Excess in Arrangements of Segments

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Abstract

Let S be a set of n line segments in the plane. The *excess* of S is the number of repetitions of segments of S along the boundary of the same face of $\mathcal{A}(S)$, summed over all segments and faces. We show that the excess of S is at most $O(n \log\log n)$, improving a previous $O(n \log n)$ bound given in [1].

In this note we study the notion, introduced in [1], of the *excess* of an arrangement $\mathcal{A}(S)$ of a set S of n line segments in the plane in general position. Intuitively, the excess counts the number of repetitions of segments along the boundary of the same face of $\mathcal{A}(S)$, summed over all faces. It is formally defined as follows. A *side* of a segment e is any one of the two halfplanes bounded by the line containing e. A *1-border* is a pair (e, R), where e is a segment and R is a side of e. A 1-border (e, R) bounds a face f of $\mathcal{A}(S)$ if some portion e' of e appears as an edge of f, so that the intersection of R with a sufficiently small neighborhood of e' is contained in f.

Let e be a segment in S. If some face f in $\mathcal{A}(S)$ has a 1-border of the form (e', R), where $e' \subseteq e$ and R is any of the two sides of e' (that is, of e), we put $p(e, R, f) = 1$ and say that (e, R) is *present* on the boundary of f; otherwise put $p(e, R, f) = 0$. In either case, we define $\kappa(e, R, f)$ to be the number of 1-borders (e', R) of f, with $e' \subseteq e$. The *excess* $\varepsilon(e, R, f)$ of f relative to (e, R) is defined as $\kappa(e, R, f) - p(e, R, f)$. The excess of a face f is $\varepsilon(f) = \sum_{e,R} \varepsilon(e, R, f)$, where the sum extends over all $e \in S$ and their sides R, and the excess of the entire arrangement is $\varepsilon(S) = \sum_f \varepsilon(f)$, summed over all faces f of the arrangement.

Besdies being an interesting combinatorial quantity that deserves to be studied for its own sake, the excess has proved useful in several applications, such as bounding the complexity of many cells in arrangements of simplices in higher dimensions (see [1]).

First of all, notice that, by definition, $\varepsilon(f)$ is bounded by the complexity of f. Therefore, $\varepsilon(f) = O(n\alpha(n))$, for any face f, where $\alpha(n)$ is the slowly-growing inverse Ackermann's function [4, 5]. Moreover, $\varepsilon(f)$ is smaller than the number of 1-borders of f by at most $2n$, so, in the worst case, $\varepsilon(f) = \Theta(n\alpha(n))$ [4, 5, 6]. This settles the question of the worst-case

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excess of a face in an arrangement of segments. The real problem, though, is to bound the overall excess of the entire arrangement. Let $\varepsilon(n) = \max_S \varepsilon(S)$, where the maximum is taken over all collections S of n segments in general position in the plane. Our main result, given below, improves a previous bound $\varepsilon(n) = O(n \log n)$ of [1].

Theorem 1 *$\varepsilon(n) = O(n \log\log n)$.*

Proof. We partition S into $t = O(\log n)$ subcollections $S_1, \ldots, S_t$ in the following rather standard 'interval-tree' manner (see, e.g., [2]). We find a vertical line ℓ, that may meet some of the segments, so that the number of segments fully to its left is roughly equal to the number of segments fully to its right. Let S_1 be the set of segments 'stabbed' by ℓ. We now consider the subset S_L of segments lying fully to the left of ℓ, find a vertical line ℓ_L that splits S_L into two subsets, as above, find a similar line ℓ_R for the subset S_R of segments lying fully to the right of ℓ, and let S_2 be the set of segments in $S_L \cup S_R$ stabbed either by ℓ_L or by ℓ_R. We continue in this manner, for $O(\log n)$ stages, until all of S is exhausted.

We first claim that if S is a set of segments that are all stabbed by some vertical line ℓ, then the excess of S is $O(n\alpha(n))$. To see this, we may assume, with no loss of generality, that ℓ is the y-axis. Let S^+, S^- be the sets of the intersections of the segments of S with the two respective halfplanes $x \geq 0$, $x \leq 0$. It is then easy to see that $\varepsilon(S) \leq \varepsilon(S^+) + \varepsilon(S^-) + 2n$, because we may lose up to 2 units of excess for each segment $e \in S$ (one on each side of e) when e is split into a left portion and a right portion at $x = 0$. It thus suffices to establish our claim for S^+ and S^- separately, so we may assume that $S = S^+$, i.e., that all the segments in S lie in the right halfplane $x \geq 0$ and have their left endpoints on the y-axis.

Let e be a segment in S, and let e_1, e_2 be two edges of $\mathcal{A}(S)$ contained in e and bounding the same face f of $\mathcal{A}(S)$ on the same side of e. Suppose that e_1 lies to the left of e_2. Then f is clearly a nonconvex face of $\mathcal{A}(S)$ and, as is easily checked, either

(i) f is a bounded face and the left endpoint, p, of e_2 is a local x-minimum of f (see Figure 1(i)), or

(ii) f is the unbounded face of $\mathcal{A}(S)$ (containing ℓ); see Figure 1(ii).

The number of local x-minima of all the nonconvex bounded faces f of $\mathcal{A}(S)$ is $O(n)$. Indeed, we can decompose all these faces into convex pieces by vertical segments drawn up and down from the right endpoints of segments in S that lie inside such faces (these are the only nonconvex vertices of the bounded faces). The overall number of resulting convex pieces is $O(n)$, and each local x-minimum of a nonconvex bounded face must be the leftmost vertex of such a piece. Hence there are only $O(n)$ edges e_2 satisfying condition (i) above. Adding the excess of the unbounded face, which, as noted above, is $O(n\alpha(n))$, it follows that the overall excess of S is $O(n\alpha(n))$.

This implies that $\varepsilon(S_1) = O(n_1\alpha(n_1))$, where $n_i = |S_i|$ for $i = 1, \ldots, t$. Actually, the same argument implies that $\varepsilon(S_i) = O(n_i\alpha(n_i))$, since each S_i is the disjoint union of subsets of S, each stabbed by a vertical line, and no segment of any one subset of S_i intersects any segment of another subset.

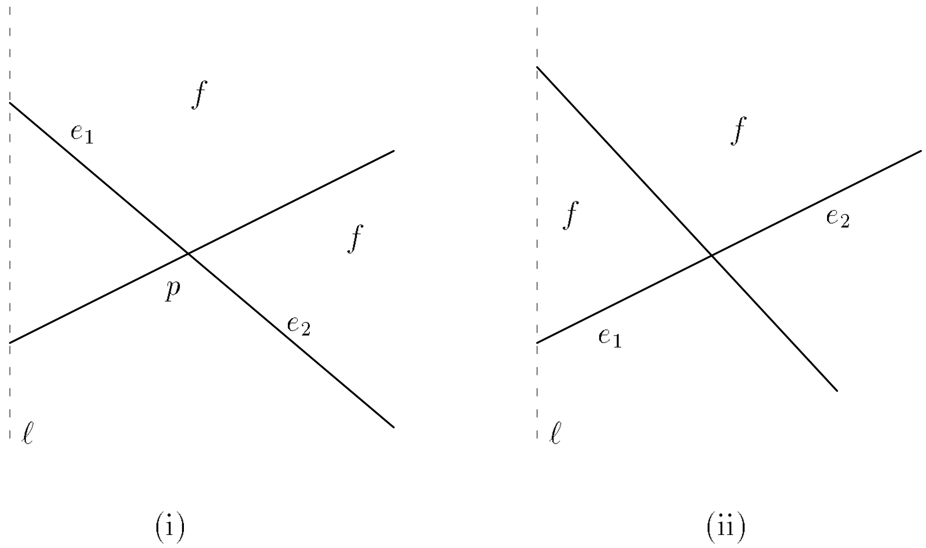

Figure 1: The two types of excess when the given segments are all stabbed by a common vertical line and have their left endpoints on that line

We now construct a minimum-height binary tree T whose leaves are associated with the sets S_i, and where each internal node u is associated with the union S_u of the sets S_i of the leaves of the subtree rooted at u. The height of T is $O(\log\log n)$. We estimate $\varepsilon(S_u)$ in a bottom-up manner. For this we need the following claim:

Claim: Let S' and S'' be two collections of a total of n line segments in the plane. Then $\varepsilon(S' \cup S'') \le \varepsilon(S') + \varepsilon(S'') + O(n)$.

To see this, we superimpose $\mathcal{A}(S')$ with $\mathcal{A}(S'')$. Let e be a segment of, say S', such that there is a face f of $\mathcal{A}(S' \cup S'')$ whose boundary contains two edges, e_1, e_2, contained in e and bounding f on the same side of e, and such that there is no third edge with this property (for the same f) contained in e between e_1 and e_2. If we remove all segments of S'' then e_1 and e_2 extend into two (equal or distinct) edges e_1^*, e_2^* of $\mathcal{A}(S')$, both bounding the same face f^* (containing f) on the same side of e. Two cases can arise:

(a) e_1^* and e_2^* are distinct edges of $\mathcal{A}(S')$.

(b) $e_1^* = e_2^*$.

In case (a), the contribution to $\varepsilon(S')$ within the face f^* by the portion of e between e_1^* and e_2^* is at least 1 (it can be larger than 1 since there might be additional edges on e between e_1^* and e_2^* bounding f^* on the same side of e), whereas the contribution by e_1 and e_2 to $\varepsilon(S' \cup S'')$ within f is exactly 1, so in this case this particular excess does not increase when the segments of S'' are added to the arrangement.

In case (b), we can charge the one unit of excess generated by e_1 and e_2 to the split of the common edge e_1^* into subedges, as caused by the addition of the segments of S''.

In this case one can easily show, arguing as in the proof of the Combination Lemma of [3], that the portion γ of ∂f between e_1 and e_2 must either consist exclusively of edges contained in segments of S'' (and then γ must contain an endpoint of some segment of S''; see Figure 2(i)), or include also edges of S' contained in other connected components of ∂f^* (which now become portions of the same connected component of ∂f; see Figure 2(ii)). Arguing as in [3], one can easily show that the number of such increases in excess is bounded by $O(n)$. This completes the proof of the claim.

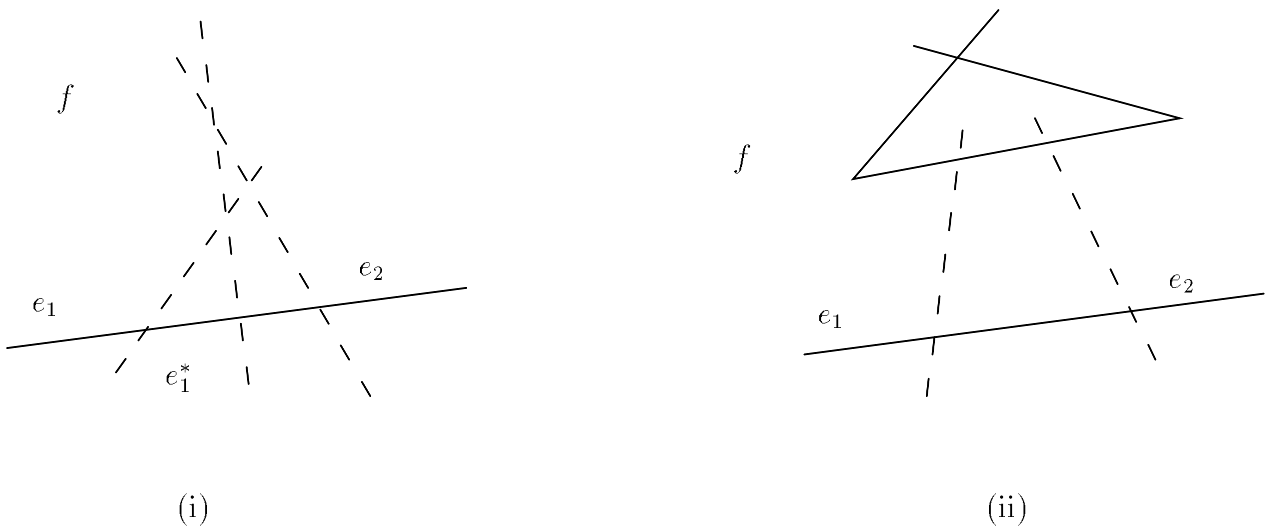

Figure 2: Illustration of case (b); dashed segments belong to S'' and solid edges to S'

It now follows that the sum of excesses of the sets S_u, over all nodes u at a fixed level of T, is larger than this sum over the nodes at the next deeper level by $O(n)$. Since the sum of excesses of the individual sets S_i is $O(n\alpha(n))$, and since T has $O(\log\log n)$ levels, it follows that

$$\varepsilon(S) = O(n\alpha(n) + n\log\log n) = O(n\log\log n)\,,$$

as asserted. $\Box$

Remark: The best known lower bound for $\varepsilon(n)$ is $\Omega(n\alpha(n))$ (as noted above, the excess of a single face can already be that large). Our result narrows the gap between the upper and lower bounds, but does not close it completely. We conjecture that the correct bound is $\Theta(n\alpha(n))$.

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