and the underlying routing graph. Using machinery from precolation theory, we show that

THEOREM 6. *The half life of any system executing a random point-to-point communication process, using any routing strategy on any routing graph, is $O(n^{1.5}\log n)$, with high probability.*

Thus, we prove that, up to a polylogarithmic factor, face routing over the Delaunay edges is asymptotically optimal for point-to-point communication.

Related work: The properties of the random *Poisson–Delaunay* tessellations have been extensively studied (for a background on this subject we refer the reader to [33]). For example, in [3] the expected maximum degree of the random Poisson–Delaunay tessellations is shown to be $O(\frac{\log n}{\log\log n})$, while in [35] the expected maximum edge length is studied, showing in particular that the expected maximum edge length in the unit disk is $\Omega(\sqrt{\frac{\log n}{n}})$. We extend this analysis by providing upper bounds on the edge lengths in the unit disk, which depend on the distances of their endpoints from the boundary. We also show how these results are applied to the problem of geometric routing in wireless networks.

Geometrical routing was originally described in [4, 27]. Exploring several possibilities, it was shown in [4] that, when the Delaunay triangulation of the given nodes is used as the connectivity graph, a simple greedy algorithm can be effectively used. Later works, however, suggested using other connectivity graphs, because the Delaunay triangulation is hard to compute distributively and because it may contain very long edges. The *Gabriel graph* was suggested in [25], and the intersection of the *unit disk graph* with the Gabriel graph was suggested in [5]. The routing algorithms used in both papers are variants of the *face routing* algorithm presented in [27], where, as described above, messages from node u to node v are routed over edges of faces intersected by the segment uv. The same routing mechanisms were used in [16] over a *restricted Delaunay graph*, which does not contain long edges, achieving better spanning properties. Increasingly more sophisticated mechanisms were then devised to improve the cost of sending a message in the distance, link or energy cost models [29, 30, 28]. For example, in [28] a combination of greedy routing and face routing is used over the Gabriel graph edges of the Clustered Backbone graph, which is a bounded degree unit graph that connects a dominating set of the original graph. Our results imply that, for randomly placed nodes (in the unit disk), such sophisticated mechanisms are not required. Furthermore, all these studies evaluate their algorithms in terms of the load *per message* and not in terms of the load *per node*, a property that defines the lifetime of the network and that can be very different [36]. We analyze the half-life of a network that uses face routing over the Delaunay triangulation, showing it to be asymptotically optimal.

The problem of minimum energy routing in general wireless networks has been considered as early as in [2, 11, 14]. The approach in these early works was to minimize the total energy necessary to reach the destination, or, more precisely, to minimize the energy consumed per unit flow or per packet. Probably the first attempt to define and analyze the lifetime of the network as the time until the first node runs out of energy, was in [37]. Further studies [6, 7, 39, 31, 23, 22, 36] used this measure to create and analyze new energy conserving techniques for wireless routing. We suggest, however, that a better measure for the network lifetime is the time until more than *half* of the network's nodes run out of energy. This measure represents (up to polylogarithmic factors) the time until any constant fraction of the nodes fail, which can be very different from the time until the first node fails.

Finally, we remark that while efficient flooding and geocasting were considered before [5, 38], we are not aware of any work that analyzes their effect on the network's life expectancy.

2. Model

Let $P = \{p_1, \ldots, p_n\}$ be a set of n points, each chosen independently from the uniform distribution over the unit disk $\mathbb{D}$ in the plane, and define the *cost over p_i* of a message transmission from p_i to p_j to be $O(\|p_i - p_j\|^2)$. As we are interested in the effects of repeated communication steps on the entire network, we define the half life of the network to be the first time when more than half of the points consume more than one unit of energy each.

Let $G = (P, E)$ be a graph defined on P. A routing strategy is a function which maps each pair of points in P to a simple path that connects them. More formally, Definition 2.1. *An (n-)*routing strategy *is a measurable function* $\mathcal{R} : \mathbb{D}^n \times \{1,\ldots,n\}^2 \to \bigcup_{l=1}^{\infty}\{1,\ldots,n\}^l$ *such that for any n points* $v_1,\ldots,v_n$ *in* $\mathbb{D}$ *and any i and j, the value* $\{k_1,k_2,k_3,\ldots,k_l\} := \mathcal{R}(v_1,\ldots,v_n,i,j)$ *encodes a simple path* $\{v_{k_1},\ldots,v_{k_l}\}$ *from* v_i *to* v_j.

Denote by $Del(P)$ the Delaunay triangulation [9] of P and by s_{ij} the line segment that connects points p_i, p_j. We regard the edges of $Del(P)$ as forming a (planar) graph on P, which we also denote by $Del(P)$.

Definition 2.2. *A* Delaunay face routing strategy, $\mathcal{F}$, *over the graph* $G = Del(P)$, *is a routing strategy* $\mathcal{R}$ *such that for every two points i, j, every edge in the path* $\mathcal{F}(v_1,\ldots,v_n,i,j)$ *belongs to a face of* $Del(P)$ *that intersects* s_{ij}.

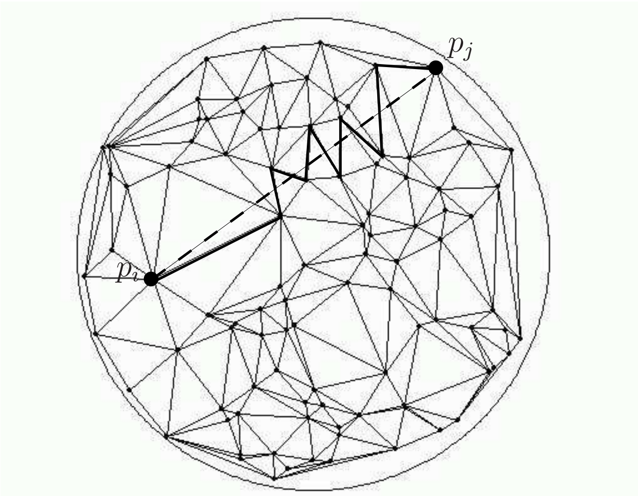

Figure 1: The Delaunay face routing strategy assigns to any two points p_i, p_j a connecting path of Delaunay edges that bound faces crossed by the line segment p_ip_j.

We note that, as defined here, there may be many instances of face routing strategies over the Delaunay graph, all of which are captured by our analysis. A specific instance of such a strategy, which follows [24], is presented in Figure 1 where all the edges of $\mathcal{F}(i,j)$, with the exception of the first and the last edges, are crossed by s_{ij}.

3. Properties of the Random Delaunay Triangulation

In this section we establish two properties of the Delaunay triangulation of a set of random points in the unit disk, involving the maximum length of Delaunay edges and the expected sum of squares of edge lengths.

We use the following properties of the Delaunay triangulation, which can be found in [9] (pages 187–188)

Theorem 3.1. *The Delaunay graph of planar point set is a plane graph.*

Theorem 3.2. *Let P be a set of points in the plane.*

1. *Three points* $p_i, p_j, p_k \in P$ *are vertices of the same face of the Delaunay graph of P if and only if the circle through* p_i, p_j, p_k *contains no point of P in its interior.*

2. *Two points* p_i *and* p_j *are connected by an edge in the Delaunay graph, if and only if there is a closed disk C that contains* p_i *and* p_j *on its boundary and does not contain any other point of P.*

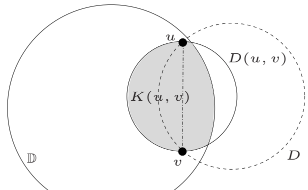

Figure 2: The region $K(u,v)$, and an empty Delaunay disk D passing through u and v.

Let $B^*(u,v)$ be the event that, for given values $p_1 = u$, $p_2 = v$, the points $p_3,\ldots,p_n$ are chosen so that uv is a Delaunay edge. Thus,

$$\mathbf{Pr}[B^*(u,v)] = \int_{B^*(u,v)} dp_3 dp_4 \cdots dp_n.$$

We can upper bound this integral as follows; see Figure 2. Let $D(u,v)$ denote the diametral disk of u, v. Consider the intersection $K(u,v) = \mathbb{D} \cap D(u,v)$, and split it into two regions by the segment uv (which clearly is fully contained in $K(u,v)$). Denote these regions as $K_1(u,v)$, $K_2(u,v)$. Let $A(u,v)$ be the area of the smaller of the two regions normalized so that the area of the entire disk is 1. If $D(u,v) \subseteq \mathbb{D}$ then we have $A(u,v) = \|u-v\|^2/8$; otherwise $A(u,v)$ is smaller.

If uv is a Delaunay edge then there exists a disk D passing through u and v and containing no other point of P in its interior. Clearly, D fully contains either $K_1(u,v)$ or $K_2(u,v)$, and both of these sets are contained in $\mathbb{D}$. Hence, a necessary condition that uv be a Delaunay edge is that either $K_1(u,v)$ or $K_2(u,v)$ does not contain any point of P. Hence, the probability of $B^*(u,v)$ (i.e., the value of the inner integral) is at most $2(1 - A(u,v))^{n-2}$.

Lemma 3.3. *For* $0 \le \|v\|, \|u\|$, *we have*

$$A(u,v) \ge$$
$$a \min\left\{\|u-v\|^2,\ \|u-v\|\left(1-\|u\|+1-\|v\|+\|u-v\|^2\right)\right\},$$

for some absolute constant $a > 0$.

Proof: Put $d = \|u-v\|$, $t_u = 1-\|u\|$ and $t_v = 1-\|v\|$. Let $c(u,v)$ be the center of $D(u,v)$ (i.e., the midpoint of uv), let $w \in \partial\mathbb{D}$ be the endpoint of the radius through $c(u,v)$

and e be the distance of $c(u,v)$ from w; see Figure 3. Now if e is $\geq d/2$ then $K(u,v) = D(u,v)$ and $A(u,v) = d^2/8$. A similar lower bound of $\Omega(d^2)$, with a smaller constant of proportionality, holds when $e \geq d/4$. Otherwise, $A(u,v)$ is at least the area of Δuvw. The height of triangle Δuvw subtended from w is at least proportional to e. This follows from the fact that, since $e < d/4$, the angle between Ow and uv must be strictly larger than $60°$, as is easily checked. Using the cosine rule in triangles $\Delta Ouc(u,v)$ and $\Delta Ovc(u,v)$ we have

$$\begin{aligned}(1-t_u)^2 &= \frac{d^2}{4} + (1-e)^2 - \frac{d^2}{2}(1-e)\cos(\theta)\\ (1-t_v)^2 &= \frac{d^2}{4} + (1-e)^2 + \frac{d^2}{2}(1-e)\cos(\theta)\end{aligned}$$

and thus

$$e = 1 - \frac{1}{2}\sqrt{(1-t_u)^2 + (1-t_v)^2 - \frac{d^2}{2}} \geq b(t_u + t_v + d^2),$$

for an appropriate absolute constant $b > 0$ ($b = 1/9$ will do, as is easily checked). It follows that in this case $A(u,v) = \Omega(d(t_u + t_v + d^2))$, as asserted. $\square$

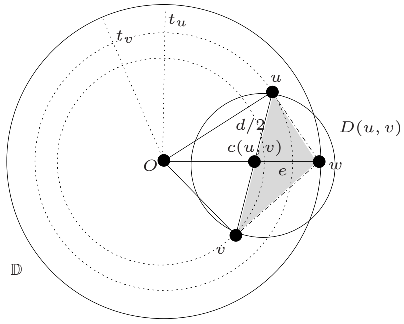

Figure 3: The lower bound on $A(u,v)$ when $e < d/4$.

We continue to use the shorthand notations $t_u = 1 - \|u\|$, $t_v = 1 - \|v\|$, $d = \|u-v\|$. Put

$$\begin{aligned}&M(u,v) =\\ &\min\{\|u-v\|^2,\ \|u-v\|(1-\|u\|+1-\|v\|+\|u-v\|^2\} =\\ &\min\{d^2,\ d(t_u+t_v)+d^3\} \quad (1)\end{aligned}$$

Lemma 3.3 and the preceding discussion thus imply:

COROLLARY 3.4.

$$\mathbf{Pr}[B^*(u,v)] \leq 2(1-aM(u,v))^{n-2},$$

where $a > 0$ is an absolute constant.

3.1 Length of Longest Edge

THEOREM 3.5. *With very high probability, the length of the longest Delaunay edge in $\mathbb{D}$ is $O\left(\sqrt[3]{\frac{\log n}{n}}\right)$.*

Proof: Since $t_u + t_v \geq 0$, we have $M(u,v) \geq \min\{d^2, d^3\} \geq d^3/2$ (since $d \leq 2$). Hence, for a fixed pair of points p_1, p_2,

$$\mathbf{Pr}\left[\|p_1p_2\| \geq c\sqrt[3]{\frac{\log n}{n}} \text{ and } p_1p_2 \text{ is a Delaunay edge}\right] =$$

$$\int_{\substack{(u,v)\in\mathbb{D}^2\\ \|u-v\|\geq c\sqrt[3]{\frac{\log n}{n}}}} \mathbf{Pr}\left[B^*(u,v)\right] dudv \leq$$

$$\int_{\substack{(u,v)\in\mathbb{D}^2\\ \|u-v\|\geq c\sqrt[3]{\frac{\log n}{n}}}} 2\left(1 - aM(u,v)\right)^{n-2} dudv \leq$$

$$2\left(1 - ac\frac{\log n}{2n}\right)^{n-2} < 2e^{-\frac{ac(n-2)\log n}{2n}} = 2\frac{1}{n^{\frac{ac(n-2)}{2n}}} \leq \frac{1}{n^{ac/4}},$$

as $n \to \infty$. The probability that at least one pair of points induces a long Deluanay edge is thus at most

$$\binom{n}{2} \cdot \frac{1}{n^{ac/4}} \leq \frac{1}{n^{ac/4-2}}.$$

The assertion now follows if we choose c to be a sufficiently large constant. $\square$

THEOREM 3.6. *With very high probability, the length of the longest Delaunay edge uv such that $t_u + t_v \geq \sqrt{\frac{\log n}{n}}$ is $O\left(\sqrt{\frac{\log n}{n}}\right)$.*

Proof: Assume that $d \geq c\sqrt{\frac{\log n}{n}}$, for some $c > 1$, and that $t_u + t_v \geq \sqrt{\frac{\log n}{n}}$. Then

$$M(u,v) = \min(d^2, d(t_u + t + v) + d^3) \geq c\frac{\log n}{n}.$$

Hence, for a fixed pair p_1, p_2,

$$\mathbf{Pr}\Bigg[\|p_1p_2\| \geq c\sqrt{\frac{\log n}{n}} \text{ and } \max\{\|p_1\|, \|p_2\|\} \leq 1 - \sqrt{\frac{\log n}{n}}$$

$$\text{and } p_1p_2 \text{ is a Delaunay edge}\Bigg] \leq$$

$$2\mathbf{Pr}\Bigg[\|p_1p_2\| \geq c\sqrt{\frac{\log n}{n}} \text{ and } \|p_2\| \leq \|p_1\| \leq 1 - \sqrt{\frac{\log n}{n}}$$

$$\text{and } p_1p_2 \text{ is a Delaunay edge}\Bigg] =$$

$$2\int_{\substack{(u,v)\in\mathbb{D}^2\\ \|u-v\|\geq c\sqrt{\frac{\log n}{n}},\ \|v\|\leq\|u\|\leq 1-\sqrt{\frac{\log n}{n}}}} \mathbf{Pr}\left[B^*(u,v)\right] dudv \leq$$

$$4\int_{\substack{(u,v)\in\mathbb{D}^2\\ \|u-v\|\geq c\sqrt{\frac{\log n}{n}},\ \|v\|\leq\|u\|\leq 1-\sqrt{\frac{\log n}{n}}}} (1 - aM(u,v))^{n-2} dudv \leq$$

$$4\left(1 - ac\frac{\log n}{n}\right)^{n-2} \leq \frac{1}{n^{ac/4}},$$

as $n \to \infty$, as above. The probability that at least one pair of points falls in the above range and induces a long Deluanay edge is thus at most

$$\binom{n}{2} \cdot \frac{1}{n^{ac/4}} \le \frac{1}{n^{ac/4-2}}.$$

The assertion now follows if we choose c to be a sufficiently large constant. □

Using the same machinery, one can obtain the following extension of both Theorems 3.5 and 3.6.

Theorem 3.7. *Depending on the distances of the endpoints u, v from the boundary, the maximum length of a Delaunay edge uv is, with high probability,*

$$\begin{cases} O\left(\sqrt[3]{\frac{\log n}{n}}\right) & t_u + t_v \le \left(\frac{\log n}{n}\right)^{2/3} \\ O\left(\frac{1}{t_u+t_v} \cdot \frac{\log n}{n}\right) & \left(\frac{\log n}{n}\right)^{2/3} \le t_u + t_v \le \left(\frac{\log n}{n}\right)^{1/2} \\ O\left(\sqrt{\frac{\log n}{n}}\right) & \left(\frac{\log n}{n}\right)^{1/2} \le t_u + t_v. \end{cases}$$

3.2 Expected Sum of Power of Edges' Lengths

Theorem 3.8. *The expected value of the sum of the squared lengths of the edges in $Del(P)$ is $\Theta(1)$.*

Proof: Since the Euclidean minimum spanning tree is a subgraph of the Delaunay triangulation, it follows e.g. from [17, 26] that the expected value is $\Omega(1)$. To prove that it is also $O(1)$, fix a pair of distinct points $u, v \in P$, say $u = p_i$ and $v = p_j$. The contribution of the pair u, v to the desired expectation is

$$E(u, v) = \int_{B_{ij}} \|u - v\|^2 dp_1 dp_2 \cdots dp_n,$$

where B_{ij} is the event that $p_i p_j$ is a Delaunay edge in $Del(P)$.

We can rewrite this integral as follows. First assume, without loss of generality, that $u = p_1$ and $v = p_2$. Then

$$E(u, v) = \int \|u - v\|^2 \left(\int_{B^*(u,v)} dp_3 dp_4 \cdots dp_n \right) dudv,$$

Using Corollary 3.4, we have

$$E(u, v) \le 2 \int_{\mathbb{D}^2} \|u - v\|^2 (1 - aM(u, v))^{n-2} dudv,$$

where the integration is normalized so that Area($\mathbb{D}$) = 1. Since $M(u, v)$ is symmetric in u and v, we can rewrite this as

$$E(u, v) \le 4 \int_{\|u\| \le 1} \left(\int_{\|v\| \le \|u\|} \|u - v\|^2 (1 - aM(u, v))^{n-2} dv \right) du. \tag{2}$$

Lemma 3.9. *Define, for any fixed $x \le 1$,*

$$S(x) = \{(u, v) \in \mathbb{D}^2 \mid \|v\| \le \|u\|, M(u, v) \le x\}.$$

Then

$$\int_{S(x)} \|u - v\|^2 dudv = O(x^2).$$

Proof: Consider the partition $S(x) = S_1(x) \cup S_2(x) \cup S_3(x)$, where

$$\begin{aligned} S_1(x) &= S(x) \cap \{d < t_u\} \\ S_2(x) &= S(x) \cap \{d^2 < t_u \le d\} \\ S_3(x) &= S(x) \cap \{t_u \le d^2\}. \end{aligned}$$

We estimate separately each of the subintegrals $\int_{S_j(x)} \|u - v\|^2 dudv$, for $j = 1, 2, 3$.

Integration over $S_1(x)$: Here we have $d(t_u + t_v) + d^3 \ge d^2 + d^3 > d^2$, so $M(u, v) = d^2$. Since $M(u, v) \le x$, we have $\|u - v\| \le x^{1/2}$ over $S_1(x)$, and so

$$\int_{S_1(x)} \|u - v\|^2 dudv \le x \int_{S_1(x)} dudv \le x \int_{\|u\| \le 1} \left(\int_{\|v-u\| \le x^{1/2}} dv \right) du = O(x^2).$$

(The inner integral is at most the normalized area of a disk of radius $x^{1/2}$, which is x.)

Integration over $S_3(x)$: Here $d(t_u + t_v) + d^3 \ge d^3$, so, arguing as above, $M(u, v) \ge d^3/2$, which implies that $d = \|u - v\| \le (2x)^{1/3}$, and $1 - \|u\| = t_u \le d^2 \le (2x)^{2/3}$. Hence

$$\begin{aligned} &\int_{S_3(x)} \|u - v\|^2 dudv \le (2x)^{2/3} \int_{S_3(x)} dudv \le \\ &(2x)^{2/3} \int_{1-(2x)^{2/3} \le \|u\| \le 1} \left(\int_{\|v-u\| \le (2x)^{1/3}} dv \right) du = \\ &O(x^{4/3}) \cdot \int_{1-(2x)^{2/3} \le \|u\| \le 1} du = O(x^{4/3}) \cdot O(x^{2/3}) = O(x^2), \end{aligned}$$

since the final integral is the normalized area of the annulus $1 - (2x)^{2/3} \le \|u\| \le 1$, which is $O(x^{2/3})$.

Integration over $S_2(x)$: Here $d(t_u + t_v) + d^3 \ge dt_u$ and $d^2 \ge dt_u$, so $M(u, v) \ge dt_u$, and thus $d \le x/t_u$. On the other hand, $d \le t_u^{1/2}$. Moreover, $x \ge dt_u \ge t_u^2$, so $t_u \le x^{1/2}$. That is, we have

$$\|v - u\| = d \le \min\left\{\frac{x}{t_u}, t_u^{1/2}\right\} = \begin{cases} \frac{x}{t_u} & x^{2/3} \le t_u \le x^{1/2}, \\ t_u^{1/2} & t_u \le x^{2/3}. \end{cases}$$

We thus have

$$\int_{S_2(x)} \|u-v\|^2 dudv \leq$$
$$\int_{0\leq 1-\|u\|\leq x^{2/3}} \left(\int_{\|v-u\|\leq (1-\|u\|)^{1/2}} \|v-u\|^2 dv \right) du$$
$$+ \int_{x^{2/3}\leq 1-\|u\|\leq x^{1/2}} \left(\int_{\|v-u\|\leq \frac{x}{1-\|u\|}} \|v-u\|^2 dv \right) du \leq$$
$$\int_{0\leq 1-\|u\|\leq x^{2/3}} (1-\|u\|)^2 du + \int_{x^{2/3}\leq 1-\|u\|\leq x^{1/2}} \frac{x^4}{(1-\|u\|)^4} du \leq$$
$$x^{4/3} \int_{0\leq 1-\|u\|\leq x^{2/3}} du + x^4 \int_{x^{2/3}\leq 1-\|u\|\leq x^{1/2}} \frac{du}{(1-\|u\|)^4}.$$

As in the case of integration over $S_3(x)$, the first integral is $O(x^2)$. The second integral, in polar coordinates, becomes

$$x^4 \int_{1-x^{1/2}}^{1-x^{2/3}} \frac{rdr}{(1-r)^4} = x^4 \int_{x^{2/3}}^{x^{1/2}} \frac{(1-z)dz}{z^4} =$$
$$x^4 \left[\frac{1}{2z^2} - \frac{1}{3z^3} \right]_{z=x^{2/3}}^{z=x^{1/2}} =$$
$$x^4 \left[\frac{1}{2x} - \frac{1}{2x^{4/3}} - \frac{1}{3x^{3/2}} + \frac{1}{3x^2} \right] = O(x^2).$$

This completes the proof of the lemma. □

Returning to the estimation of (2), we have

$$E(u,v) \leq$$
$$4 \int_{\|u\|\leq 1,\ \|v\|\leq\|u\|} \|u-v\|^2 (1-aM(u,v))^{n-2} dudv =$$
$$4 \left(\sum_{k=1}^{n} \int_{S(k/n)\setminus S((k-1)/n)} + \int_{\|v\|\leq\|u\|,\ (u,v)\notin S(1)} \right)$$
$$\|u-v\|^2 (1-aM(u,v))^{n-2} dudv \leq$$
$$4 \sum_{k=1}^{n} \left(1 - \frac{a(k-1)}{n} \right)^{n-2}$$
$$\int_{S(k/n)} \|u-v\|^2 dudv + 4 \int_{\mathbb{D}\times\mathbb{D}} (1-a)^{n-2} dudv =$$
$$O\left(\sum_{k=1}^{n} e^{-ak} \left(\frac{k}{n}\right)^2 + (1-a)^n \right) = O\left(\frac{1}{n^2}\right).$$

That is, we have thus shown that

$$E(u,v) = O\left(\frac{1}{n^2}\right),$$

for every pair $u, v \in P$. Hence, the expected sum of the squared lengths of the Delaunay edges is $\sum_{u,v} E(u,v) = O(1)$. This completes the proof of Theorem 3.8. □

4. Applications of the Geometric Bounds

The preceding geometric bounds have immediate implications for the analysis of the performance of random wireless networks.

We use the following terminology in the next theorem. For every edge e in the Delaunay triangulation there is a disk $D(e)$ that contains the edge e and no other point in P. If the area of $D(e) \cap \mathbb{D}$ is less than πr^2 we say that the edge e is an *r-local edge*. If all the edges in the Delaunay triangulation are r-local we say that the Delaunay triangulation is an *r-local triangulation*.

Theorem 4.1. *With very high probability the Delaunay triangulation over P is an $O\left(\sqrt[3]{\frac{\log n}{n}}\right)$-local triangulation.*

Proof: An immediate consequence of Theorem 3.5. □

Thus, with high probability, any pair of points whose distance is larger than $O\left(\sqrt[3]{\frac{\log n}{n}}\right)$, do not form a Delaunay edge, and for any pair of points u, v within that distance, only a small neighborhood of uv need be tested to determine whether uv is a Delaunay edge. This suggests an efficient local and distributed mechanism for constructing the Delaunay triangulation, as the network is set in operation.

Theorem 4.2. *The half life of the iterated flooding process over $Del(P)$ is $\Theta(n)$.*

Proof: It follows from Theorem 3.8 that there exists some constant c, such that with high probability, the total energy used for n floods, is cn. Since this cost is divided among the n nodes, there must be $\frac{n}{2}$ nodes which expend at most $2c$ energy units each during the n floods. Therefore, it follows that the number of floods that can be executed until more than half of the nodes fail is $\Theta(n)$. □

Theorem 4.2 implies that using the Delaunay edges to flood information should be feasible in random graphs.

4.1 Load of the Random PtP Process

Analyzing the load of the random point-to-point communication process, however, requires a more elaborate analysis.

For a given set $P = \{p_i\}$ of n random points in $\mathbb{D}$, let $(s_1, d_1), \ldots, (s_k, d_k)$ be k source-destination pairs, where $s_j \neq d_j$, for $j = 1, \ldots, k$, are randomly chosen points of P. For any node p_i, routing strategy $\mathcal{R}$, and any pair (s_j, d_j), there are at most two edges in the path $\mathcal{R}(s_j, d_j)$ that are incident to p_i. Let $e_i^j(\mathcal{R})$ be the longest edge of $\mathcal{R}(s_j, d_j)$ that is incident to p_i (if p_i is not on $\mathcal{R}(s_j, d_j)$ then $e_i^j(\mathcal{R}) = 0$).

Let $\{X_i^j(\mathcal{R})\}, 1 \leq i \leq n, 1 \leq j \leq k$ be a set of random variables defined as $X_i^j(\mathcal{R}) = \|e_i^j(\mathcal{R})\|^2$. Variable X_i^j represents the distribution of energy consumption on node p_i by the communication between s_j and d_j with routing strategy $\mathcal{R}$.

Lemma 4.3. *Let $\mathbf{C}$ be a disk of radius l fully contained in the unit disk $\mathbb{D}$. Let p_1, p_2 be two randomly chosen points in $\mathbb{D}$. Then the probability that the segment $p_1 p_2$ intersects $\mathbf{C}$ is at most $10l$.*

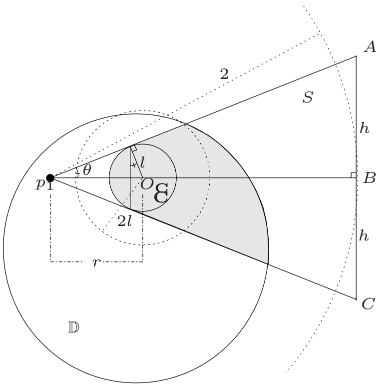

Figure 4: For B'_l to occur, p_2 has to lie in the shaded region, which is always contained in the triangle $\Delta p_1 AC$.

Proof: Denote by s the segment connecting p_1, p_2 and by B_l the event that s intersects $\mathbf{C}$. Denote by r the distance of p_1 from the center, $O_{\mathbf{C}}$, of $\mathbf{C}$; see Figure 4.

Assume that $r > 2l$. Let B'_l denote the event that B_l occurs and $\|p_1 O_{\mathbf{C}}\| > 2l$. Having chosen p_1 (at distance larger than $2l$ from $O_{\mathbf{C}}$), B'_l occurs if p_2 is chosen in the "shadow" $\mathbf{C}(p_1)$ of $\mathbf{C}$ cast within $\mathbb{D}$ by p_1. More precisely, $\mathbf{C}(p_1)$ is the portion of the wedge W formed by the two rays that emerge from p_1 and are tangent to $\mathbf{C}$, consisting of all points that lie in $\mathbb{D}$ and are hidden from p_1 by $\partial\mathbf{C}$.

Let $\Delta p_1 AC$ denote the isosceles triangle whose apex is p_1, whose sides lie on the tangent rays from p_1 to $\mathbf{C}$, and whose height $p_1 B$ is 2. It is easily verified that $\mathbf{C}(p_1)$ is fully contained in $\Delta p_1 AC$. It follows that

$$\mathbf{Pr}[B'_l] = \int_{\|p_1 O_{\mathbf{C}}\| \geq 2l} \text{Area}(\mathbf{C}(p_1)) dp_1 \leq \int_{\|p_1 O_{\mathbf{C}}\| \geq 2l} \text{Area}(\Delta p_1 AC) dp_1.$$

We rewrite the integral using polar coordinates about $O_{\mathbf{C}}$. Recalling that we use normalized areas, the integral is at most

$$\mathbf{Pr}[B'_l] \leq 2 \int_{2l}^{2} \text{Area}(\Delta p_1 AC) r dr. \tag{3}$$

We estimate the area of $\Delta p_1 AC$ as follows. Denote by h its half-base. Since the height from p_1 is 2, we have

$$\frac{h}{\sqrt{h^2+4}} = \frac{l}{r},$$

or

$$\frac{h^2+4}{h^2} = \frac{r^2}{l^2},$$

or $h = \frac{2l}{\sqrt{r^2-l^2}}$, and since $l < r/2$, we have $h \leq \frac{4l}{\sqrt{3}r}$. Thus the (normalized) area of $\Delta p_1 AC$ is $\frac{2h}{\pi} \leq \frac{8l}{\pi\sqrt{3}r}$. Substituting this in (3), we obtain

$$\mathbf{Pr}[B'_l] \leq \frac{16l}{\pi\sqrt{3}} \int_{2l}^{2} dr < \frac{32l}{\pi\sqrt{3}} < 6l,$$

and since the probability that $r \leq 2l$ is $4l^2 < 4l$ we are done. □

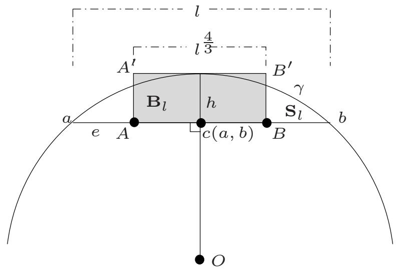

Figure 5: The rectangle $ABB'A'$ is the boundary box $\mathbf{B}_l$ defined for the circular cap $\mathbf{S}_l$.

Let $\mathbf{S}_l$ be a circular cap, bounded by a straight segment e of length l and arc γ of $\partial\mathbb{D}$, and for simplicity, assume that $l \leq 1$ and that $\mathbf{S}_l$ is the smaller of the circular caps defined by e. Denote by a and b the two endpoints of e, by $c(a,b)$ the midpoint of e and by h the distance from $c(a,b)$ to γ. As can be easily checked, $\frac{1}{8}l^2 \leq h \leq l^2$. Let A, B be two points on e such that $\|A - c(a,b)\| = \|B - c(a,b)\| = \frac{1}{2}l^{4/3}$, where A, B are in opposite directions from $c(a,b)$. The *boundary box* of $\mathbf{S}_l$, denoted $\mathbf{B}_l$, is defined to be the rectangle $ABB'A'$ formed by subtending two parallel and equal line segments from A and B towards γ that are perpendicular to e and equal in length to h; see Figure 5.

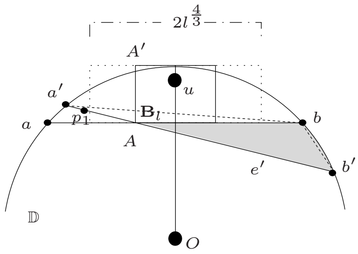

Figure 6: For B'_l to occur, point p_2 must lie in the shaded region, which is fully contained in the circular cap defined by $a'b'$.

LEMMA 4.4. *Let $\mathbf{B}_l$ be the boundary box of the circular cap $\mathbf{S}_l$ and let p_1, p_2 be two randomly chosen points in $\mathbb{D}$. Then the probability that the segment p_1p_2 intersects $\mathbf{B}_l$ is at most $131l^{10/3}$.*

Proof: Denote by s the segment connecting p_1, p_2 and by B_l the event that s intersects $\mathbf{B}_l$.

Assume first that both points are in $\mathbf{S}_l$. Since $\text{Area}(\mathbf{S}_l) \leq lh \leq l^3$ the probability that both points are in $\mathbf{S}_l$ is less than $l^6 < l^{10/3}$. Next, assume that at least one point, say p_1, is inside $\mathbf{S}_l$ but that the distance from p_1 to the center of $\mathbf{B}_l$ is at most $l^{4/3}$. I.e, point p_1 is within the rectangle that is twice the size of $\mathbf{B}_l$ and contains $\mathbf{B}_l$ (marked by dotted lines in Figure 6. As can be easily verified, the probability for this event is $\leq 2l^2 l^{4/3} = 2l^{10/3}$.

Finally, assume that one point, say p_1, is in $\mathbf{S}_l$ yet at distance bigger than $l^{4/3}$ from the center of $\mathbf{B}_l$ while the other point is outside $\mathbf{S}_l$ and denote by B'_l the intersection of this event and event B_l, i.e., that s intersects $\mathbf{B}_l$. Event B'_l occurs if p_2 is chosen in the "shadow", $\mathbf{B}_l(p_1)$, of $\mathbf{B}_l$ cast by p_1 within $\mathbb{D} \setminus \mathbf{S}_l$. In other words, $\mathbf{B}_l(p_1)$ is the portion of $\mathbb{D} \setminus \mathbf{S}_l$ that is cut out by a ray that emerge from p_1 and is tangent to $\mathbf{B}_l$, consisting of all the points in $\mathbb{D} \setminus \mathbf{S}_l$ that are hidden from p_1 by $\partial\mathbf{B}_l$; see Figure 6.

Since $\mathbf{B}_l$ divides $\mathbf{S}_l$ into two equal parts we have

$$\mathbf{Pr}[B'_l] = 2\mathbf{Pr}[B'_l | A \text{ is the tangent point}].$$

Assume that p_1 is chosen so that the tangent point is A. Let $\mathbf{S}_{p_1}$ be the circular cap of $\mathbb{D}$ whose bounding straight edge, e', passes through p_1 and A, and denote by a' and b' its endpoints on $\partial\mathbb{D}$. Notice that $\mathbf{B}_l(p_1)$ (the shaded area in Figure 6) is contained in $\mathbf{S}_{p_1}$. Consider now the triangle $\Delta a'b'b$ and denote $\alpha := \angle ba'b'$. First, notice that $\|a'b\| \leq \|ab\| = l$. Now, since the distance of a' from AA' is at least $\frac{1}{2}l^{4/3}$, $\alpha \leq \arctan(\frac{1}{2}l^{2/3}) \leq l^{2/3}$. But that means that the arc bb' must be subtended by a central angle of size 2α and thus, for a disk $\mathbb{D}$ of size 1, $\|bb'\| < \frac{2}{\sqrt{\pi}}\alpha \leq 2l^{2/3}$. Using the triangular inequality in triangle $\Delta a'b'b$ it follows that $\|e'\| \leq \|a'b\| + \|bb'\| \leq 4l^{2/3}$.

$$\mathbf{Pr}[B'_l] \leq 2\int_{p_1 \in \mathbf{S}_l}\int_{p_2 \in \mathbf{S}(p_1)} dp_2 dp_1 \leq$$
$$2\|e'\|^3 \int_{p_1 \in \mathbf{S}_l} dp_1 \leq 128 l^5 \leq 128 l^{10/3}.$$

□

For a point p near the boundary of $\mathbb{D}$, a boundary box can be used as an estimation of the union of Delaunay triangles incident to a p.

Definition 4.5. *A circular cap $\mathbf{S}_l$ is said to be* symmetric around point u *if* **(1)** *u is contained in $\mathbf{S}_l$ and* **(2)** *the radius that passes through point u crosses e at its midpoint, c.*

Lemma 4.6. *Let u be a point in $\mathbb{D}$ such that $t_u = 1 - \|u\| < \sqrt{\frac{\log n}{n}}$. Then for some constant $c > 1$ and n sufficiently large there exists a circular cap $\mathbf{S}_l$, $l = c\sqrt[4]{\frac{\log n}{n}}$ such that $\mathbf{S}_l$ is symmetric around u and with high probability all the Delaunay triangles incident to u are contained in the boundary box of $\mathbf{S}_l$.*

Proof: Since for any such $\mathbf{S}_l$, $h \geq \frac{l^2}{8} = \frac{c^2}{8}\sqrt{\frac{\log n}{n}}$, it is possible to choose such a circular cap that contains u, where Ou crosses e at its midpoint, so the first part is correct; see Figure 5. To prove the second part of the lemma divide $\mathbb{D}$ into three nonequal parts, L, M, R by two lines that extend the two short sides of $\mathbf{B}_l$, AA' and BB', and further divide M into two equal parts, M_1 and M_2, by a line that passes through O and is parallel to AB and where M_1 contains u; see Figure 7. First, we note that the length of any Delaunay edge uv such that $v \in M_2$ would be constant and hence by Theorem 3.7 the probability of such an event is 0. Otherwise, assume that there is some Delaunay edge uv such that v is either in L or in R. since $\mathbf{B}_l$ is symmetric around u it must be that $\|u - v\| > \frac{1}{2}c\sqrt[3]{\frac{\log n}{n}}$ and, from Theorem 3.7 the probability for this event is again negligible. Thus, we only have to consider the case where v is in M_1.

Noting that for all points in M_1, $t_A = t_B$ is minimal we can conclude from the Pythagorean Theorem on triangle ΔOAc (Figure 5) that

$$t_A \geq \frac{t_A^2}{2} - \frac{c^4}{128}\frac{\log n}{n} + \frac{c}{2}\sqrt{\frac{\log n}{n}} - \frac{c^{8/3}}{2}\left(\frac{\log n}{n}\right)^{2/3} > \frac{c}{3}\sqrt{\frac{\log n}{n}}$$

for large enough n. Thus, for any point $v \in M_1$, $t_u + t_v \geq \left(\frac{\log n}{n}\right)^{1/2}$. Since for all $v \in M_1$, $\|u - v\| \geq c^2\sqrt{\frac{\log n}{n}}$, it follows from Theorem 3.7 that we can choose c to be such that probability for such edges is negligible. □

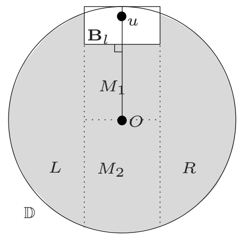

Figure 7: The short sides of the symmetric boundary box $\mathbf{B}_l$ around u define a partition of the unit disk into three, unequal parts. The middle part is then partitioned into two equal parts.

Lemma 4.7. *The load on the most loaded node in P, when executing k random point to point communication steps using face routing over $Del(P)$, is $O\left(k\left(\frac{\log n}{n}\right)^{1.5}\right)$ with high probability.*

Proof: Let $M_n^k(\mathcal{R})$ be the maximal energy consumption incurred at any node by all the communication steps:

$$M_n^k(\mathcal{R}) = \max_{1 \leq i \leq n}\left(\sum_{j=1}^{k} X_i^j(\mathcal{R})\right).$$

Denote by $u \in P$, a node that experiences this load. Denote by F_u the union of all the Delaunay triangles that are incident to u. By the definition of face routing, if the routing path that connects a pair of nodes p_i, p_j goes through u then $s_{ij} = p_i p_j$ must intersect F_u.

Assume first that $t_u = 1 - \|u\| \geq \sqrt{\frac{\log n}{n}}$. By Theorem 3.7, with very high probability, all Delaunay edges incident to u are of length at most $O\left(\sqrt{\frac{\log n}{n}}\right)$, and hence with high probability F_u is fully contained in a disk $\mathbf{C}$ of radius $O\left(\sqrt{\frac{\log n}{n}}\right)$ centered at u. It follows from Lemma 4.3 that the expected number of routing paths that intersect $\mathbf{C}$ is $O\left(k\sqrt{\frac{\log n}{n}}\right)$. Hence, using

Markov's inequality, it also follows that the maximal number of routing paths intersecting F_u is with high probability $O\left(k\sqrt{\frac{\log n}{n}}\right)$ and thus in this case,

$$M_n^k = O\left(k\left(\frac{\log n}{n}\right)^{1.5}\right)$$

Otherwise, by Lemma 4.6, F_u is contained in the boundary box of the symmetric circular cap around u, $\mathbf{S}_l, l = c\sqrt[4]{\frac{\log n}{n}}$. But from Lemma 4.4 and Theorem 3.7 and by using the same argument as before it also follows that in this case

$$M_n^k = c'^2\left(\frac{\log n}{n}\right)^{2/3}(57c)^{10/3}\left(\frac{\log n}{n}\right)^{\frac{1}{4}\frac{10}{3}} k = O\left(k\left(\frac{\log n}{n}\right)^{1.5}\right)$$

and the claim holds. □

Corollary 4.8. *The number of steps taken by a system executing a random point to point communication process using Delaunay face routing until the first node fails is with high probability* $\Omega((\frac{n}{\log n})^{1.5})$.

Proof: Since the energy consumption of the most loaded node after k steps is with high probability $O(k(\frac{\log n}{n})^{1.5})$, it is possible to find a constant, c, such that after $k = c(\frac{n}{\log n})^{1.5}$ steps the energy consumption of the most loaded node would asymptotically be 1.□

The half life of a system is obviously longer than the time it takes for the first node to fail and hence

Theorem 4.9. *The half life of a system executing a random point to point communication process using Delaunay face routing is with high probability* $\Omega((\frac{n}{\log n})^{1.5})$.

5. Asymptotic Upper bounds

To show that face routing on the Delaunay triangulation is asymptotically optimal up to poly-logarithmic factors, we estimate the expected load on a node of k transmissions when using any routing strategy. Obviously, the maximal load is larger than the expected load.

Theorem 5.1. *For any routing strategy $\mathcal{R}$ on n random points in the unit disk $\mathbb{D}$,*

$$E[\sum_{j=1}^{k} X_i^j(\mathcal{R})] = \Omega\left(\frac{k}{n^{1.5}\log n}\right)$$

The proof requires some standard tools from continuum percolation. We refer the reader to [32] for a general introduction to the topic. The first is the λ-Poisson process P_λ, which is a random process that gives random points in $\mathbb{R}^2$ (generally in $\mathbb{R}^d$ but we are now interested in $d = 2$) with density λ. See [32, page 11] for a precise definition.

Lemma 5.2. *The number of points P_λ has in the unit disk is a random variable with the distribution*

$$\mathbf{Pr}[\#(P_\lambda \cap \mathbb{D}) = n] = e^{-\lambda\pi}\frac{(\lambda\pi)^n}{n!}.$$

This follows immediately from the definition of a Poisson process. See [32, eq. 1.3].

Lemma 5.3. *A Poisson process conditioned to have n points inside the unit disk has the same distribution as n independent uniform random points.*

This follows from proposition 1.2 in [32]. Notice that it is true for any λ! We will use this lemma with $\lambda = n/\pi$ and in this case the event we are conditioning on has the probability $n^n/(n!e^n) \approx n^{-0.5}$.

A λ,ρ-Continuum percolation is the process of coloring by white a disk of radius ρ around every point of the λ-Poisson process. We get a random set, "the white region", which we denote by $W(\mathbb{R}^2)$. Generally, if $A \subset \mathbb{R}^2$ is some set, we define $W(A)$ as the collection of all connected components of $W(\mathbb{R}^2)$ which intersect A. In [32] the radii may also be random, independent variables, but we will not need this generalization.

Lemma 5.4. *There exists an $\alpha > 0$ with the following property: if $\rho \leq \alpha\lambda^{-0.5}$ and if A is a disk of radius $\lambda^{-0.5}$ then*

$$\mathbf{Pr}[\mathrm{diam}(W(A)) > m\lambda^{-0.5}] \leq Ce^{-cm}$$

where C and c are some constants.

This follows theorem 2.4 and 3.2 in [32]. Actually, theorem 2.4 is formulated for A being one point, but the proof holds for a small disk with no change. It seems that theorem 3.4 in [32] is also used, implicitly, in the proof of theorem 2.4 ibid.

We remark that the supremum of the α's satisfying this requirement is called the critical α. However, we will not need here any of the delicate results concerning this quantity.

Lemma 5.5. *Let $v_1, \ldots, v_n$ be n independent, uniform points in the unit disk $\mathbb{D}$, and for each v_i let V_i be the set of points connectable to v_i by paths with edges of length $\leq 2\alpha\sqrt{\pi/n}$ where α is from lemma 5.4. Then for some K,*

$$\mathbf{Pr}\left[\exists i, \mathrm{diam}\, V_i > K\frac{\log n}{\sqrt{n}}\right] \leq \frac{C}{n}.$$

Proof: Define $\lambda := n/\pi$ and $\rho := \alpha\lambda^{-0.5}$ and examine λ,ρ-Continuum percolation. Lemma 5.4 gives that if K is sufficiently large then

$$\mathbf{Pr}\left[W(A) > K\frac{\log n}{\sqrt{n}}\right] \leq Ce^{-cK\log n} \leq Cn^{-2.5}$$

for any disk A with radius $\lambda^{-0.5}$. Obviously, we could have put any exponent instead of the 2.5 by only changing the value of K. Cover $\mathbb{D}$ by disks $A_1, \ldots, A_m$ of radius $\lambda^{-0.5}$ (hence $m \leq Cn$) and get that

$$\mathbf{Pr}\left[\exists v \in \mathbb{D}, W(\{v\}) > K\frac{\log n}{\sqrt{n}}\right] \leq \sum_{i=1}^{m}\mathbf{Pr}\left[W(A_i) > K\frac{\log n}{\sqrt{n}}\right] \leq Cn^{-1.5}.$$

Let N be the number of points the λ-Poisson process defining our Continuum percolation has in the unit disk. As already remarked (lemma 5.2), $\mathbf{Pr}[N = n] \approx n^{-0.5}$ and then

$$\mathbf{Pr}\left[\exists v \in \mathbb{D},\, W(\{v\}) > K\frac{\log n}{\sqrt{n}}\middle| N = n\right] \leq \frac{\mathbf{Pr}\left[\exists v \in \mathbb{D},\, W(\{v\}) > K\frac{\log n}{\sqrt{n}}\right]}{\mathbf{Pr}[N = n]} \leq Cn^{-1}.$$

However, λ, ρ-Continuum percolation conditioned to have exactly n points inside the unit disk is identical to n independent uniform random points (lemma 5.3) and V_i is obviously a subset of the white region created by coloring disks of radius ρ around every v_i. Therefore

$$\mathbf{Pr}\left[\exists i,\, \mathrm{diam}\, V_i > K\frac{\log n}{\sqrt{n}}\right] \leq \mathbf{Pr}\left[\exists v \in \mathbb{D},\, W(\{v\}) > K\frac{\log n}{\sqrt{n}}\middle| N = n\right] \leq Cn^{-1},$$

and the lemma is proved.□

Proof of Theorem 5.1. Using lemma 5.5 we may assume that $\mathrm{diam}\, V_i \leq K\frac{\log n}{\sqrt{n}}$ for all i. Let v_α and v_β satisfy $||v_\alpha - v_\beta|| > \frac{1}{2}$. There have to be at least $\frac{\sqrt{n}}{2K\log n}$ different V_is on the path $\mathcal{R}(v_\alpha, v_\beta)$ hence there have to be at least $\frac{\sqrt{n}}{2K\log n} - 1$ edges with length $> 2\alpha\sqrt{\pi/n}$. Since with high probability there are $\Omega(k)$ such source-destination pairs, the expected energy used for k transmissions is $\Omega(\frac{k}{\sqrt{n}\log n})$ and so the expected load on a single node is

$$E[\sum_{j=1}^{k} X_i^j(\mathcal{R})] = \Omega(\frac{k}{n^{1.5}\log n})$$

□

CONJECTURE 1. *The $\log n$ factor in theorem 5.1 is not necessary, that is that the theorem holds with $E = \Omega(\frac{k}{n^{1.5}})$.*

There is some constant c such that after $k = cn^{1.5}\log n$ steps there is at least one node p_i, such that $\sum_{j=1}^{k} X_i^j(\mathcal{R}) = 1$ and therefore the next theorem follows.

THEOREM 5.6. *The half life of any system executing a random point to point communication using any routing strategy is $k = O(n^{1.5}\log n)$ with high probability.*

6. Conclusions and Future Research

In this paper, we analyzed the asymptotic behavior of the Delaunay triangulation of n random points uniformly distributed in the unit disk, and showed that the length of the Delaunay edges is bounded by $O\left(\left(\frac{\log n}{n}\right)^{1/3}\right)$ with high probability, and that the expected sum of squares of the edge lengths is $\Theta(1)$. We used these theoretic results to show that the Delaunay triangulation is a viable solution as a connectivity graph for wireless communication between nodes uniformly placed in the unit disk. As we showed, the triangulation is easy to compute locally, it is simple to use as a routing graph and it is energetically efficient both for routing and flooding.

One obvious direction of future research is to investigate other distribution of the sensors, and in particular uniform distributions within other kinds of convex planar regions. We believe that our results hold (with appropriate modifications) for any bounded convex region whose boundary has everywhere strictly positive curvature. On the other hand, we believe that for the square, the energy cost of flooding operations on a random Delaunay triangulation diverges logarithmically (this is also supported by simulations). Another interesting question is to explore probability measures other than uniform. We believe our results could be extended to any measure co-regular (that is, absolutely continuous in both directions) with the Lebesgue measure on a convex set whose boundary has strictly positive curvature. It would be nice to know what happens when the points are normally distributed, since we expect this distribution to model actual scenarios when sensors are scattered by airdrop from a single point. It is easy to see that there will always be highly isolated sensors, so the first question is actually what are the correct requirements from the communication model. Further research could be done to answer questions such as the maximal degree or the minimal and maximal angle in the bounded random Delaunay triangulation which have been solved for the Poisson case and seem still to be open for the uniform distribution in bounded regions.

Although the focus of this work has been the Delaunay triangulation, other connectivity graphs were suggested for wireless networks. For example the intersection of the unit graph with the Gabriel graph was suggested in [5] while the restricted Delaunay graph was suggested in [16] where in both cases the resulting routing graph is guaranteed not to have long edges. It would be interesting to repeat our analysis for these graphs particularly since for nodes randomly placed in the unit square the Delaunay triangulation (in particular the convex hull) contains an edge of constant length with high probability.

Finally, we remark that to simplify the analysis of the communication processes we assumed that the Delaunay triangulation is computed only at the beginning of the process. It makes sense that, as nodes fail, the triangulation should be recomputed, perhaps changing the traffic patterns. Analyzing the network half-life when the routing graph is dynamic is an interesting challenge. Also, our bounds for geometric routing are tight only up to polylogarithmic factors. Making these bounds tight up to constant factors is a challenge.

Acknowledgments

We would like to thank Yishai Mansour for many helpful discussions.

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