Combinatorial Geometry with Algorithmic Applications

The Alcala Lectures

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Preface and Apology

These lecture notes are a compilation of surveys of the topics that are presented in a series of talks at Alcala, August 31 – September 5, 2006, by János Pach and Micha Sharir. To a large extent, these surveys are adapted from earlier papers written by the speakers and their collaborators. In their present form, the notes aptly describe both the history and the state of the art of these topics.

The notes are arranged in an order that roughly parallels the order of the talks. Chapter 1 describes the beginnings of combinatorial geometry: starting with Sylvester's problem on the existence of "ordinary lines," we introduce a number of exciting problems on *incidences* between points and lines in the plane and in space. This chapter uses the material in Pach, Pinchasi, Sharir [9]. In Chapter 2 we survey many aspects of the theory of *arrangements of surfaces* in higher dimensions. It is adapted from Agarwal and Sharir [2]. Readers that have some familiarity with the basic theory of arrangements can start their reading on this topic from this chapter, while those that are complete novices may find it useful to look first at Chapter 3, which studies arrangements of *curves* in the plane, with special emphasis on *Davenport–Schinzel sequences* and the role they play in the theory of arrangements. This chapter is adapted from Agarwal and Sharir [1].

Chapter 4 covers the topic of incidences between points and curves and its many relatives, where a surge of activity has taken place in the past five years. It is adapted from two similar surveys by Pach and Sharir [10, 11]. The study of combinatorial and topological properties of planar arrangements of curves has become a separate discipline in discrete and computational geometry, under the name of *Graph Drawing.* Some basic aspects of this emerging discipline are discussed in Chapter 5, which is based on the survey by Pach [8]. Some classical questions of Erdős on repeated interpoint distances in a finite point set can be reformulated as problems on the maximum number of incidences between points and circles, spheres, etc. In fact, these questions motivated and strongly influenced the early development of the theory of incidences a quarter of a century ago and they led to the discovery of powerful new combinatorial and topological tools. Many of Erdős's questions can be naturally generalized to problems on larger repeated subpatterns in finite point sets. Based on Brass and Pach [4], in Chapter 6 we outline some of the most challenging open problems of this kind, whose solution would have interesting consequences in *pattern matching* and recognition.

Chapter 7 treats the special topic of lines in three-dimensional space, which is a nice application (or showpiece, if you will) of the general theory of arrangements on one hand, and shows up in a variety of only loosely related topics, ranging from *ray shooting* and *hidden surface removal* in computer graphics to geometric *transversal theory.* This chapter partially builds upon a somewhat old paper by Chazelle, Edelsbrunner, Guibas, Sharir and Stolfi [6], but its second half is new, and presents (some of) the recent developments. Some combinatorial properties of arrangements of spheres, boxes, etc., are discussed in Chapter 8. They raise difficult questions on the *chromatic numbers* and other similar parameters of certain geometrically defined graphs and hypergraphs, with possible applications to *frequency allocation* in cellular telephone networks. Here we borrowed some material from Pach, Tardos, Tóth [12].

An old and rich area of applications of Davenport–Schinzel sequences and the theory of geometric arrangements is *Motion Planning.* Starting with Sam Loyd's coin puzzles, in Chapter 9 we discuss a number of problems that can be regarded as discrete variants of the "piano movers' problem" on graphs and grids. Some of the results have applications to the reconfiguration of metamorphic *robotic systems.* This chapter is based on recent joint papers with Bereg, Călinescu, and Dumitrescu [3], [5], [7].

While we have made our best attempts to make these notes comprehensive, they are not at the level of a polished monograph, nor do they provide full coverage of all the relevant recent results and developments. It is our hope that they be a useful source of reference to the rich and extensive theory presented at the Alcala series of talks.

Apart from those friends and coauthors whose names were mentioned above, we freely borrowed from joint work with D. Pálvölgyi and R. Wenger. We are extremely grateful to all of them for the enjoyable and fruitful collaboration and for their kind permission to reproduce their ideas and "plagiarize" their words.

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Contents

Chapter 1

Sylvester-Gallai Problem: The Beginnings of Combinatorial Geometry

1.1 James Joseph Sylvester and the Beginnings

James Joseph Sylvester (1814–1897) was one of the most colorful figures of nineteenth century British mathematics. He started his studies at the University of London at the age of 14, where he was a student of the logician, Augustus de Morgan. In spite of his brilliant achievements in Cambridge, he was not granted a degree there until 1872, because as a Jew, he declined to take the Thirty-Nine Articles of the Church of England. In 1838, however, he became Professor of Natural Philosophy at UCL, but he could not obtain a teaching position until 1855. In 1841, he was awarded a BA and an MA by Trinity College Dublin. While working in London as an actuary, together with his lifelong friend, Arthur Cayley, he made important contributions to matrix theory and invariant theory. In 1877, Sylvester became the inaugural professor of mathematics at the new Johns Hopkins University, and one year later he founded the American Journal of Mathematics. He did pioneering work in combinatorics, in number theory, and in the theory of partitions.

"The early study of Euclid made me a hater of geometry," said Sylvester [35]. It is somewhat ironic that most likely the hatred would not be mutual! The following innocent looking question of Sylvester was first proposed as a problem in the *Educational Times* [45]. Euclid would have probably loved this question, because to formulate it we need only three notions: points, lines, and incidences, the three basic elements of his geometry! *Is it true that any finite set of points in the Euclidean plane, not all on a line, has two elements whose connecting line does not pass through a third?* Such a connecting line is called an *ordinary* line.

In the same year, the journal published an incorrect solution by Woodall and an other argument, which was characterized as "equally incomplete, but may be worth notice." In the early 1930s, the question was rediscovered by Erdős, and shortly thereafter, an affirmative answer was given by T. Grünwald (alias Gallai). In 1943, Erdős [20] posed the problem in the *American Mathematical Monthly*. In the following year, it was solved by Steinberg [43] and others. However, the oldest published proof is due to Melchior [36], who established the dual statement, as a corollary to a more general inequality: any finite family of lines in the plane, not all of which pass through the

same point, determines a *simple* intersection point, i.e., a point that belongs to precisely two lines. Both the primal and dual forms of the result have become known as the *Sylvester–Gallai theorem*. Many alternative proofs and generalizations have been found by de Bruijn and Erdős [9], Coxeter [12], Motzkin [37], Lang [33], Williams [48], Lin [34], Edelstein et al. [19], [18], [4], [5], Giering [23], Herzog and Kelly [29], Kupitz [32], Watson [47]. A survey of these results was given by P. Borwein and W. Moser [7]. The history of this problem is somewhat complicated; much of the confusion can be explained by interruptions to academic life caused by World War II. In fact, it is not even clear whether originally Sylvester had any solution in mind.

Theorem 1.1.1 (Sylvester–Gallai theorem). *Any finite set of points in the plane has two elements whose connecting line does not pass through a third.*

The following "book proof" – using Erdős' terminology – was found by Kelly. Take a set P of n points in the plane, not all collinear. Consider all pairs (ℓ, p), where ℓ is a line passing through at least two elements of P, and $p \in P$ does not lie on ℓ. Among all these pairs, pick one, for which the distance between p and ℓ is minimal. Denoting such a pair by (ℓ, p) – with a slight abuse of notation –, we claim that ℓ cannot pass through more than two points. Assume for contradiction that it does. Then at least two points $q, r \in P \cap \ell$ lie on the same side of the foot p' of the perpendicular from p to ℓ. Assume without loss of generality that r is closer to p' than q is. Clearly, the distance between $r \in P$ and the line $\ell' := qp$ is smaller than the distance between p and ℓ, contradicting the minimality of (ℓ, p). □

The next natural task is to find the minimum number $ol(n)$ of ordinary lines (passing through precisely two points) determined by n noncollinear points in the plane.

Conjecture 1.1.2 (Dirac [17], Motzkin). *For every $n \neq 7, 13$, the number of ordinary lines determined by n noncollinear points in the plane is at least $\lceil \frac{1}{2}n \rceil$.*

Kelly and W. Moser [KeM58] proved that $ol(n) \geq \frac{3}{7}n$. The dual statement also holds for pseudo-line arrangements [KeR72]. The best known lower bound, $ol(n) \geq \frac{6}{13}n$, was found by Csima and Sawyer [15], [14]. The following exact values were determined by Crowe and McKee [13], and by Brakke [8]:

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19–21	22
$ol(n)$	3	3	4	3	3	4	6	5	6	6	6	7	?	8	?	9	?–?	11

Observe that the function $ol(n)$ is not monotonically increasing. There are two configurations with 7 and 13 points with exceptionally few ordinary lines. Apart from these values, the general upper bound is $ol(n) \leq \frac{n}{2}$ if n is even and $ol(n) \leq 3\lfloor \frac{n}{4} \rfloor$ if n is odd.

For even n, the construction consists of a regular $\frac{n}{2}$-gon, which determines $\frac{n}{2}$ directions, and the $\frac{n}{2}$ projective points corresponding to these directions. If $n \equiv 1 \pmod 4$, the best known example can be obtained by adding the center of the polygon to the construction for $n-1$. If $n \equiv 3 \pmod 4$, one of the projective points has to be deleted from the construction for $n+1$. The substantial difference between the cases n even and n odd is quite unusual. It is especially strange in view of the fact that there are several other configurations of both parities that determine very few different directions and may serve as bases for constructions of point sets with few ordinary lines.

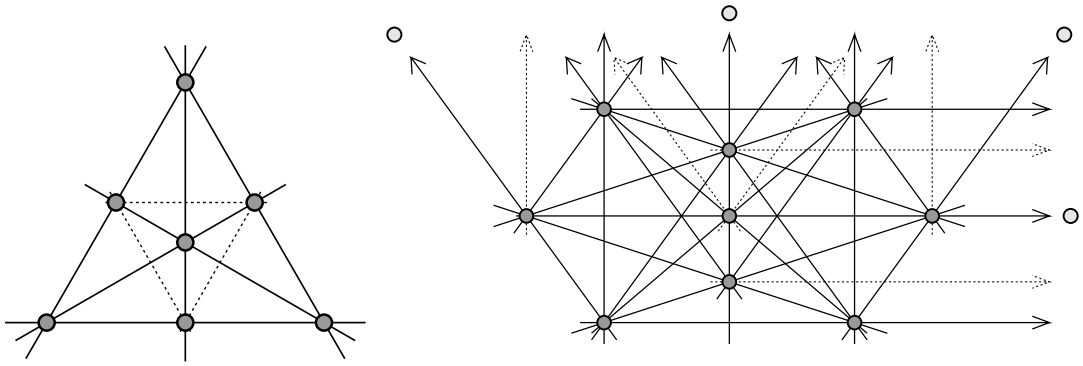

Figure 1.1: Exceptional sets with 7 and 13 points and few ordinary lines.

Problem 1.1.3. *For odd values of n, find sets of n points in the plane that determine only $\frac{1}{2}n + O(1)$ ordinary lines.*

A ghost reference related to the Dirac–Motzkin conjecture is the dissertation of S. Hansen from 1981. It claims to prove Conjecture 1.1.2, but one of its key lemmas turned out to be false [15].

1.2 Connecting Lines and Directions

Erdős pointed out the following immediate consequence of the Gallai-Sylvester theorem on ordinary lines, i.e., lines passing through precisely two elements of the set.

Theorem 1.2.1 (Erdős theorem). *Any set of n noncollinear points in the plane determines at least n different connecting lines. Equality is attained if and only if all but one of the points are collinear.*

Proof. To establish the first statement, it is enough to notice that if we delete one of the points that lie on an ordinary line, then the number of connecting lines induced by the remaining point set will decrease by at least one. Hence, the result follows by induction, unless the remaining point set is collinear. However, in the latter case, our set determines precisely n connecting lines.

The easy proof of the second statement is left to the reader. □

Erdős' theorem was the starting point of many important investigations that led to the birth of the theory of *finite projective planes* and, more generally, to the theory of *block designs*.

Here we recall a far reaching combinatorial generalization of the theorem, together with its "book proof," due to Motzkin and Conway.

Theorem 1.2.2. *Let P be a set of $n \geq 3$ elements and let $\ell_1, \ldots, \ell_m$ be proper subsets of P such that any pair of elements of P is contained in precisely one set ℓ_i. Then we have $m \geq n$.*

Proof. Suppose $m < n$. For any $p \in P$, let $d(p)$ denote the number of ℓ_i's containing p.

If $p \notin \ell_i$, then $d(p) \geq |\ell_i|$, which, in turn, implies that $m|\ell_i| < nd(p)$. From here, we get

$$n(n - |\ell_i|) > n(m - d(p)).$$

Thus, we have

$$1 = \sum_{p \in P} \sum_{p \notin \ell_i} \frac{1}{n(m - d(p))} > \sum_{\ell_i} \sum_{p \notin \ell_i} \frac{1}{m(n - |\ell_i|)} = 1.$$

This contradiction completes the proof of the theorem. □

In the same spirit, Scott [42] posed two similar questions in 1970: Is it true that the minimum number of different directions assumed by the connecting lines of

(1) $n \geq 4$ noncollinear points in the plane is $2\lfloor n/2 \rfloor$?

(2) $n \geq 6$ noncoplanar points in 3-space is $2n - 3$ if n is even and $2n - 2$ if n is odd?

Twelve years later, the first question was answered in the affirmative by Ungar.

Theorem 1.2.3 (Ungar's theorem [46]). *Any set of $n \geq 4$ points in the plane, not all on a line, determine at least $2\lfloor n/2 \rfloor$ different directions.*

Ungar's proof is a real gem, a brilliant application of the method of *allowable sequences* invented by Goodman and Pollack [24], [25]. Moreover, it solves the problem in an elegant combinatorial setting, for "pseudolines", as was suggested independently by Goodman and Pollack and by Cordovil [11]. For even n, Ungar's theorem generalizes Erdős' above mentioned result. However, in contrast to Erdős' result, here there is an overwhelming diversity of extremal configurations, for which equality is attained. Four infinite families and more than one hundred sporadic configurations were catalogued by Jamison and Hill [31] (see also [30] for an excellent survey).

Progress on the second question of Scott has been much slower. As Jamison [30] noticed, unless we impose some further restriction on the point set, for odd n, the number of directions determined by n points in 3-space can be as small as $2n - 5$. Indeed, equality is attained, e.g., for the n-element set obtained from the vertex set of a regular $(n-3)$-gon P_{n-3} (or from any other centrally symmetric extremal configuration for the planar problem) by adding its center c and two other points whose midpoint is c and whose connecting line is orthogonal to the plane of P_{n-3}. Blokhuis and Seress [3] introduced a natural condition excluding the above configurations: they assumed that no three points are collinear. Under this assumption, they proved that every noncoplanar set of n points in 3-space determines at least $1.75n - 2$ different directions.

Following [40] and [41], one can answer Scott's second question in the affirmative, using the same assumption as Blokhuis and Seress, and in almost full generality.

Theorem 1.2.4. *Every set of $n \geq 6$ points in $\mathbb{R}^3$, not all of which are on a plane and no three are on a line, determine at least $n + 2\lceil n/2 \rceil - 3$ different directions. This bound is sharp.*

Theorem 1.2.5. *Every set of $n \geq 6$ points in $\mathbb{R}^3$, not all of which are on a plane, determine at least $2n - 5$ different directions if n is odd, and at least $2n - 7$ different directions if n is even. This bound is sharp for every odd $n \geq 7$.*

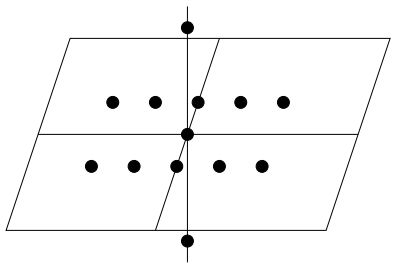

Figure 1.2: n noncoplanar points in space with minimum number of directions.

In the next two sections, we outline a proof of Theorem 1.2.4.

According to a beautiful result of Motzkin [38], Rabin, and Chakerian [10] (see also [1]), any set of n noncollinear points in the plane, colored with two colors *red* and *green*, determines a monochromatic line. Motzkin and Grünbaum [27] initiated the investigation of *biased* colorings, i.e., colorings without monochromatic red lines. Their motivation was to justify the intuitive feeling that if there are many red points in such a coloring and not all of them are collinear, then the number of green points must also be rather large. Denoting the sets of red and green points by R and G, respectively, it is a challenging unsolved question to decide whether the "surplus" $|R| - |G|$ of the coloring can be arbitrarily large. We do not know any example where this quantity exceeds 6 [28].

The problem of biased colorings was rediscovered by Erdős and Purdy [21], who formulated it as follows: What is the

Problem 1.2.6 (Grünbaum, Erdős–Purdy). *Let $m(n)$ denote the smallest number of points necessary to represent all lines spanned by n noncollinear points in the plane, if the generating points cannot be used. Is it true that $n - m(n) \to \infty$?*

An $\Omega(n)$ lower bound on $m(n)$ follows from the "weak Dirac conjecture" proved by Szemerédi and Trotter [44] and Beck [2], according to which there is a point that lies on $\Omega(n)$ different connecting lines. Each of these connecting lines has to be represented by a different point.

Conjecture 1.2.7 (Dirac's conjecture [17]). *There exists a constant c such that any set of n points in the plane, not all on a line, has an element incident to at least $\frac{n}{2} - c$ connecting lines.*

It was believed that, apart from some small examples listed in [26], this statement is true with $c = 0$, until Felsner exhibited an infinite series of configurations, showing that $c \geq 3/2$.

In Section 1.3, we reduce Theorem 1.2.4 to a statement (Theorem 1.3.2) showing that under some further restrictions the surplus is indeed bounded. More precisely, if there is no connecting line whose leftmost and rightmost points are both red, then we have $|G| \geq 2\lfloor |R|/2 \rfloor$, so in particular $|R| - |G| \leq 1$.

Another way of rephrasing Ungar's theorem is that from all closed segments whose endpoints belong to a noncollinear set of n points in the plane, one can always select at least $2\lfloor n/2 \rfloor$ such that no two of them are parallel. Unless we explicitly state it otherwise, every *segment* used in this chapter is assumed to be *closed*. Our proof of Theorem 1.3.2 is based on a far-reaching generalization

of Ungar's result. To formulate this statement, we need to relax the condition of two segments being *parallel*.

Definition 1.2.8. *Two segments belonging to distinct lines are called* avoiding *if one of the following two conditions is satisfied:*

(i) they are parallel, or

(ii) the intersection of their supporting lines does not belong to any of the segments.

An alternative definition is that two segments are avoiding if and only if they are disjoint and their convex hull is a quadrilateral.

The following strengthening of Ungar's theorem, which is of independent interest, implies Theorem 1.2.4 (and Theorem 1.3.2 stated in the next section).

Theorem 1.2.9 (Generalized Ungar theorem). *From all closed segments determined by a set of n noncollinear points in the plane, one can always select at least $2\lfloor n/2 \rfloor$ pairwise nonavoiding ones, lying on distinct lines.*

Theorem 1.2.9 is established in Section 1.4.

1.3 Directions in Space vs. Points in the Plane

In this section, we reduce Theorem 1.2.4 to Theorem 1.2.9.

Let P be a set of n points in $\mathbb{R}^3$ such that not all of them lie in a common plane and no three of them are collinear. Let p_0 be an *extreme* point of P, i.e., a vertex of the convex hull of P. Consider a supporting plane to P at p_0, and translate it to the side that contains P. Let π denote the resulting plane. Project from p_0 all points of $P \setminus \{p_0\}$ onto π. We obtain a set R of $n-1$ distinct points in π, not all on a line, and we will refer to the elements of R as *red* points. Each red point corresponds to a direction determined by p_0 and some other point of P.

For each pair of elements $p, p' \in P \setminus \{p_0\}$, take a line parallel to pp' that passes through p_0. Color with *green* the intersection point of this line with π, unless it has already been colored red. The set of all green points is denoted by G. By definition, we have $R \cap G = \emptyset$.

We need the following simple property of the sets R and G, which implies that along every line passing through at least two red points either the leftmost or the rightmost point belonging to $R \cup G$ is green.

Lemma 1.3.1. *Every line connecting two red points $r, r' \in R$ passes through at least one green point $g \in G$ that does not belong to the (closed) segment rr'.*

Proof. Let ℓ be a line in π passing through at least two red points $r, r' \in R$. Assume without loss of generality that r and r' are the leftmost and rightmost red points along ℓ. Let p and p' denote those elements of P whose projections to π are r and r', respectively. Observe that in the plane induced by p_0 and ℓ, the direction of pp' does not belong to the convex cone enclosed by the rays p_0p and p_0p', so the line through p_0 parallel to pp' will cross ℓ in a green point g meeting the requirements. □

To establish Theorem 1.2.4, it is sufficient to verify the following result.

Theorem 1.3.2. *Let R be a set of n red points in the plane, not all collinear, and let G be a set of m green points such that $R \cap G = \emptyset$ and every line ℓ connecting at least two red points in R passes through a green point $g \in G$ that does not belong to any segment rr', for $r, r' \in R \cap \ell$.*

Then we have $m \geq 2\lfloor n/2 \rfloor$.

Indeed, to prove Theorem 1.2.4 it is enough to notice that in our setting we have $|R| = n - 1$ and that the number of different directions determined by P is equal to

$$|R| + |G| \geq n - 1 + 2\left\lfloor \frac{n-1}{2} \right\rfloor = n + 2\left\lceil \frac{n}{2} \right\rceil - 3.$$

Thus, applying Theorem 1.3.2, Theorem 1.2.4 immediately follows.

It is interesting to note that Theorem 1.3.2 also implies Ungar's above-mentioned theorem. To see this, regard the elements of our given planar point set as *red*, and the directions determined by them as green points on the line at infinity, and apply Theorem 1.3.2. (If we wish, we can perform a projective transformation and avoid the use of points at infinity.)

It remains to prove Theorem 1.3.2. However, as mentioned in the introduction, this result can be easily deduced from Theorem 1.2.9, which is a further extension of Ungar's theorem.

Proof. (Proof Theorem 1.3.2 using Theorem 1.2.9.) Applying Theorem 1.2.9 to the set R, we obtain $2\lfloor n/2 \rfloor$ segments with red endpoints that lie in distinct lines and no pair of them are avoiding. By the condition in Theorem 1.3.2, the continuation of each of these segments passes through a green point. Assign such a green point to each segment. Observe that these points are all distinct. Indeed, if we can assign the same green point to two different segments, then they must be avoiding, by definition. This completes the proof of Theorem 1.3.2 and hence of Theorem 1.2.4. □

1.4 Proof of the Generalized Ungar Theorem

The aim of this section is to prove Theorem 1.2.9, the Generalized Ungar theorem.

Fix an (x, y)-coordinate system in the plane. We apply a standard duality transform that maps a point $p = (p_1, p_2)$ to the line p^* with equation $y + p_1 x + p_2 = 0$. Vice versa, a nonvertical line l with equation $y + l_1 x + l_2 = 0$ is mapped to the point $l^* = (l_1, l_2)$. Consequently, any two parallel lines are mapped into points having the same x-coordinate. It is often convenient to imagine that the dual picture lies in another, so-called *dual*, plane, different from the original one, which is referred to as the *primal* plane.

The above mapping is incidence and order preserving, in the sense that p lies above, on, or below ℓ if and only ℓ^* lies above, on, or below p^*, respectively. The points of a segment $e = ab$ in the primal plane are mapped to the set of all lines in the closed *double wedge* e^*, which is bounded by a^* and b^* and does not contain the vertical direction. All of these lines pass through the point $q = a^* \cap b^*$, which is called the *apex* of the double wedge e^*. All double wedges used here are assumed to be closed, and they never contain the vertical direction.

Definition 1.4.1. *We call two double wedges* avoiding *if their apices are distinct and the apex of neither of them is contained in the other.*

It is easy to see that, according to this definition, two noncollinear segments in the primal plane are avoiding if and only if they are mapped to avoiding double wedges.

Switching to the dual plane, Theorem 1.2.9 can now be reformulated as follows.

Theorem 1.4.2. *Let L be a set of n pairwise nonparallel lines in the plane, not all of which pass through the same point. Then the set of all double wedges bounded by pairs of lines in L has at least $2\lfloor n/2 \rfloor$ pairwise nonavoiding elements with different apices.*

Note that the definition of double wedges depends on the choice of the coordinate system, so *a priori* Theorem 1.4.2 gives a separate statement in each coordinate frame. However, each of these statements is equivalent to Theorem 1.2.9, and that result does not depend on coordinates. Therefore, we are free to use whatever coordinate system we like. In the final part of the analysis (given in Section 4), we will exploit this property. But until then, no restriction on the coordinate system is imposed.

Suppose that a set of $2\lfloor n/2 \rfloor$ double wedges meets the conditions in Theorem 1.4.2. Clearly, we can replace each element of this set, bounded by a pair of lines $\ell_1, \ell_2 \in L$, by the *maximal* double wedge with the same apex, i.e., the double wedge bounded by those lines through $\ell_1 \cap \ell_2$ which have the *smallest* and *largest* slopes. If every pair of double wedges in the original set were nonavoiding, then this property remains valid after the replacement.

It is sufficient to prove Theorem 1.4.2 for the case when n is even, because for odd n the statement trivially follows.

The proof is constructive. Let $\mathcal{A}(L)$ denote the *arrangement* of L, consisting of all vertices, edges, and faces of the planar map induced by L. We will construct a set of n vertices of $\mathcal{A}(L)$ with distinct x-coordinates, and show that the maximal double wedges whose apices belong to this set are pairwise non-avoiding.

We start by defining a sequence J of vertices $v_1, v_2, \ldots$, which will be referred to as *junctions.* Let L^- (resp., L^+) denote the subset of L consisting of the $n/2$ lines with the smallest (resp., largest) slopes. If we wish to simplify the picture, we can apply an affine transformation that keeps the vertical direction fixed and carries the elements of L^- and L^+ to lines of negative and positive slopes, respectively (whence the choice of notation). However, we will never use this property explicitly (although the figures will reflect this convention).

The construction proceeds as follows.

Step 1: Set $i := 1$ and $L_1^- := L^-$, $L_1^+ := L^+$.

Step 2: If $L_i^- = L_i^+ = \emptyset$, the construction of J terminates. Otherwise, as we will see, neither set is empty. Let v_i be the *leftmost* intersection point between a line in L_i^- and a line in L_i^+. Let d_i^- (and d_i^+) denote the number of elements of L_i^- (and L_i^+, respectively) incident to v_i, and put $d_i = \min\{d_i^-, d_i^+\}$. Define L_{i+1}^- (and L_{i+1}^+) as the set of lines obtained from L_i^- (resp., L_i^+) by deleting from it the d_i elements that are incident to v_i and have the smallest (resp., largest) slopes among those incident lines. (That is, if $d_i^- = d_i^+$, then all lines incident to v_i are deleted; otherwise, if, say, $d_i^- > d_i^+$, we are left with $d_i^- - d_i^+$ lines through v_i that belong to L_i^+ and separate the deleted elements of L_i^- from the deleted elements of L_i^+.) Set $i := i + 1$, and repeat Step 2.

Let $J = \langle v_1, v_2, \ldots, v_k \rangle$ denote the resulting sequence.

It is easy to verify the following properties of this construction.

Claim 1.4.3. *(i)* $|L_i^-| = |L_i^+|$, *for each* $i = 1, \ldots, k$.

(ii) For every $1 \le i < j \le k$, *the junction* v_i *lies in the left unbounded face* f_j *of* $\mathcal{A}(L_j^- \cup L_j^+)$ *which separates* L_j^- *and* L_j^+ *at* $x = -\infty$ *(whose rightmost vertex is* v_j*).* v_i *lies in the interior of* f_j *if* $d_i^- = d_i^+$*; otherwise it may lie on the boundary of* f_j.

(iii) $\sum_{i=1}^{k} d_i = n/2$. □

Next, between any two consecutive junctions v_i and v_{i+1}, for $1 \le i < k$, we specify $d_i + d_{i+1} - 1$ further vertices of $\mathcal{A}(L)$, called *stations*.

Fix an index $1 \le i < k$, and consider the vertical slab between v_i and v_{i+1}. By Claim 1.4.3 (ii), v_i lies inside or on the boundary of the face f_{i+1} of $\mathcal{A}(L_{i+1}^- \cup L_{i+1}^+)$, whose rightmost vertex is v_{i+1}. Hence, the segment $e = v_i v_{i+1}$ is contained in the closure of f_{i+1}. Now at least one of the following two conditions is satisfied: (a) all the d_i lines removed from L_i^+ and all the d_{i+1} lines removed from L_{i+1}^- pass above e, or (b) all the d_i lines removed from L_i^- and all the d_{i+1} lines removed from L_{i+1}^+ pass below e. (We caution the reader that this statement is not totally obvious when e belongs to the boundary of f_{i+1}.)

Assume, by symmetry, that (a) holds. Denote the lines removed from L_i^+ by $\ell_1^+, \ldots, \ell_{d_i}^+$, listed according to increasing slopes, and those removed from L_{i+1}^- by $\ell_1^-, \ldots, \ell_{d_{i+1}}^-$, listed according to decreasing slopes. Define the set of *stations* S_i in the vertical slab between v_i and v_{i+1} as the collection of all intersection points of $\ell_{d_i}^+$ with the lines $\ell_1^-, \ldots, \ell_{d_{i+1}}^-$, and all intersection points of $\ell_{d_{i+1}}^-$ with the lines $\ell_1^+, \ldots, \ell_{d_i}^+$. Clearly, we have $|S_i| = d_i + d_{i+1} - 1$ such points.

Finally, we have to consider the portions of the plane to the left of v_1 and to the right of v_k and collect there a set S_k of $d_k + d_1 - 1$ additional *stations*. Actually, exploiting the fact that we can (almost) freely select the coordinate system used for the duality transform, we will be able to select $d_k + d_1 - 1$ suitable stations, so that all of them, or all but one, lie to the left of v_1. The proper choice of the coordinate system as well as the details of the construction of S_k are described in the next section.

Let $Q = J \cup \left(\cup_{i=1}^{k} S_i\right)$. In view of Claim 1.4.3 (iii), the total number $|Q|$ of junctions and stations equals

$$|Q| = |J| + \sum_{i=1}^{k} |S_i| = k + \sum_{i=1}^{k-1} (d_i + d_{i+1} - 1) + (d_k + d_1 - 1) = 2 \sum_{i=1}^{k} d_i = n.$$

All elements of Q are distinct. To complete the proof of Theorem 1.4.2 (and hence of Theorem 1.2.9), we need to verify

Claim 1.4.4. *Associate with each element* $q \in Q$ *the* maximal *double wedge* $W(q)$ *(not containing the vertical line through* q*), which is bounded by a pair of lines passing through* q. *Then the resulting set of* n *double wedges has no two avoiding elements.*

□

1.5 Colored Versions of the Sylvester-Gallai Theorem

Da Silva and Fukuda [16] suggested the following attractive bipartite version (and strengthening) of the Sylvester-Gallai theorem.

Conjecture 1.5.1. *Let R be a set of red points and let B be a set of blue points in the plane, not all on a straight line. Assume that*

(i) R and B are separated by a straight line, and

(ii) $|R|$ and $|B|$ differ by at most one.

Then there exists a bichromatic *ordinary line, i.e., a line passing through precisely one red and one blue point.*

It is not hard to see that this statement does not remain true if we drop any of the above assumptions. For instance, it is easy to see that the conjecture is false if we drop condition (i).

Suppose that n is even, say, $n = 2k$. Let P be the vertex set of a regular n-gon, and let Q be the set of intersection points of the line at infinity with all lines determined by two elements of P. Clearly, we have $|P| = |Q| = n$. Color $P \cup Q$ with two colors, red and blue, so that the number of red points equals the number of blue points and every *ordinary line* is *monochromatic*.

Notice that every ordinary line determined by $P \cup Q$ passes through one element of Q. Furthermore, for every pair of opposite vertices $p_1, p_2 \in P$, there is a unique point $q \in Q$, such that p_1q and p_2q are ordinary lines. Pick $\lfloor \frac{k}{2} \rfloor$ pairs of opposite vertices of P and the corresponding points on the line at infinity, and color them red. Color the remaining $\lceil \frac{k}{2} \rceil$ pairs of opposite vertices of P and the corresponding points on the line at infinity blue. Finally, color the k uncolored points of Q so that to balance the number of red and blue points. Obviously, all ordinary lines determined by $P \cup Q$ are monochromatic. If we wish to avoid using points at infinity, we can modify this construction by applying a suitable projective transformation. Similar constructions can be given in the case when $n > 3$ is odd.

By generating all possible combinatorial types of small point configurations, Finschi and Fukuda [22] managed to find a nine-element counterexample to Conjecture 1.5.1. However, it is still possible that the conjecture is true for every sufficiently large n, or for all even values of n.

Pach and Pinchasi [39] showed that without making any special assumption it is still true that there always exist bichromatic lines containing relatively few points of $R \cup B$.

Theorem 1.5.2 ([39]). *Given n red and n blue points in the plane, not all on a line,*

(i) there exist more than $n/2$ bichromatic lines that pass through at most two red points and at most two blue points;

(ii) the number of bichromatic lines passing through at most six points is at least one tenth of the total number of connecting lines.

The proof is based on the following simple consequence of Euler's Polyhedral Formula (see e.g. [1]), which immediately implies the Sylvester-Gallai theorem.

Lemma 1.5.3. *Let P be a finite non-collinear point set in the plane, and let l_i $(i = 2, 3, \ldots)$ denote the number of lines passing through precisely i elements of P. Then we have*

$$\sum_{i=2}^{n-1}(i-3)l_i \leq -3.$$

Now we prove Theorem 1.5.2

Proof. Let R and B be two disjoint n-element point sets in the plane, and assume that not all elements of $R \cup B$ are on the same line. We will refer to the elements of R and B, as *red* points and *blue* points, respectively.

For any ordered pair of non-negative integers (i, j), $i+j \geq 2$, let l_{ij} denote the number of lines passing through precisely i red and j blue points. In particular, the number of bichromatic lines is $\sum_{i,j\geq 1} l_{ij}$. Set $l_{ij} := 0$, whenever $i + j \leq 1$.

The number of monochromatic point pairs is equal to

$$\sum_{i,j\geq 0}\left[\binom{i}{2}+\binom{j}{2}\right]l_{ij} = 2\binom{n}{2} = n^2 - n.$$

The number of bichromatic point pairs is $\sum_{i,j\geq 0} ijl_{ij} = n^2$. Thus, we have

$$\sum_{\substack{i,\,j\,\geq\,0\\ i\,+\,j\,\geq\,2}}\left[\binom{i}{2}+\binom{j}{2}-ij\right]l_{ij} = -n. \tag{1.5.1}$$

The Lemma at the end of the previous section implies that

$$\sum_{\substack{i,\,j\,\geq\,0\\ i\,+\,j\,\geq\,2}}(i+j-3)l_{ij} \leq -3. \tag{1.5.2}$$

Adding up twice (1.5.1) and $1 + \varepsilon$ times (1.5.2), for some positive ε, we obtain

$$\sum_{\substack{i,\,j\,\geq\,0\\ i\,+\,j\,\geq\,2}}\left[(i-j)^2+\varepsilon(i+j-3)-3\right]l_{ij} \leq -2n-3(1+\varepsilon). \tag{1.5.3}$$

For any $i, j \geq 0$ $(i + j \geq 2)$, let γ_{ij} denote the coefficient of l_{ij} in the above inequality, so that $\sum_{i,j}\gamma_{ij}l_{ij} < 0$.

First, set $\varepsilon = 1$. It is easy to verify that $\gamma_{11} = -4$, $\gamma_{12} = \gamma_{21} = \gamma_{22} = -2$, and that all other coefficients γ_{ij} are non-negative. Therefore, (1.5.3) yields that

$$-4l_{11} - 2l_{12} - 2l_{21} - 2l_{22} \leq -2n - 6.$$

Consequently,

$$2l_{11} + l_{12} + l_{21} + l_{22} \geq n + 3,$$

which proves part (i).

To establish part (ii), set $\varepsilon = 3/5$. Then

$$\gamma_{11} = -\frac{18}{5},\ \gamma_{12} = \gamma_{21} = -2,\ \gamma_{22} = -\frac{12}{5},\ \gamma_{23} = \gamma_{32} = -\frac{4}{5},\ \gamma_{33} = -\frac{6}{5},$$

and all other coefficients are at least $2/5$. Hence,

$$\sum_{\substack{i,\,j\,\geq\,0\\ i+j\,\geq\,2}} \gamma_{ij} l_{ij} + 4l_{11} + \frac{12}{5}l_{12} + \frac{12}{5}l_{21} + \frac{14}{5}l_{22} + \frac{6}{5}l_{23} + \frac{6}{5}l_{32} + \frac{8}{5}l_{33} \geq \frac{2}{5} \cdot \sum_{\substack{i,\,j\,\geq\,0\\ i+j\,\geq\,2}} l_{ij}.$$

Comparing the last inequality with (1.5.3), we obtain

$$-2n - 3\left(1 + \frac{3}{5}\right) + 4l_{11} + \frac{12}{5}l_{12} + \frac{12}{5}l_{21} + \frac{14}{5}l_{22} + \frac{6}{5}l_{23} + \frac{6}{5}l_{32} + \frac{8}{5}l_{33} \geq \frac{2}{5} \cdot \sum_{\substack{i,\,j\,\geq\,0\\ i+j\,\geq\,2}} l_{ij}.$$

That is,

$$\sum_{\substack{i,\,j\,\geq\,0\\ 2\,\leq\,i+j\,\leq\,6}} l_{ij} \geq l_{11} + \frac{3}{5}l_{12} + \frac{3}{5}l_{21} + \frac{7}{10}l_{22} + \frac{3}{10}l_{23} + \frac{3}{10}l_{32} + \frac{2}{5}l_{33}$$

$$\geq \frac{1}{10} \cdot \sum_{i,j\geq 0} l_{ij} + \frac{n}{2} + \frac{6}{5},$$

which completes the proof of part (ii) of the theorem. □

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Chapter 2

Arrangements and Their Applications

2.1 Introduction

The *arrangement* of a finite collection Γ of geometric objects in $\mathbb{R}^d$, denoted as $\mathcal{A}(\Gamma)$, is the decomposition of the space into relatively open connected cells of dimensions $0, \ldots, d$ induced by Γ, where each cell is a maximal connected set of points lying in the intersection of a fixed subset of Γ. Besides being interesting in their own right, because of the rich geometric, combinatorial, algebraic, and topological structures that arrangements possess, they also lie at the heart of several geometric problems arising in a wide range of applications including robotics, computer graphics, molecular modeling, and computer vision. Before proceeding further, we present a few such examples.

(a) Assume that we have a robot system B with d degrees of freedom, i.e., we can represent each placement of B as a point in $\mathbb{R}^d$, and we call the space of all placements the *configuration space* of B. Suppose the three-dimensional workspace of B is cluttered with polyhedral obstacles whose shapes and locations are known. B is allowed to move freely in a motion that traces a continuous path in the configuration space, but B has to avoid collision with the obstacles. For each combination of a geometric feature (vertex, edge, face) of an obstacle and a similar feature (face, edge, vertex) of B, define their *contact surface* as the set of all points in $\mathbb{R}^d$ that represent placements of B at which contact is made between these specific features. Let Γ be the set of all contact surfaces. Let Z be a point corresponding to a given initial *free* placement of B, i.e., a placement at which it does not intersect any obstacle. Then the set of all free placements of B that can be reached from Z via a collision-free continuous motion corresponds to the cell containing Z in the arrangement of the contact surfaces. Thus, the problem of determining whether there exists a collision-free path from an initial configuration I to a final configuration F is equivalent to determining whether I and F lie in the same cell of $\mathcal{A}(\Gamma)$. This close relationship between arrangements and motion planning has led to considerable work on arrangements; see, for example, [28, 62, 226, 238, 260, 261, 275, 279, 359]. If we want to compute the set of all placements reachable from the initial placement I, the *combinatorial complexity* of the cell in $\mathcal{A}(\Gamma)$ containing I, i.e., the total number of lower-dimensional faces appearing on its boundary, serves as a trivial lower bound for the running time. It turns out that in many instances one can design motion-planning algorithms whose performance almost matches this bound.

(b) A molecule can be modeled as an arrangement of spheres, where the radius of each sphere depends on the atom that it models and the position of each sphere depends on the molecular structure. In

the *Van der Waals model*, a molecule is a family of possibly overlapping spheres, where the radius of each sphere is determined by the van der Waals radius of the corresponding atom in the molecule. Lee and Richards [276] proposed another model, called *solvent accessible* model, which is used to study the interaction between the protein and solvent molecules. Also in this model, a molecule is modeled as a sphere, but the balls representing the solvent molecules are shrunk to points and the balls representing atoms in the protein are inflated by the same amount [341]. Even though these models ignore various properties of molecules, they have been useful in a variety of applications. Many problems in molecular modeling can be formulated as problems related to arrangements of spheres. For example, computing the "outer surface" of the molecule corresponds to computing the unbounded cell of the corresponding arrangement of spheres. See [161, 162, 232, 239, 305] for more details on applications of arrangements in molecular biology.

(c) Arrangements are also attractive because of their relationship with several other structures. For example, using the *duality* transform, a point $p = (p_1, \ldots, p_d)$ in $\mathbb{R}^d$ can be mapped to the hyperplane $\sum_{i=1}^{d} x_i p_i = 1$, and vice versa. This facilitates the formulation of several problems related to point configurations in terms of arrangements of hyperplanes. See [126, 158] for a small sample of such problems. The Grassmann-Plücker relation transforms[1] k-flats in $\mathbb{R}^d$ to hyperplanes or points in $\mathbb{R}^u$, for $u = \binom{d+1}{k+1} - 1$ [86, 242]; e.g., lines in $\mathbb{R}^3$ can be mapped to hyperplanes or points in $\mathbb{R}^5$. Therefore many problems involving lines in $\mathbb{R}^3$ have been solved using hyperplane arrangements in $\mathbb{R}^5$ [123, 155, 331, 364]. The well-known combinatorial structure *oriented matroids* of rank $k+1$ are closely related to arrangements of pseudo-hyperplanes in $\mathbb{R}^k$ [86, 342], and *zonotopes* in $\mathbb{R}^d$ correspond to hyperplane arrangements in $\mathbb{R}^{d-1}$ [158, 390]. Several applications of arrangements in singularity theory, algebraic group theory, and other fields of mathematics can be found in [317, 319, 320].

Study of arrangements of lines and hyperplanes has a long, rich history. The first paper on this topic is perhaps by J. Steiner in 1826 [365], in which he obtained bounds on the number of cells in arrangements of lines and circles in the plane and of planes and spheres in $\mathbb{R}^3$. His results have since been extended in several ways [40, 41, 42, 100, 343, 383, 388, 389]. A summary of early work on arrangements of hyperplanes can be found in the monograph and the survey paper by Grünbaum [213, 214]. Most of the work on hyperplane arrangements until the 1980s dealt with the combinatorial structure of the entire arrangement or of a single cell in the arrangement (i.e., a convex polyhedron), and with the algebraic and topological properties of the arrangement [192, 318, 319, 320]. Various substructures and algorithmic issues of hyperplane arrangements, motivated by problems in computational and combinatorial geometry, have received attention mostly during the last twenty years.

Although hyperplane arrangements possess a rich structure, many applications (e.g., the motion-planning problem and the molecular models described above) call for a systematic study of higher-dimensional arrangements of patches of algebraic surfaces. For more than a century, researchers in algebraic geometry have studied arrangements of algebraic surfaces, but their focus has largely been on algebraic and combinatorial issues rather than on algorithmic ones. Considerable progress has been made on all fronts during the last fifteen years.

It is beyond the scope of a survey chapter, or even of a book, to cover all aspects of arrangements. In this chapter we will survey combinatorial and algorithmic problems on arrangements of (hyper)surfaces (or of surface patches) in real affine space $\mathbb{R}^d$. (Hyperplane arrangements in complex space have also been studied; see, e.g., [87, 320].) We will assume that d is a small constant, that the surfaces are algebraic and their degree is bounded by a constant, and that any surface patch

[1] With some care, excluding some lower-dimensional variety of flats, to achieve representation in real, rather than projective, space.

is a semialgebraic set defined by a Boolean combination of a constant number of polynomial equalities and inequalities of constant maximum degree. We refer to such a collection of surfaces or of surface patches ad having *constant description complexity*. There has also been some recent work on combinatorial and algorithmic issues involving arrangements of more general surfaces, known as semi-pfaffian sets, which include graphs of trigonometric, exponential, or logarithmic functions on bounded domains [203, 263]. We also note that a study of algebraic and topological problems on arrangements of algebraic surfaces can be found in [88]. In this survey we will mostly review the known results on the combinatorial complexity of various substructures of arrangements, the known algorithms for computing these substructures, and the geometric applications that benefit from these results. Many other combinatorial problems related to arrangements are discussed in [86, 193, 206, 322, 323, 363, 390]. An excellent source on combinatorial and algorithmic results on arrangements of hyperplanes is the book by Edelsbrunner [158]. The book of Sharir and Agarwal [359] covers some of the topics discussed here in more detail. Other survey papers on arrangements include [220, 228, 237, 356], as well as the earlier versions of this chapter and Chapter 3 [32, 31].

In this chapter we review the basic theory of arrangements in higher dimensions, where the front of the research nowadays lies. However, readers unfamiliar with the earlier reults on planar arrangements and Davenport-Schinzel sequences should read first Chapter 3, which reviews those earlier results.

This survey is organized as follows. In Section 2.2 we define arrangements formally, state the assumptions we will be making in this survey, and discuss the known bounds on the complexity of entire arrangements. Sections 2.3–2.10 discuss combinatorial complexities of various substructures of arrangements. Section 2.11 discusses several methods for representing arrangements. Section 2.12 describes algorithms for computing the entire arrangement, and Section 2.13 reviews algorithms for computing various substructures of arrangements. We discuss a few applications of arrangements in Section 2.14.

2.2 Preliminaries

Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a collection of n (hyper)surfaces or surface patches in $\mathbb{R}^d$. The set Γ induces a decomposition of $\mathbb{R}^d$ into connected cells (or faces), called the *arrangement* of Γ and denoted $\mathcal{A}(\Gamma)$, so that each cell is a maximal connected subset of the intersection of a fixed (possibly empty) subset of surface patches that avoids all other surface patches. Thus a d-dimensional cell is a maximal connected region that does not meet any surface patch of Γ. The *combinatorial complexity* of $\mathcal{A}(\Gamma)$ is the total number of cells, of all dimensions, in $\mathcal{A}(\Gamma)$. The combinatorial complexity of a k-dimensional cell C in $\mathcal{A}(\Gamma)$ is the number of cells of $\mathcal{A}(\Gamma)$ of dimension less than k that are contained in the boundary of C.

We assume that Γ satisfies the following assumptions.

(A1) Each $\gamma_i \in \Gamma$ is a semialgebraic set of constant description complexity. The local dimension of every point in γ_i is $d-1$.[2]

(A2) Each $\gamma_i \in \Gamma$ is of the form $(Q_i = 0) \wedge F_i(P_{i_1} \geq 0, P_{i_2} \geq 0, \ldots, P_{i_u} \geq 0)$. Here u is a constant; F_i

[2] A subset of $\mathbb{R}^d$ is called a *real semialgebraic set* if it is obtained as a finite Boolean combination of sets of the form $\{f = 0\}$ or $\{f > 0\}$ for d-variate polynomials f. As defined above, a semialgebraic set has *constant description complexity* if it can be described in terms of a constant number of polynomials, with a constant bound on the degrees of the corresponding polynomials.

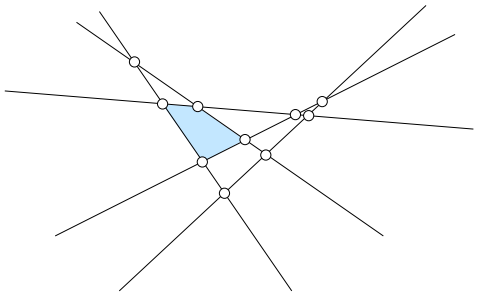

Figure 2.1: An arrangement of lines.

is a Boolean formula; $Q_i, P_{i_1}, \ldots, P_{i_u} \in \mathbb{R}[x_1, \ldots, x_d]$; and the degrees of Q_i, P_{i_j} are at most b, for some constant b. Let $\mathcal{Q} = \{Q_1, \ldots, Q_n\}$.

Note that (A2) implies (A1), but we will mention both assumption for the sake of clarity. We will refer to a semialgebraic set satisfying (A1) and (A2) a *(hyper)surface patch* in $\mathbb{R}^d$ (of constant description complexity). If γ_i is simply the zero set of Q_i, we will call γ_i a *(hyper)surface*. Using a stratification algorithm [88, 244], we can decompose each γ_i into a constant number connected surface patches so that the interior of each patch is *smooth* and each of them satisfies (A1) and (A2) with a different, possibly larger, value of b. We can also assume that each resulting patch is *monotone* in $x_1, \ldots, x_{d-1}$ (i.e., any line parallel to the x_d-axis intersects it in at most one point). In some cases, the resulting collection may also include *vertical* surface patches, namely, patches whose projection on the hyperplane $x_d = 0$ has dimension $\leq d-2$. However, in most of the presentation we will assume that no vertical patches exist.

An arrangement of hyperplanes is called *simple* if any d of the hyperplanes intersect in exactly one point, and no $d+1$ of them have a nonempty intersection. In a simple arrangement, a k-dimensional cell is contained in exactly $d-k$ hyperplanes. We will also need a similar concept for arrangements of surface patches. An arrangement $\mathcal{A}(\Gamma)$ satisfying assumptions (A1) and (A2) is said to be in *general position* if the coefficients of the polynomials defining the surface patches in Γ and their boundaries are algebraically independent[3] over the rationals; otherwise, $\mathcal{A}(\Gamma)$ is called *degenerate*. This condition ensures that no degeneracy occurs among the surface patches, such as too many surface patches with a common point, tangencies or overlaps between different intersections of subsets of the surface patches, etc. We note that this definition of general position is quite strong (e.g., surfaces defined by polynomials with integer coefficients are not in general position in this strong sense). In all the applications much weaker versions of general position are required, which rule out a specific list of forbidden degenerate situations. If $\mathcal{A}(\Gamma)$ is in general position, then any d surface patches of Γ intersect in at most s points for some constant s depending on d and b. By Bezout's theorem [242], $s \leq b^d$.

If $\mathcal{A}(\Gamma)$ is degenerate, we can perturb the coefficients of the polynomials in $\mathcal{Q}$ by various infinitesimals so that the coefficients of the perturbed polynomials are in extension fields of the reals that are fields of *Puiseux Series* in these infinitesimals, and so that the resulting surface patches are in general position. Moreover, it can be shown that, as far as worst-case bounds are concerned, the perturbation may reduce the combinatorial complexity of any cell of the arrangement by at most a

[3]A set $\{x_1, \ldots, x_k\}$ of real numbers is *algebraically independent* (over the rationals) if no k-variate polynomial with integer coefficients vanishes at $(x_1, x_2, \ldots, x_k)$.

constant factor [333, 357, 359]. Actually, in many cases the size of a substructure of Γ has maximum complexity when $\mathcal{A}(\Gamma)$ is in general position. This observation allows us to restrict our attention to arrangements in general position while investigating the combinatorial complexity of substructures of arrangements.

However, in order to achieve the general position defined above, the perturbation scheme has to introduce a different infinitesimal for each coefficient, which makes any algorithm based on this perturbation scheme impractical. Fortunately, most of the algorithms involving arrangements either work for any degenerate arrangement or require a considerably weaker definition of general position, e.g., the intersection of any k surface patches is either empty or a $(d-k)$-dimensional set, all surface patches intersect "properly," etc. The perturbation scheme required by an algorithm depends on the degenerate situations that it wants to rule out. Several constructive perturbation schemes have been proposed that use only a few infinitesimals [79, 170, 184, 185, 195]. Although these schemes cannot handle all the cases, they work for a wide range of applications. The paper by Seidel [354] contains a detailed discussion on "linear" perturbation and its applications in geometric algorithms. A few algorithms have also been proposed to handle degeneracies directly without resorting to perturbations; see e.g. [70, 101]. We will, nevertheless, use the strong definition of general position, defined above, in order to simplify the exposition, and refer the reader to original papers for specific *general-position assumptions* required by different algorithms.

In the light of the preceding discussion, and since we are mainly interested in asymptotic bounds, we will make the following additional assumptions on the surface patches in Γ, without any real loss of generality, whenever required.

(A3) Each surface patch in Γ is connected and monotone in $x_1, \ldots, x_{d-1}$, and its relative interior is smooth.

(A4) The surface patches in Γ are in general position.

(A5) Any d surface patches in Γ intersect in at most s points, for some constant s. (This is a consequence of the preceding assumptions, but is stated to introduce s explicitly.)

Generally, we will be stating assumptions (A1) and (A2), but most of the proofs and algorithms sketched in the survey will also make assumptions (A3)–(A5).

Assumptions (A1)–(A3) imply that we can regard each surface patch γ as the graph of a partially defined $(d-1)$-variate function $x_d = \gamma(x_1, \ldots, x_{d-1})$ of constant description complexity. We will refer to the projection of γ onto the hyperplane $x_d = 0$ as the *domain*, denoted γ^*, of γ (over which the function γ is defined). The boundary of γ^*, called the *domain boundary*, is a collection of $O(1)$ $(d-2)$-dimensional surface patches in $\mathbb{R}^{d-1}$ satisfying assumptions corresponding to (A1)–(A2). Abusing the notation slightly, we will not distinguish between the surface patch γ and the underlying function $\gamma(x_1, \ldots, x_{d-1})$.

The most fundamental question in the combinatorial study of an arrangement $\mathcal{A}(\Gamma)$ of surfaces is to prove a bound on the combinatorial complexity, $f(\Gamma)$, of $\mathcal{A}(\Gamma)$.

In 1826, Steiner [365] studied the complexity of arrangements of lines and circles in $\mathbb{R}^2$ and of planes and spheres in $\mathbb{R}^3$. In particular, he has shown that an arrangement of n planes in $\mathbb{R}^3$ in general position has $\binom{n}{3}$ vertices, $\binom{n}{2} + 3\binom{n}{3}$ edges, $n^2 + 3\binom{n}{3}$ 2-faces, and $1 + n + \binom{n}{2} + \binom{n}{3}$ 3-cells. Later Roberts [343] extended Steiner's formula to count the number faces in arbitrary arrangements of planes (allowing all kinds of degeneracies) in $\mathbb{R}^3$, using the inclusion-exclusion principle.

Brousseau [99] used a plane-sweep argument to count the number of faces in arrangements of planes in $\mathbb{R}^3$. (A similar argument was used by Hadwiger [224] to derive Euler's formula for convex polytopes.) His method was later extended by Alexanderson and Wetzel [42].

Buck [100] was the first to bound the combinatorial complexity of hyperplane arrangements in higher dimensions. In more recent work, Zaslavsky [388, 389] studied hyperplane arrangements; he used the Möbius inversion formula and lattice theory to count the number of cells of all dimensions in (possibly degenerate) hyperplane arrangements. Let Γ be a set of n hyperplanes in $\mathbb{R}^d$. Let $\varphi_k(\Gamma)$ denote the number of k-cells in $\mathcal{A}(\Gamma)$. Zaslavsky [388] and Las Vergnas [274] proved that for nonsimple arrangements, $\varphi_k(\Gamma)$ depends on the underlying matroid structure. There are several results on bounding $\varphi_k(\Gamma)$ in nonsimple hyperplane arrangements. For example, Fukuda *et al.* [202] proved that the mean number of $(k-1)$-cells bounding a k-cell in an arrangement of n hyperplanes is less than $2k$, which implies that $\varphi_k(\Gamma) \leq \binom{d}{k}\varphi_d(\Gamma)$. See [202, 301, 345] for some other results of this type.

In summary, the following theorem gives a bound on the combinatorial complexity of hyperplane arrangements. (See [32] for a proof.)

Theorem 2.2.1 (Buck [100]). *Let Γ be a set of n hyperplanes in $\mathbb{R}^d$. For any $0 \leq k \leq d$,*

$$\varphi_k(\Gamma) \leq \binom{n}{d-k} \sum_{i=0}^{k} \binom{n-d+k}{i}.$$

The equality holds when $\mathcal{A}(\Gamma)$ is simple.

For arrangements $\mathcal{A}(\Gamma)$ of a set Γ of surfaces satisfying assumptions (A1) and (A2), obtaining a sharp bound on $f(\Gamma)$, the combinatorial complexity of $\mathcal{A}(\Gamma)$, is not easy. If the surface patches are in general position, in the sense defined above, it is obvious that $f(\Gamma) = O(n^d)$. However, it is not easy to argue that the arrangements have maximum complexity when the surface patches are in general position (this is due to the complicated algebraic structures that can arise in degenerate settings). Heintz *et al.* [245] proved that $f(\Gamma) = (nb)^{O(d)}$. A lower bound of $\Omega((nb/d)^d)$ is not difficult to prove. Warren [379] had proved that the number of d-dimensional cells in an arrangement of n hypersurfaces, each of degree $\leq b$, in $\mathbb{R}^d$ is $O((nb/d)^d)$. This bound also follows from the results by Milnor [308], Petrovskiĭ and Oleĭnik [332], and Thom [372]. Using a perturbation argument, Pollack and Roy [333] generalized Warren's result and proved that the number of cells of all dimensions in an arrangement of n hypersurfaces is $(O(nb)/d)^d$. An easy consequence of their result is the following theorem.

Theorem 2.2.2. *Let Γ be a set of n surface patches in $\mathbb{R}^d$ satisfying assumptions (A1) and (A2). Then*

$$f(\Gamma) = \left(\frac{O(nb)}{d}\right)^d.$$

An extension of this theorem by Basu *et al.* [78] shows that if Σ be a k-dimensional algebraic varierty of degree at most b in $\mathbb{R}^d$, then the number of cells in the subdivision of Σ induced by Γ is at most $O((n/k)^k b^d)$.

Improved bounds on the complexity of the arrangement can be proved in some special cases. For example, if Γ is a set of n $(d-1)$-simplices in $\mathbb{R}^d$ that form the boundaries of k convex polytopes, then $f(\Gamma) = O(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil})$ [53]. See [146] for improved bounds in a few other cases. A concept closely related to the combinatorial complexity of arrangements is the number of *realizable sign*

sequences of a family of polynomials. Let $\mathcal{Q} = \{Q_1, \ldots, Q_n\}$ be a set of d-variate polynomials as defined above, and let Γ be the family of the zero-sets of the polynomials in $\mathcal{Q}$. We can define $\sigma_i(\mathbf{x})$, for a point $\mathbf{x} \in \mathbb{R}^d$, as follows.

$$\sigma_i(\mathbf{x}) = \begin{cases} -1 & Q_i(\mathbf{x}) < 0, \\ 0 & Q_i(\mathbf{x}) = 0, \\ +1 & Q_i(\mathbf{x}) > 0. \end{cases}$$

Since $\sigma_i(\mathbf{x})$ remains the same for all points $\mathbf{x}$ in a single cell of $\mathcal{A}(\Gamma)$, we can define the *sign sequence* for each cell $\sigma(C) = \langle \sigma_1(\mathbf{x}), \sigma_2(\mathbf{x}), \ldots, \sigma_n(\mathbf{x}) \rangle$ for any point $\mathbf{x} \in C$. A sign sequence σ is *realized* by $\mathcal{A}(\Gamma)$ if there is a cell $C \in \mathcal{A}(\Gamma)$ with $\sigma = \sigma(C)$. A well-studied question in algebraic geometry is to bound the number of sign sequences that can be realized by a set of polynomials [43]. Obviously, $f(\Gamma)$ is an upper bound on this quantity.

2.3 Lower Envelopes

Definitions and preliminary results. In Chapter 3 we will review lower envelopes of arcs in the plane, and will show the relationship between such envelopes and *Davenport–Schinzel sequences*, a powerful combinatorial construct with rather surprising properties. Here we assume familiarity of the reader with the basic results concerning Davenport–Schinzel sequences and univariate lower envelopes, but we will not need too much of that theory. We use the standard notation $\lambda_q(n)$ for the maximum length of Davenport–Schinzel sequences of order q on n symbols, and recall that $\lambda_q(n)$, for $q \geq 3$, is nearly linear (ever so slightly super-linear) in n for any fixed q. We also recall that the complexity of the lower envelope of n univariate continuous fully defined (resp., partially defined) functions, each pair of which intersect in at most q points, is at most $\lambda_q(n)$ (resp., $\lambda_{q+2}(n)$). We refer the reader to Chapter 3 for many additinal details, which can also be found in [295, 359, 31].

In this section we study lower envelopes of surface patches in higher dimensions. Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a collection of surface patches in $\mathbb{R}^d$ satisfying assumptions (A1)–(A3). If we regard each surface patch as the graph of a partially defined function, the *lower envelope* of Γ, denoted $L(\Gamma)$ (or L for brevity), is defined as the graph of the partially defined function

$$L_\Gamma(\mathbf{x}) = \min_{1 \leq i \leq n} \gamma_i(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^{d-1};$$

$\gamma_i(\mathbf{x})$ is set to $+\infty$ if $\mathbf{x} \notin \gamma_i^*$. The *upper envelope* $U(\Gamma)$ of Γ is defined as the graph of the partially defined function

$$U_\Gamma(\mathbf{x}) = \max_{1 \leq i \leq n} \gamma_i(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^{d-1};$$

$\gamma_i(\mathbf{x})$ is set to $-\infty$ if $\mathbf{x} \notin \gamma_i^*$. We can extend the definitions of lower and upper envelopes even if Γ satisfies only (A1) and (A2). We can decompose each γ_i into $O(1)$ connected patches, each of which is monotone in $x_1, \ldots, x_{d-1}$ directions and satisfies (A1) and (A2). Let Γ' denote the resulting set of surface patches. We define $L(\Gamma) = L(\Gamma')$ and $U(\Gamma) = U(\Gamma')$.

$L(\Gamma)$ induces a partition of $\mathbb{R}^{d-1}$ into maximal connected ($(d-1)$-dimensional) regions so that $L(\Gamma)$ is attained by a fixed (possibly empty) subset of Γ over the interior of each such region. The boundary of such a region consists of points at which $L(\Gamma)$ is attained by at least two of the surface patches or by the relative boundary of at least one surface. Let $\mathcal{M}(\Gamma)$ denote this subdivision of $\mathbb{R}^{d-1}$, which we call the *minimization diagram* for the collection Γ. A *face* of $\mathcal{M}(\Gamma)$ is a maximal connected region over which $L(\Gamma)$ is attained by the same set of functions and/or relative boundaries of function

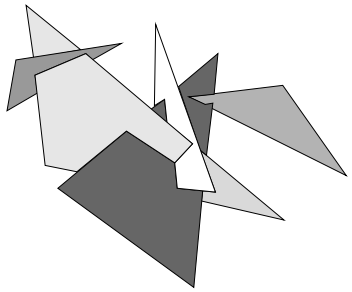

Figure 2.2: Lower envelope of triangles in $\mathbb{R}^3$, as viewed from below.

graphs in Γ. Note that if a face $f \in \mathcal{M}(\Gamma)$ lies on the boundary of the domain of a surface in Γ, then f may not correspond to any face of $L(\Gamma)$. However, if f lies in the relative interior of the domains of all the relevant surface patches, f is the projection of a face $\hat{f}$ of $L(\Gamma)$. The *combinatorial complexity* of $L(\Gamma)$, denoted $\kappa(\Gamma)$, is the number of faces of all dimensions in $\mathcal{M}(\Gamma)$. For an infinite family $\mathbf{G}$ of surface patches satisfying assumptions (A1)–(A2), we define $\kappa(n, d, \mathbf{G}) = \max \kappa(\Gamma)$, where the maximum is taken over all subsets Γ of $\mathbf{G}$ of size n. If $\mathbf{G}$ is the set of all surface patches satisfying (A1)–(A2) or if $\mathbf{G}$ is obvious from the context, we will simply write $\kappa(n, d)$. The *maximization diagram* is defined as the subdivision of $\mathbb{R}^{d-1}$ induced, in the same manner, by the upper envelope $U(\Gamma)$ of Γ.

As will be discussed in Chapter 3, the complexity of the lower envelope of n arcs in the plane, each pair of which intersects in at most s points, is at most $\lambda_{s+2}(n)$, the maximum length of an (n, s)-Davenport–Schinzel sequence (see also [359]). Extending to higher dimensions, it was conjectured that the complexity of the lower envelopes of a family of n surface patches satisfying (A1)–(A2) is $O(n^{d-2}\lambda_q(n))$ for a constant $q \geq 0$. If Γ is a set of n hyperplanes in $\mathbb{R}^d$, then the Upper Bound Theorem implies that the complexity of $L(\Gamma)$ is $\Theta(n^{\lfloor d/2 \rfloor})$ [390]. Let Δ be the set of all $(d-1)$-simplices in $\mathbb{R}^d$. Extending the lower-bound construction by Wiernik and Sharir [384] to higher dimensions, one can prove that $\kappa(n, d, \Delta) = \Omega(n^{d-1}\alpha(n))$. This suggests we cannot hope to aim for an $o(n^{d-1})$ bound on $\kappa(n, d)$ for general surface patches. At the end of this section we will discuss some cases in which better bounds on $\kappa(n, d)$ can be proved.

Using a divide-and-conquer approach, Pach and Sharir [325] proved that, for a set Γ of n simplices in $\mathbb{R}^d$, the number of $(d-1)$-dimensional faces in $\mathcal{M}(\Gamma)$ is $O(n^{d-1}\alpha(n))$. Roughly speaking, they divide Γ into subsets Γ_1, Γ_2, each of size at most $\lceil n/2 \rceil$, and bound the number of $(d-1)$-dimensional faces of $\mathcal{M}(\Gamma_1), \mathcal{M}(\Gamma_2)$ recursively. They prove that the number of $(d-1)$-dimensional faces in $\mathcal{M}(\Gamma)$ is $|\mathcal{M}(\Gamma_1)| + |\mathcal{M}(\Gamma_2)| + O(n^{d-1}\alpha(n))$, thereby obtaining the claimed bound. Edelsbrunner [159] extended their result to give the same asymptotic bound for the number of faces of all dimensions. Simpler proofs for this bound were proposed by Sharir and Agarwal [359] and Tagansky [369]. Roughly speaking, both proofs proceed by induction on d, and they bound the change in the complexity of the minimization diagram as a simplex is inserted into Γ.

The main complexity bound. All the aforementioned proofs rely crucially on the fact that if Γ is a set of surface patches in general position, then any triple of surface patches intersect in at most one point. These proofs do not extend to the case when a tripe intersects in two or more points. Halperin and Sharir [235] proved a near-quadratic bound on $\kappa(n, 3)$ for the case when $s \leq 2$. Sharir [357] extended their approach to higher values of s and d. Their results are stated in the

following theorem.

Theorem 2.3.1 (Halperin and Sharir [235]; Sharir [357]). *Let Γ be a set of n surface patches in $\mathbb{R}^d$ satisfying assumptions (A1)–(A2). Then $\kappa(n,d) = O(n^{d-1+\varepsilon})$, for any $\varepsilon > 0$. The constant of proportionality depends on ε, d, b (and s).*

Proof: We will sketch the proof for a set of bivariate surface patches in $\mathbb{R}^3$ satisfying assumptions (A1)–(A5) with $s = 2$, i.e., a triple of surface patches intersect in at most two points. For a pair of surface patches $\gamma_i, \gamma_j \in \Gamma$, let β_{ij} denote the intersection arc $\gamma_i \cap \gamma_j$. If β_{ij} is not x_1-monotone, we decompose it at its x_1-extremal points; each intersection arc is thereby decomposed into $O(1)$ pieces. If any of these points appears on the lower envelope, we regard it as a vertex on the envelope and its projection as a vertex on the minimization diagram.

Since Γ is in general position, it suffices to bound the number of vertices in $\mathcal{M}(\Gamma)$. Indeed, a higher-dimensional face f of $\mathcal{M}(\Gamma)$ must be incident to a vertex v of $\mathcal{M}(\Gamma)$, and we can charge f to v. By the general-position assumption, each vertex is charged only a constant number of times. For a subset $R \subseteq \Gamma$, let $\phi^*(R)$ denote the number of vertices in $\mathcal{M}(R)$; set $\phi^*(r) = \max \phi^*(R)$, where the maximum is taken over all subsets of Γ of size r.

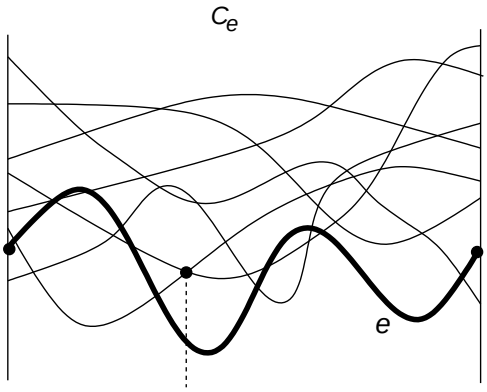

Figure 2.3: Vertical cylinder C_e and the vertical cross-section Γ_e of Γ.

We call a vertex of $\mathcal{M}(\Gamma)$ a *boundary* vertex if it lies on the boundary $\partial\gamma_i^*$ of the domain of a surface γ_i; otherwise, we call it an *inner* vertex. The number of boundary vertices is $O(n\lambda_q(n))$, where q is a constant depending on b, the maximum degree of surface patches and their boundaries. Indeed, let C_e be the vertical cylinder erected on an edge e of the boundary $\partial\gamma_i^*$, i.e., $C_e = e \times \mathbb{R}$. Define $\Gamma_e = \{\gamma \cap C_e \mid \gamma \in \Gamma \setminus \{\gamma_i\}\}$, which is a collection of $O(n)$ arcs; see Figure 2.3. Each arc in Γ_e satisfies assumptions (A1)–(A3) (with $d = 2$, and with larger, but still constant, parameters b and s). It is easily seen that a boundary vertex of $\mathcal{M}(\Gamma)$ appearing on e is a vertex of $\mathcal{M}(\Gamma_e)$. If the arcs in Γ_e intersect in at most $q-2$ points, $O(\lambda_q(n))$ boundary vertices lie on e. Summing over all $O(n)$ edges of domain boundaries of Γ, we obtain the desired bound on the number of boundary vertices.

We call an inner vertex *regular* if it is not an x_1-extremal vertex of any of the three intersection curves. The number of irregular vertices is obviously $O(n^2)$. For a subset $R \subseteq \Gamma$, let $\phi(R)$ denote the number of regular (inner) vertices in $\mathcal{M}(R)$, and let $\phi(r) = \max_{|R|=r} \phi(R)$. The above discussion implies that

$$\phi^*(\Gamma) \leq \phi(\Gamma) + O(n\lambda_q(n)).$$

Next, we derive a recurrence for $\phi(\Gamma)$, which will solve to $O(n^{2+\varepsilon})$. Fix a regular vertex v of $\mathcal{M}(\Gamma)$. Let $\hat{v}$ be the corresponding vertex of $L(\Gamma)$ (since v is an inner vertex, $\hat{v}$ is well defined). Suppose $\hat{v}$ is one of the two intersection points of three surface patches, say, $\gamma_1, \gamma_2, \gamma_3$. Assume, without loss of generality, that if $|\gamma_1 \cap \gamma_2 \cap \gamma_3| = 2$, then the x_1-coordinate of the other intersection point of γ_1, γ_2, and γ_3 is larger than that of $\hat{v}$. Since $\hat{v}$ is a regular vertex, one of the three pairwise-intersection curves β_{ij}, say β_{12}, lies above $L(\Gamma)$ in the halfspace $x_1 < x_1(v)$ in a sufficiently small neighborhood of $\hat{v}$. We mark on β_{12} the intersection points of β_{12} with other surface patches of Γ and the points that lie above the boundaries of other surface patches in Γ.

We fix a parameter $t = t(\varepsilon)$ and follow β_{12} in the $(-x_1)$-direction, starting from $\hat{v}$, until one of the following three events occurs:[4]

(C1) we reach the left endpoint of β_{12};

(C2) β_{12} appears on $L(\Gamma)$; or

(C3) we crossed t marked points on β_{12}.

We call v a vertex of type (Ci), for $i = 1, 2, 3$, if we first reach an event of type (Ci). If (C1) occurs, we charge v to the left endpoint of β_{12}. Since each endpoint is charged at most twice, the total number of regular vertices of type (C1) is $O(n^2)$. If (C2) occurs, then we must have passed above the boundary of γ_3 while following β_{12} because β_{12} lies strictly above γ_3 in the halfspace $x_1 < x_1(v)$. Let w be the marked point on β_{12} lying above $\partial\gamma_3^*$ that we have visited. We charge v to w. Suppose w lies above an edge e of $\partial\gamma_3^*$. We can define C_e and Γ_e as above; then w is a vertex of $\mathcal{A}(\Gamma_e)$. Since (C2) occurred before (C3), at most t marked points lie on β_{12} between v and w, which implies that less than t arcs of Γ_e lie below w. As shown in [355], the number of vertices of $\mathcal{A}(\Gamma_e)$ that lie above at most t arcs is $O(t\lambda_q(n))$. Summing over all edges of domain boundaries, the number of marked points on intersection arcs to which a vertex of type (C2) is charged is $O(nt\lambda_q(n))$. Since each marked point is charged $O(1)$ times, the number of type (C2) vertices is $O(nt\lambda_q(n))$.

Finally, if (C3) occurs, then we charge $1/t$ to each marked point on β_{12} that we visited. Each marked point will be charged only $O(1/t)$ units, and each such marked point lies above at most t surface patches of Γ. Theorem 2.6.1 in Section 2.6 implies that the number of such marked points, summed over all intersection curves, is $O(t^3\phi^*(n/t))$. The total number of vertices of type (C3) is thus

$$O(1/t) \cdot O(t^3\phi^*(n/t)) = O(t^2\phi(n/t) + n\lambda_q(n)).$$

Hence, we obtain the following recurrence for $\phi(n)$:

$$\phi(n) \le At^2\phi\left(\frac{n}{t}\right) + Btn\lambda_q(n),$$

where A and B are constants (depending on b). The solution of the above recurrence is

$$\phi(n) = O(tn^{1+\log_t A}\lambda_q(n)).$$

If $t = t(\varepsilon)$ is chosen sufficiently large, then $\phi(n) = O(n^{2+\varepsilon})$. This proves the theorem for $d = 3, s = 2$.

For $s > 2$, Sharir [355] introduces the notion of *index* of a regular vertex. The index of a vertex $v \in \bigcap_{i=1}^3 \gamma_i$ is the number of points of $\bigcap_{i=1}^3 \gamma_i$ whose x_1 coordinates are less than that of v. For $0 \le j < s$, let $\phi^{(j)}(\Gamma)$ be the number of regular vertices in $L(\Gamma)$ of index j. Then $\phi(\Gamma) = \sum_{j=0}^{s-1} \phi^{(j)}(\Gamma)$.

[4] If the x_1-coordinate of the other intersection point of γ_1, γ_2, and γ_3 were smaller than that of $\hat{v}$, we would have traced β_{12} in the $(+x_1)$-direction.

Modifying the above argument slightly, Sharir derived a system of recurrences for the quantities $\phi^{(j)}(\Gamma)$, for $j < s$. There are three main differences. First, the tracing of β_{12} is always done in the decreasing x_1-direction. Second, the value of the parameter t now depends on j and is denoted by t_j, Third, there is one more stopping criterion:

(C4) β_{12} intersects γ_3; let z be the (first) intersection point.

Using the fact that the index of z is $\leq j-1$ and that at most t_j surface patches lie below z, Sharir derives the following recurrence for $\phi^{(j)}(n) = \max_{|\Gamma|=n} \phi^{(j)}(\Gamma)$.

$$\phi^{(j)}(n) \leq A_j t_j^2 \phi^* \left(\frac{n}{t_j}\right) + B_j \left(t_j^3 \phi^{(j-1)} \left(\frac{n}{t_j}\right) + n t_j \lambda_q(n)\right).$$

By expanding this system of recurrences and by choosing the values of t_j carefully, Sharir proved that the solution of this system satisfies

$$\phi^*(n) = O(n^{2+\varepsilon}).$$

The theorem is proved in higher dimensions by induction on d, using a similar charging scheme. See the original paper by Sharir for details. □

Problem 2.3.2. *Let Γ be a set of n surface patches in $\mathbb{R}^d$ satisfying assumptions (A1) and (A2). Is $\kappa(n,d) = O(n^{d-2}\lambda_q(n))$, where q is a constant depending on d and b?*

Bounds in special cases. As noted above, sharper bounds are known on the complexity of lower envelopes in some special cases; see [348, 359]. For example, if Γ is a set of pseudo-planes in $\mathbb{R}^3$, i.e., each triple of surfaces intersects in at most one point and the intersection of a pair of surfaces in Γ is a single (closed or unbounded) Jordan curve, then $\kappa(\Gamma) = O(n)$. On the other hand, if Γ is a set of pseudo-spheres, i.e., each triple intersects in at most two points and the intersection curve of any pair is a single Jordan curve, then $\kappa(\Gamma) = O(n^2)$. If Γ is a family of hypersurfaces in $\mathbb{R}^d$, a sharper bound on $\kappa(\Gamma)$ can be proved using the so-called *linearization* technique. Here is a sketch of this technique.

Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a collection of hypersurfaces of degree at most b, i.e., each γ_i is the zero set of a d-variate polynomial Q_i of degree at most b. Let $\mathcal{Q} = \{Q_1, \ldots, Q_n\}$. We say that Γ admits a *linearization* of dimension k if, for some $p > 0$, there exists a $(d+p)$-variate polynomial

$$g(\mathbf{x}, \mathbf{a}) = \psi_0(\mathbf{a}) + \psi_1(\mathbf{a})\varphi_1(\mathbf{x}) + \psi_2(\mathbf{a})\varphi_2(\mathbf{x}) + \cdots + \psi_{k-1}(\mathbf{a})\varphi_{k-1}(\mathbf{x}) + \varphi_k(\mathbf{x}),$$

for $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{a} \in \mathbb{R}^p$, so that, for each $1 \leq i \leq n$, we have $Q_i(\mathbf{x}) = g(\mathbf{x}, \mathbf{a}_i)$ for some $\mathbf{a}_i \in \mathbb{R}^p$. Here each $\psi_j(\mathbf{a})$, for $0 \leq j \leq k$, is a p-variate polynomial, and each $\varphi_j(\mathbf{x})$, for $1 \leq j \leq k+1$, is a d-variate polynomial. It is easily seen that such a polynomial representation always exists for $p \leq d^{b+1}$—let the φ's be the monomials that appear in at least one of the polynomials of $\mathcal{Q}$, and let $\psi_j(\mathbf{a}) = a_j$ (where we think of $\mathbf{a}$ as the vector of coefficients of the monomials).

We define a transform $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}^k$ that maps each point in $\mathbb{R}^d$ to the point

$$\varphi(\mathbf{x}) = (\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x}), \ldots, \varphi_k(\mathbf{x}));$$

the image $\varphi(\mathbb{R}^d)$ is a d-dimensional algebraic surface Σ in $\mathbb{R}^k$. For each function $Q_i(\mathbf{x}) = g(\mathbf{x}, \mathbf{a}_i)$, we define a k-variate linear function

$$h_i(\mathbf{y}) = \psi_0(\mathbf{a}_i) + \psi_1(\mathbf{a}_i)y_1 + \cdots \psi_{k-1}(\mathbf{a}_i)y_{k-1} + y_k.$$

Let $H = \{h_i = 0 \mid 1 \leq i \leq n\}$ be a set of n hyperplanes in $\mathbb{R}^k$. Let ξ be a vertex of $L(\Gamma)$. If ξ is incident to $\gamma_1, \ldots, \gamma_d$, then $Q_1(\xi) = \cdots = Q_d(\xi) = 0$ and $Q_{d+1}(\xi)\sigma_{d+1}0, \ldots, Q_n(\xi)\sigma_n 0$, where $\sigma_i \in \{>, <\}$. By construction, $Q_i(\xi) = h_i(\varphi(\xi))$. Let $Q \in \mathbb{R}[x_1, \ldots, x_d]$ be a d-variate polynomial. If we regard Q as a univariate polynomial in x_d and the coefficient of the leading term in Q is a positive constant, then we call Q a *positive polynomial*. If all Q_i's are positive, then, by the definition of lower envelopes, $Q_i(\xi) < 0$ for every $i > d$. Hence, $h_1(\varphi(\xi)) = \cdots h_d(\varphi(\xi)) = 0$ and $h_{d+1}(\varphi(\xi)) < 0, \ldots, h_n(\varphi(\xi)) < 0$. That is, $\varphi(\xi)$ is a vertex of $L(H) \cap \Sigma$. Since each h_i is a hyperplane in $\mathbb{R}^k$ and the degree of Σ depends only on d and b, the Upper Bound Theorem for convex polyhedra (see McMullen and Shephard [300] and Ziegler [390]) implies that the number of vertices in $\Sigma \cap L(H)$ is $O(n^{\lfloor k/2 \rfloor})$. Hence, we can conclude the following.

Theorem 2.3.3. *Let Γ be a collection of n hypersurfaces in $\mathbb{R}^d$, of constant maximum degree b. If Γ admits a linearization of dimension k and each surface Γ is the zero set of a positive polynomial, then $\kappa(\Gamma) = O(n^{\lfloor k/2 \rfloor})$, where the constant of proportionality depends on k, d, and b.*

We illustrate the linearization technique by giving an example. A sphere in $\mathbb{R}^d$ with center $(a_1, \ldots, a_d)$ and radius a_{d+1} can be regarded as the zero set of the polynomial $g(\mathbf{x}, \mathbf{a})$, where

$$\begin{aligned} g(\mathbf{x}, a_1, \ldots, a_{d+1}) \quad = \quad & [a_1^2 + \cdots + a_d^2 - a_{d+1}^2] - [2a_1 \cdot x_1] - [2a_2 \cdot x_2] - \cdots - \\ & [2a_d \cdot x_d] + [x_1^2 + \cdots x_d^2] \end{aligned}$$

Thus, setting

$$\begin{aligned} \psi_0(\mathbf{a}) &= \sum_{i=1}^{d} a_i^2 - a_{d+1}^2, \quad \psi_1(\mathbf{a}) = -2a_1, \quad \cdots \quad \psi_d(\mathbf{a}) = -2a_d, \quad \psi_{d+1}(\mathbf{a}) = 1, \\ & \varphi_1(\mathbf{x}) = x_1, \quad \cdots \quad \varphi_d(\mathbf{x}) = x_d, \quad \varphi_{d+1}(\mathbf{x}) = \sum_{i=1}^{d} x_i^2, \end{aligned}$$

we obtain a linearization of dimension $d + 1$. We can therefore conclude the following.

Corollary 2.3.4. *Let Γ be a set of n spheres in $\mathbb{R}^d$. Then $\kappa(\Gamma) = O(n^{\lceil d/2 \rceil})$.*

The overlay of minimization diagrams. Motivated by several applications, researchers have studied the complexity of the overlay of two minimization diagrams. That is, let Γ and Γ' be two families of surface patches satisfying assumptions (A1)–(A2); set $n = |\Gamma| + |\Gamma'|$. The *overlay* of $\mathcal{M}(\Gamma)$ and $\mathcal{M}(\Gamma')$ is the decomposition of $\mathbb{R}^{d-1}$ into maximal connected regions so that each region lies within a fixed pair of faces of $\mathcal{M}(\Gamma)$ and $\mathcal{M}(\Gamma')$. It is conjectured that the complexity of the overlay of the two diagrams is also close to $O(n^{d-1})$. Although this conjecture is obviously true for the minimization diagrams of arcs in the plane, it is not intuitive even in $\mathbb{R}^3$ because the overlay of two planar maps with m edges each may have $\Omega(m^2)$ vertices. Edelsbrunner *et al.* [166] proved an $O(n^{d-1}\alpha(n))$ upper bound if Γ and Γ' are sets of a total of n simplices in $\mathbb{R}^d$.

Agarwal *et al.* [26] proved that the overlay of two minimization diagrams, defined for a total of n surface patches, in $\mathbb{R}^3$ has $O(n^{2+\varepsilon})$ complexity, for any $\varepsilon > 0$. Note that in $\mathbb{R}^3$, each vertex of the

overlay is a vertex of $\mathcal{M}(\Gamma)$, a vertex of $\mathcal{M}(\Gamma')$, or an intersection point of an edge of $\mathcal{M}(\Gamma)$ with an edge of $\mathcal{M}(\Gamma')$. The proof in [26] establishes an upper bound on the number of intersection points by generalizing the proof technique of Theorem 2.3.1.

Recently, Koltun and Sharir [270] have extended the result to $\mathbb{R}^4$, showing that the complexity of the ovrlay of two or three minimization diagrams of a total of n surface patches in $\mathbb{R}^4$ satisfying assumptions (A1)–(A5) is $O(n^{3+\varepsilon})$, for any $\varepsilon > 0$. A simple recent observation by Koltun and Sharir [272] reduces the overlay of k minimization diagrams in $\mathbb{R}^d$, for $k \leq d-1$, into a substructure of a *single* minimization diagram in $\mathbb{R}^{d+k-1}$, implying that the complexity of the overlay is $O(n^{d+k-2+\varepsilon})$, for any $\varepsilon > 0$. While these bounds are not sharp (compare, e.g., with the bounds cited above for $d = 3$, $k = 2$), they improve upon previous naive estimates. Moreover, for the case of minimization diagrams of *hyperplanes*, Koltun and Sharir obtain improved bounds that turn out to be tight in the worst case. Their results are related to, and inspired by, a similar recent observation due to Chan [112].

Problem 2.3.5. *What is the complexity of the overlay of two (or any number up to $d-1$) minimization diagrams in $\mathbb{R}^d$, for $d \geq 5$?*

The following problem is closely related to the overlay of minimization diagrams. Let Γ, Γ' be two sets of surface patches in $\mathbb{R}^d$ satisfying (A1)–(A5). Regarding each surface as the graph of a partially defined function, define

$$\mathcal{S}(\Gamma, \Gamma') = \left\{ \mathbf{x} \;\middle|\; L_\Gamma(x_1, \ldots, x_{d-1}) \geq x_d \geq U_{\Gamma'}(x_1, \ldots, x_{d-1}) \right\},$$

i.e., $\mathcal{S}(\Gamma, \Gamma')$, called the *sandwich region*, is the set of points lying above all surface patches of Γ' and

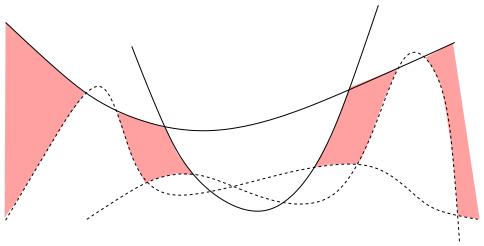

Figure 2.4: $\mathcal{S}(\Gamma, \Gamma')$; solid arcs are in Γ, and dashed arcs are in Γ'.

below all surface patches of Γ; see Figure 2.4. It can be shown that the combinatorial complexity of $\mathcal{S}(\Gamma, \Gamma')$ is proportional to the complexity of the overlay of the minimization diagram of Γ and the maximization diagram of Γ'. The results of Agarwal *et al.* [26] and of Koltun and Sharir [270] imply that the complexity of $\mathcal{S}(\Gamma, \Gamma')$ is $O(n^{2+\varepsilon})$ in 3-space, and $O(n^{3+\varepsilon})$ in 4-space. In general, the complexity of the overlay of the minimization diagram of Γ and the maximization diagram of Γ' may be larger than that of $\mathcal{S}(\Gamma, \Gamma')$. As an application, which also illustrates this discrepancy, consider the following example. Let $S = \{S_1, \ldots, S_n\}$ be a set of n spheres in $\mathbb{R}^3$. A line in $\mathbb{R}^3$ can be parameterized by four real parameters. We can therefore define the set of lines tangent to a sphere S_i and lying above (resp. below) S_i as a surface patch γ_i (resp. γ_i') in $\mathbb{R}^4$. Define $\Gamma = \{\gamma_i \mid 1 \leq i \leq n\}$ and $\Gamma' = \{\gamma_i' \mid 1 \leq i \leq n\}$. If the lines are parameterized carefully, Agarwal *et al.* [11] showed that $\mathcal{S}(\Gamma, \Gamma')$ is the set of lines intersecting all the spheres of S and that the combinatorial complexity of $\mathcal{S}(\Gamma, \Gamma')$ is $O(n^{3+\varepsilon})$, for any $\varepsilon > 0$. However, a construction of Pellegrini [330] implies that the combinatorial complexity of the overlay of the two diagrams can be $\Omega(n^4)$.

For the special case of arrangements of $(d-1)$-simplices in $\mathbb{R}^d$, the proofs of [325, 159] actually imply that the complexity of the sandwich region is also $O(n^{d-1}\alpha(n))$. A recent result of Koltun and Wenk [273] establishes a slightly weaker bound on the complexity of the overlay of minimization diagrams in arrangements of $(d-1)$-simplices in $\mathbb{R}^d$.

2.4 Single Cells

Lower envelopes are closely related to other substructures in arrangements, notably *cells* and *zones*. The lower envelope is a portion of the boundary of the bottommost cell of the arrangement, though the worst-case complexity of $L(\Gamma)$ can be larger than that of the bottommost cell of $\mathcal{A}(\Gamma)$. In two dimensions, it was shown in [221] that the complexity of a single face in an arrangement of n arcs, each pair of which intersect in at most s points, is $O(\lambda_{s+2}(n))$, and so has the same asymptotic bound as the complexity of the lower envelope of such a collection of arcs. The same holds more or less in higher dimensions, although it did take some time to get there: The complexity of a single cell in an arrangement of n surface patches in $\mathbb{R}^d$ satisfying the assumptions (A1) and (A2) is $O(n^{d-1+\varepsilon})$, for any $\varepsilon > 0$—see below. The Upper Bound Theorem implies that the complexity of a single cell in an arrangement of hyperplanes in $\mathbb{R}^d$ is $O(n^{\lfloor d/2 \rfloor})$, and the linearization technique described in Section 2.3 implies that the complexity of a single cell in an arrangement of n spheres is $O(n^{\lceil d/2 \rceil})$. However, the lower-bound construction for lower envelopes implies a lower bound of $\Omega(n^{d-1}\alpha(n))$ for the complexity of a single cell for arrangements of simplices.

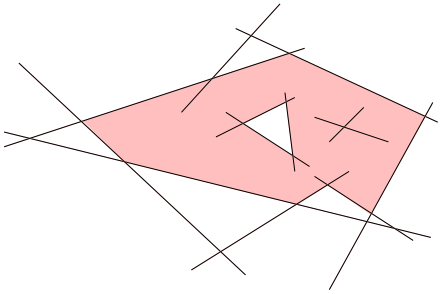

Figure 2.5: A single cell in an arrangement of segments.

Pach and Sharir [325] were the first to prove a subcubic upper bound on the complexity of a single cell in arrangements of triangles in $\mathbb{R}^3$. This bound was improved by Aronov and Sharir [60] to $O(n^{7/3})$, and subsequently to $O(n^2 \log n)$ [62]. The latter approach extends to higher dimensions; that is, the complexity of a single cell in an arrangement of n $(d-1)$-simplices in $\mathbb{R}^d$ is $O(n^{d-1} \log n)$. A simpler proof was given by Tagansky [369]. These approaches, however, do not extend to nonlinear surfaces even in $\mathbb{R}^3$.

Halperin [226, 227] proved near-quadratic bounds on the complexity of a single cell in arrangement of certain classes of n bivariate surface patches, which arise in motion-planning applications. Halperin and Sharir [238] proved a near-quadratic bound on the complexity of a single cell in an arrangement of the contact surfaces that arise in a rigid motion of a simple polygon amid polygonal obstacles in the plane, i.e., the surfaces that represent the placements of the polygon at which it touches one of the obstacles. The proof borrows ideas from the proof of Theorem 2.3.1.

A near-optimal bound on the complexity of a single cell in the arrangement of an arbitrary

collection of surface patches in $\mathbb{R}^3$ satisfying assumptions (A1) and (A2) was finally proved by Halperin and Sharir [236]. It took some time to extend to higher dimensions the main topological property that is needed in the proof. This has been done by Basu [77], who has thus obtained the general result:

Theorem 2.4.1 (Halperin and Sharir [236], Basu [77]). *Let Γ be a set of surface patches in $\mathbb{R}^d$ satisfying assumptions (A1) and (A2). For any $\varepsilon > 0$, the complexity of a single cell in $\mathcal{A}(\Gamma)$ is $O(n^{d-1+\varepsilon})$, for any $\varepsilon > 0$, where the constant of proportionality depends on ε, d, and on the maximum degree of the surface patches and of their boundaries.*

The proof (in $\mathbb{R}^3$, say) proceeds along the same lines as the proof of Theorem 2.3.1. However, the following two additional results are established to "bootstrap" the recurrences that the proof derives. Let C be the cell of $\mathcal{A}(\Gamma)$ whose complexity we want to bound.

(a) There are only $O(n^2)$ vertices v of the cell C that are locally x-extreme (that is, there is a neighborhood N of v and a connected component C' of the intersection of N with the interior of C, such that v lies to the left (in the x-direction) of all points of C', or v lies to the right of all these points).

(b) There are only $O(n^{2+\varepsilon})$ vertices on *popular faces* of C, that is, 2-faces f for which C lies locally near f on both sides of f.

Property (a) is proved by an appropriate decomposition of C into $O(n^2)$ subcells, in the style of a *Morse decomposition* of C (see [307]), so that each subcell has at most two points that are locally x-extreme in C. Property (b) is proved by applying the machinery of the proof of Theorem 2.3.1, where the quantity to be analyzed is the number of vertices of popular faces of C, rather than all inner vertices. Once these two results are available, the proof of Theorem 2.3.1 can be carried through, with appropriate modifications, to yield a recurrence for the number of vertices of C, whose solution is $O(n^{2+\varepsilon})$. We refer the reader to the original paper [236] for more details. Basu's paper [77] follows the same recurrence scheme, and provides an appropriate extension of (a) and (b) to higher dimensions.

The linearization technique in the previous section can be extended to bound the complexity of a cell as well, namely, one can prove the following.

Theorem 2.4.2. *Let Γ be a collection of n hypersurfaces in $\mathbb{R}^d$, of constant maximum degree b. If Γ admits a linearization of dimension k, then the combinatorial complexity of a cell of $\mathcal{A}(\Gamma)$ is $O(n^{\lfloor k/2 \rfloor})$, where the constant of proportionality depends on k, d, and b.*

2.5 Zones

Let Γ be a set of n surfaces in $\mathbb{R}^d$. The *zone* of a variety σ (not belonging to Γ), denoted as $zone(\sigma; \Gamma)$, is defined to be the set of d-dimensional cells in $\mathcal{A}(\Gamma)$ that intersect σ. The complexity of $zone(\sigma; \Gamma)$ is defined to be the sum of complexities of the cells of $\mathcal{A}(\Gamma)$ that belong to $zone(\sigma; \Gamma)$, where the complexity of a cell in $\mathcal{A}(\Gamma)$ is the number of faces of all dimensions that are contained in the closure of the cell.

The complexity of a zone was first studied by Edelsbrunner *et al.* [171]; see also [126]. The "classical" zone theorem [158, 173] asserts that the maximum complexity of the zone of a hyperplane in an arrangement of n hyperplanes in $\mathbb{R}^d$ is $\Theta(n^{d-1})$, where the constant of proportionality depends on d. The original proof given by Edelsbrunner *et al.* [171] had some technical problems. A correct, and simpler, proof was given by Edelsbrunner *et al.* [173]. Their technique is actually quite general and can also be applied to obtain several other interesting combinatorial bounds involving arrangements. For example, the proof by Aronov and Sharir for the complexity of a single cell in arrangements of simplices [62] used a similar approach. Other results based on this technique can be found in [4, 58, 59]. We therefore describe the technique, as applied in the proof of the zone theorem:

Theorem 2.5.1 (Edelsbrunner, Seidel, and Sharir [173]). *The maximum complexity of the zone of a hyperplane in an arrangement of n hyperplanes in $\mathbb{R}^d$ is $\Theta(n^{d-1})$.*

This result is easy to prove for $d = 2$; see Chapter 3 for details. For a set Γ of n hyperplanes in $\mathbb{R}^d$ and another hyperplane b, let $\tau_k(b;\Gamma)$ denote the total number of k-faces contained on the boundary of cells in $zone(b;\Gamma)$; each such k-face is counted once for each cell that it bounds. Let

$$\tau_k(n,d) = \max \tau_k(b;\Gamma)\,,$$

where the maximum is taken over all hyperplanes b and all sets Γ of n hyperplanes in $\mathbb{R}^d$. The maximum complexity of $zone(b;\Gamma)$ is at most $\sum_{k=0}^{d} \tau_k(n,d)$. Thus the following lemma immediately implies the upper bound in Theorem 2.5.1.

Lemma 2.5.2. *For each d and $0 \le k \le d$,*

$$\tau_k(n,d) = O(n^{d-1})\,,$$

where the constants of proportionality depend on d and k.

Proof: We use induction on d. As just noted, the claim holds for $d = 2$. Assume that the claim holds for all $d' < d$, let Γ be a set of n hyperplanes in $\mathbb{R}^d$, and let b be some other hyperplane. Without loss of generality, we can assume that the hyperplanes in $\Gamma \cup \{b\}$ are in general position. We define a *k-border* to be a pair (f, C), where f is a k-face incident to a d-dimensional cell C of $\mathcal{A}(\Gamma)$. Thus $\tau_k(b;\Gamma)$ is the total number of k-borders (f, C) for which $C \in zone(b;\Gamma)$.

We pick a hyperplane $\gamma \in \Gamma$ and count the number of all k-borders (f, C) in $zone(b;\Gamma)$ such that f is not contained in γ. If we remove γ from Γ, then any such k-border is contained in a k-border (f', C') of $zone(b;\Gamma \setminus \{\gamma\})$ (i.e., $f \subseteq f'$ and $C \subseteq C'$). Our strategy is thus to consider the collection of k-borders in $zone(b;\Gamma \setminus \{\gamma\})$ and to estimate the increase in the number of k-borders as we add γ back to Γ. Observe that we do not count k-borders that lie in γ.

Let $\Gamma|_\gamma = \{\gamma' \cap \gamma \mid \gamma' \in \Gamma \setminus \{\gamma\}\}$; the set $\Gamma|_\gamma$ forms a $(d-1)$-dimensional arrangement of $n-1$ hyperplanes within γ. Let (f, C) be a k-border of $zone(b;\Gamma \setminus \{\gamma\})$, and consider what happens to it when we reinsert γ. The following cases may occur:

$\gamma \cap C = \emptyset$: In this case the k-border (f, C) gives rise to exactly one k-border in $zone(b;\Gamma)$, namely itself.

$\gamma \cap C \neq \emptyset$, $\gamma \cap f = \emptyset$: Let γ^+ be the open halfspace bounded by γ that contains f, and let $C^+ = C \cap \gamma^+$. If C^+ intersects b, then (f, C) gives rise to one k-border in $zone(b;\Gamma)$, namely (f, C^+) (this is the case for the edge $f = e$ in Figure 2.6); otherwise it gives rise to no k-border in $zone(b;\Gamma)$.

$\gamma \cap f \neq \emptyset$: Let γ^+ and γ^- be the two open halfspaces bounded by γ and let $C^+ = C \cap \gamma^+$ and $C^- = C \cap \gamma^-$. If the closure of only one of C^+ and C^- intersects b, say, C^+, then (f, C) gives rise to only one k-border in $zone(b; \Gamma)$, namely $(f \cap \gamma^+, C^+)$ (this is the case for the edge $f = e'$ in Figure 2.6). If both C^+ and C^- intersect b, then (f, C) gives rise to two k-borders in $zone(b; \Gamma)$, namely $(f \cap \gamma^+, C^+)$ and $(f \cap \gamma^-, C^-)$ (this is the case for the edge $f = e''$ in Figure 2.6). In this case, however, we can charge uniquely this increase in the number of k-borders to $(f \cap \gamma, C \cap \gamma)$, which, as easily seen, is a $(k-1)$-border in $zone(b \cap \gamma; \Gamma|_\gamma)$.

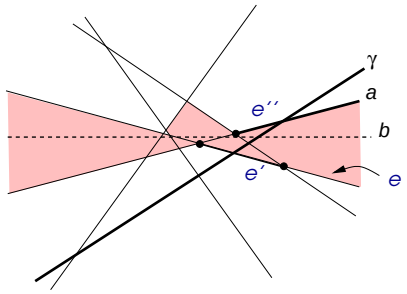

Figure 2.6: Inserting γ into $zone(b; \Gamma \setminus \{\gamma\})$.

If we repeat this process over all k-borders of $zone(b; \Gamma \setminus \{\gamma\})$, we obtain that the total number of k-borders (f, C) in $zone(b; \Gamma)$, for f not contained in γ, is at most

$$\begin{aligned} \tau_k(b; \Gamma \setminus \{\gamma\}) + \tau_{k-1}(b \cap \gamma; \Gamma|_\gamma) &\leq \tau_k(n-1, d) + \tau_{k-1}(n-1, d-1) \\ &= \tau_k(n-1, d) + O(n^{d-2}) , \end{aligned}$$

where the last inequality follows from the induction hypothesis. Repeating this analysis for all hyperplanes $\gamma \in \Gamma$, summing up the resulting bounds, and observing that each k-border of $zone(b; \Gamma)$ is counted exactly $n - d + k$ times, we obtain

$$\tau_k(n, d) \leq \frac{n}{n-d+k} \left(\tau_k(n-1, d) + O(n^{d-2}) \right) .$$

Edelsbrunner *et al.* [173] showed that this recurrence solves to $O(n^{d-1})$ for $k \geq 2$. Using Euler's formula for cell complexes, one can show that $\tau_k(n, d) = O(n^{d-1})$ for $k = 0, 1$ as well. This completes the proof of the theorem. For the lower bound, it is easily checked that the complexity of the zone of a hyperplane b in an arrangement of n hyperplanes in $\mathbb{R}^d$ in general position is $\Omega(n^{d-1})$. In fact, the complexity of the cross-section of the arrangement within b is already $\Omega(n^{d-1})$. □

The above technique can be extended to bound the quantity $\sum_{C \in \mathcal{A}(\Gamma)} |C|^2$, where Γ is a set of hyperplanes, C ranges over all d-dimensional cells of the arrangement, and $|C|$ denotes the number of lower-dimensional faces incident to C. For $d \leq 3$, an easy application of the zone theorem (see Edelsbrunner [158]) implies that $\sum_C |C|^2 = O(n^d)$; this bound is obviously tight if the lines or planes of Γ are in general position. For $d > 3$, the same application of the zone theorem yields only $\sum_C |C| f_C = O(n^d)$, where f_C is the number of hyperplanes of Γ meeting the boundary of C. Using the same induction scheme as in the proof of Theorem 2.5.1, Aronov *et al.* [58] showed that

$$\sum_{C \in \mathcal{A}(\Gamma)} |C|^2 = O(n^d \log^{\lfloor d/2 \rfloor - 1} n).$$

A different proof and a slight strengthening of the bound has recently been given by Aronov and Sharir [65]. It is believed that the right bound is $O(n^d)$. Note that such a result does not hold for arrangements of simplices or of surfaces because the complexity of single cell can be $\Omega(n^{d-1})$.

The zone theorem for hyperplane arrangements can be extended as follows.

Theorem 2.5.3 (Aronov, Pellegrini, and Sharir [59]). *Let Γ be a set of n hyperplanes in $\mathbb{R}^d$. Let σ be a p-dimensional algebraic variety of some fixed degree, or the relative boundary of any convex set with affine dimension $p+1$, for $0 \le p \le d$. The complexity of the zone$(\sigma;\Gamma)$ is $O(n^{\lfloor (d+p)/2 \rfloor} \log^{\beta} n)$, where $\beta = d + p \pmod 2$, and the bound is almost tight (up to the logarithmic factor) in the worst case.*

In particular, for $p = d-1$, the complexity of the zone is $O(n^{d-1}\log n)$, which is almost the same as the complexity of the zone of a hyperplane in such an arrangement.

The proof proceeds along the same lines of the inductive proof of Theorem 2.5.1. However, the removal and re-insertion of a hyperplane $\gamma \in \Gamma$ splits a face f of $zone(\sigma;\Gamma \setminus \{\gamma\})$ into two subfaces, both lying in $zone(\sigma;\Gamma)$, the charging scheme used in the proof of Theorem 2.5.1 becomes inadequate, because $f \cap \gamma$ need not belong to the zone of $\sigma \cap \gamma$ in the $(d-1)$-dimensional cross-section of $\mathcal{A}(\Gamma)$ along γ. What is true, however, is that $f \cap \gamma$ is a face incident to a *popular facet* of $zone(\sigma;\Gamma)$ along γ, that is, a facet $g \subseteq \gamma$ whose two incident cells belong to the zone. Thus the induction proceeds not by decreasing the dimension of the arrangement (as was done in the proof of Theorem 2.5.1), but by reapplying the same machinery to bound the number of vertices of popular facets of the original $zone(\sigma;\Gamma)$. This in turn requires similar bounds on the number of vertices of lower-dimensional popular faces. We refer the reader to Aronov *et al.* [59] for more details.

In general, the zone of a surface in an arrangement of n surfaces in $\mathbb{R}^d$ can be transformed to a single cell in another arrangement of $O(n)$ surface patches in $\mathbb{R}^d$. For example, Let Γ be a set of n $(d-1)$-simplices in $\mathbb{R}^d$, and let σ be a hyperplane. We split each $\gamma \in \Gamma$ into two polyhedra at the intersection of Δ and σ (if the intersection is nonempty), push these two polyhedra slightly away from each other, and, if necessary, retriangulate each polyhedron into a constant number of simplices. In this manner, we obtain a collection Γ' of $O(n)$ simplices, and all cells of the zone of σ in $\mathcal{A}(\Gamma)$ now fuse into a single cell of $\mathcal{A}(\Gamma')$. Moreover, by the general position assumption, the complexity of the zone of σ in Γ is easily seen to be dominated by the complexity of the new single cell of $\mathcal{A}(\Gamma')$. (The same technique has been used earlier in [165], to obtain a near-linear bound on the complexity of the zone of an arc in a two-dimensional arrangement of arcs.) Hence, the following theorem is an easy consequence of the result by Aronov and Sharir [62].

Theorem 2.5.4. *The complexity of the zone of a hyperplane in an arrangement of n $(d-1)$-simplices in $\mathbb{R}^d$ is $O(n^{d-1}\log n)$.*

Using a similar argument one can prove the following.

Theorem 2.5.5 (Halperin and Sharir [236], Basu [77]). *Let Γ be a collection of n surface patches in $\mathbb{R}^d$, satisfying assumptions (A1) and (A2). The combinatorial complexity of the zone in $\mathcal{A}(\Gamma)$ of an algebraic surface σ of some fixed degree is $O(n^{d-1+\varepsilon})$, for any $\varepsilon > 0$, where the constant of proportionality depends on ε, d, on the maximum degree of the given surfaces and their boundaries, and on the degree of σ.*

2.6 Levels

The *level* of a point $p \in \mathbb{R}^d$ in an arrangement $\mathcal{A}(\Gamma)$ of a set Γ of surface patches satisfying (A1)–(A3) is the number of surfaces of Γ lying vertically below p. For $0 \leq k < n$, we define *k-level* (resp. *$\leq k$-level*), denoted by $\mathcal{A}_k(\Gamma)$ (resp. $\mathcal{A}_{\leq k}(\Gamma)$), to be the closure of all points on the surfaces of Γ whose level is k (resp. at most k). A face of $\mathcal{A}_k(\Gamma)$ or $\mathcal{A}_{\leq k}(\Gamma)$ is a maximal connected portion of a face of $\mathcal{A}(\Gamma)$ consisting of points having a fixed subset of surfaces lying below them. For totally defined functions, any such face coincides with a face of $\mathcal{A}(\Gamma)$. Note that $\mathcal{A}_0(\Gamma)$ is the same as $L(\Gamma)$. If the surfaces in Γ are graphs of totally defined functions, then the level of all points on a face of $\mathcal{A}(\Gamma)$ is the same and $\mathcal{A}_k(\Gamma)$ is a connected monotone surface; otherwise $\mathcal{A}_k(\Gamma)$ may have discontinuities. See Figure 2.7 for an example of levels in arrangements of lines and segments.

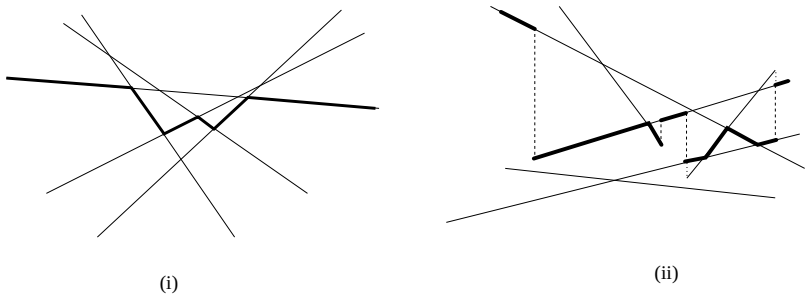

Figure 2.7: The 2-level in (i) an arrangement of lines, and (ii) in an arrangement of segments.

Levels in hyperplane arrangements in $\mathbb{R}^d$ are closely related to *k-sets* of point sets in $\mathbb{R}^d$. Let S be a set of n points in $\mathbb{R}^d$, and let S^* be the set of hyperplanes dual to S. A subset $A \subset S$ is called a *k-set* (resp. *$\leq k$-set*) if $|A| = k$ (resp. $|A| \leq k$) and A can be strictly separated from $S \setminus A$ by a hyperplane h. The level of point h^*, dual to h, in $\mathcal{A}(S^*)$ is k or $n-k$. The *k-set problem* is to bound the maximum number of k-sets of S (in terms of k and n). It is easy to see that the maximum number of k-sets in a set of n points in $\mathbb{R}^d$ is bounded by the maximum number of facets in the k-level and the $(n-k)$-level in an arrangement of n hyperplanes in $\mathbb{R}^d$.

Let $\psi_k(\Gamma)$ (resp. $\psi_{\leq k}(\Gamma)$) be the total number of faces in $\mathcal{A}_k(\Gamma)$ (resp. $\mathcal{A}_{\leq k}(\Gamma)$). Let $\mathbf{G}$ be a (possibly infinite) family of surfaces in $\mathbb{R}^d$ satisfying assumptions (A1)–(A3). We define $\psi_k(n, d, \mathbf{G}) = \max \psi_k(\Gamma)$ and $\psi_{\leq k}(n, d, \mathbf{G}) = \max \psi_{\leq k}(\Gamma)$, where the maximum in both cases is taken over all subsets $\Gamma \subseteq \mathbf{G}$ of size n. If $\mathbf{G}$ is not important or follows from the context, we will omit the argument $\mathbf{G}$.

The following theorem follows from a result by Clarkson and Shor [136].

Theorem 2.6.1 (Clarkson and Shor [136]). *Let $\mathbf{G}$ be an infinite family of surfaces satisfying assumptions (A1)–(A3). Then for any $0 \leq k < n - d$,*

$$\psi_{\leq k}(n, d, \mathbf{G}) = O\left((k+1)^d \kappa\left(\frac{n}{k+1}, d, \mathbf{G}\right)\right),$$

where $\kappa(n, d, \mathbf{G})$ is the maximum complexity of the lower envelope of n surfaces in $\mathbf{G}$.

Proof: Let $\Gamma \subset \mathbf{G}$ be a set of n surface patches satisfying assumptions (A1)–(A5). For a subset $X \subseteq \Gamma$ and an integer $0 \leq k \leq |X| - d$, let $V_k(X)$ denote the set of vertices at level k in $\mathcal{A}(\Gamma)$. As

is easily seen, $\psi_{\leq k}(\Gamma)$ is proportional to $\sum_{j=0}^{k} |V_j(\Gamma)|$, which we thus proceed to bound. We bound below only the number of vertices in the first k levels that lie in the interior of d surface patches; the other types of vertices are easier to analyze, and the same bound applies to them as well. We choose a random subset $R \subseteq \Gamma$ of size $r = \lfloor n/(k+1) \rfloor$ and bound the expected number of vertices in $V_0(R)$. A vertex $v \in V_j(\Gamma)$ is in $V_0(R)$ if and only if the d surfaces defining v are in R and none of the j surfaces of Γ lying below v are chosen in R, so the probability that $v \in V_0(R)$ is $\binom{n-j-d}{r-d}/\binom{n}{r}$. Hence, easy manipulation of binomial coefficients implies that

$$\begin{aligned}
\mathbf{E}\big[|V_0(R)|\big] &= \sum_{j=0}^{n-d} |V_j(\Gamma)| \frac{\binom{n-j-d}{r-d}}{\binom{n}{r}} \\
&\geq \sum_{j=0}^{k} |V_j(\Gamma)| \frac{\binom{n-j-d}{r-d}}{\binom{n}{r}} \\
&= \Omega\left(\frac{1}{(k+1)^d}\right) \sum_{j=0}^{k} |V_j(\Gamma)|.
\end{aligned}$$

Thus

$$\sum_{j=0}^{k} |V_j(\Gamma)| \leq c(k+1)^d \mathbf{E}\big[|V_0(R)|\big], \tag{2.6.1}$$

for some constant c. Since every vertex in $V_0(R)$ lies on the lower envelope of R, the assertion now follows from the definition of κ. □

Corollary 2.6.2. *(i)* $\psi_{\leq k}(n,d) = O((k+1)^{1-\varepsilon} n^{d-1+\varepsilon})$.

(ii) Let $\mathbf{H}$ *be the set of all hyperplanes in* $\mathbb{R}^d$. *Then* $\psi_{\leq k}(n,d,\mathbf{H}) = \Theta(n^{\lfloor d/2 \rfloor}(k+1)^{\lceil d/2 \rceil})$.

Proof: Part (i) follows from Theorems 2.3.1 and 2.6.1. Part (ii) follows from the fact that $\kappa(n,d,\mathbf{H}) = \Theta(n^{\lfloor d/2 \rfloor})$. □

There is even a tighter upper bound of $kn+1$ on the number of $\leq k$-sets of n points in the plane, $k \leq n/2$ [45, 329]; see also [204].

In contrast to these bounds on the complexity of $\leq k$-levels, which are tight or almost tight in the worst case, much less is known about the complexity of a single k-level, even for the simplest case of arrangements of lines in the plane. For example, Corollary 2.6.2(ii), for $d = 2$, implies that the complexity of an average level in an arrangement of lines in the plane is linear, but no upper bound that is even close is known. For a set Γ of n lines in the plane, Lovász [283] proved that $\psi_{\lfloor n/2 \rfloor}(\Gamma) = O(n^{3/2})$.[5] Erdős *et al.* [187] extended his argument to prove that $\psi_k(\Gamma) = O(n\sqrt{k+1})$. Since the original proof many different proofs have been proposed for obtaining the same bound on $\psi_k(\Gamma)$ [9, 223]. Goodman and Pollack [205] proved a similar bound on the maximum complexity of the k-level in an arrangement of pseudo-lines. Erdős *et al.*'s bound was slightly improved by Pach *et al.* [328] to $o(n\sqrt{k+1})$, using a rather complicated argument.

A major breakthrough in this direction was recently made by Dey, who obtained the following improvement.

[5] According to L. Lovász [283], the $(n/2)$-set problem was originally posed by A. Simmons, and E. Strauss had constructed a set of points in the plane in which the number of $(n/2)$-sets was $\Omega(n \log n)$.

Theorem 2.6.3 (Dey [150]). *Let Γ be a set of n lines in the plane. Then for any $0 \leq k < n$, $\psi_k(\Gamma) = O(n(k+1)^{1/3})$.*

Dey's proof is quite simple and elegant. It uses the following result on geometric graphs, which was independently proved by Ajtai *et al.* [39] and by Leighton [277].[6]

Lemma 2.6.4 (Crossing Lemma). *Let G be a geometric graph with n vertices and $m \geq 4n$ edges. Then there are $\Omega(m^3/n^2)$ pairs of edges in G whose relative interiors cross.*

Proof of Theorem 2.6.3: For simplicity we assume that n is even and prove the bound for $k = n/2$. We argue in the dual plane, where we have a set S of n points in general position and we wish to establish the asserted bound for the number of *halving segments* of S, where a halving segment is a straight segment connecting a pair of points $u, v \in S$ so that the line passing through u and v has exactly $(n/2) - 1$ points of S below it. Let H denote the set of halving segments.

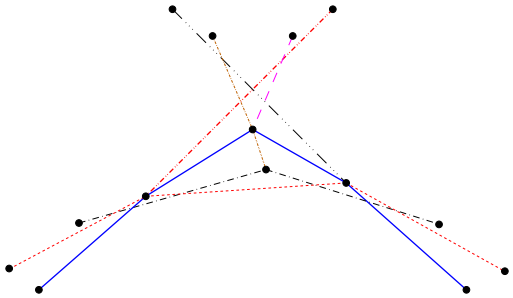

Figure 2.8: A set of 14 points with 14 halving segments, split into 7 convex x-monotone chains.

The segments in H are decomposed into $n/2$ convex x-monotone chains as follows. Let uv be an edge of H, with u lying to the right of v. We rotate the line that passes through u and v clockwise about v and stop as soon as the line overlaps another halving segment vw incident to v. It is easy to check that w lies to the right of v and that uvw is a right turn. We now rotate about w, and continue in this manner until our line becomes vertical. We apply the same procedure "backwards" by turning the line uv counterclockwise around u and keep iterating until the line becomes vertical. The halving segments that we have encountered during the whole process constitute one convex polygonal chain. By applying this procedure repeatedly, we obtain the desired decomposition of the entire H into convex x-monotone polygonal chains. Using the properties of halving segments proved by Lovász [283], we can conclude that the segments are partitioned into $n/2$ convex chains. (These convex chains are in a certain sense dual to the concave chains in the dual line arrangement that were defined by Agarwal *et al.* [9]; see also [223].)

The number of crossing points between two convex chains is bounded by the number of upper common tangents between the same two chains. Any line passing through two points of S is an upper common tangent of at most one pair of chains. Thus there are $O(n^2)$ crossings between the segments in H. By Lemma 2.6.4, any graph with n vertices and crossing number $O(n^2)$ has at most $O(n^{4/3})$ edges, so S has at most $O(n^{4/3})$ halving segments. A similar, slightly more detailed, argument proves the bound for arbitrary values of k. □

[6] A *geometric graph* $G = (V, E)$ is a graph drawn in the plane so that its vertices are points and its edges are straight segments connecting pairs of these points. A geometric graph need not be planar.

Tamaki and Tokuyama generalized Dey's proof to prove a similar bound on the complexity of the k-level in arrangements of pseudo-lines [370]. A simplified proof has recently been given in [360]. Combining the ideas from an old result of Welzl [382] with Dey's proof technique, one can obtain the following generalization. See also [50] for some other generalizations of Dey's result.

Corollary 2.6.5. *Let Γ be a set of n lines in the plane. Then for any $0 \leq k < n$, $0 < j < n - k$, we have*

$$\sum_{t=k}^{k+j} \psi_t(\Gamma) = O(n(k+1)^{1/3} j^{2/3}).$$

Bárány and Steiger proved a linear upper bound on the expected number of k-sets in a random planar point set [75]; the points are chosen uniformly from a convex region. Edelsbrunner *et al.* [175] proved that if S is a set of points in the plane so that the ratio of the maximum and the minimum distance in S is at most $c\sqrt{n}$ (a so-called *dense* set), then the number of k-sets in S is $O(c\sqrt{n}\psi_k(c\sqrt{n}))$. Applying Dey's result, the number of k-sets in a dense point set is $O(n^{7/6})$. Alt *et al.* [46] have proved that if the points in S lie on a constant number of pairwise disjoint convex curves, then the number of k-sets in S is $O(n)$.

Lower bounds. Erdős *et al.* [187] constructed, for any n and $0 \leq k < n$, a set Γ of n lines so that $\psi_k(\Gamma) = \Omega(n \log(k+1))$; see Edelsbrunner and Welzl for another construction that gives the same lower bound [176]. A fairly old unpublished result of Klawe *et al.* [265] constructs a set Γ of n *pseudo-lines* so that $\psi_{n/2}(\Gamma)$ has $n2^{\Omega(\sqrt{\log n})}$ vertices. In a fairly recent breakthrough development, Tóth [373] has established the same asymptotic bound for the case of lines (i.e., for the number of halving sets for a set of points). A recent work of Nivasch [316] simplifies the construction and improves the constant in the exponent.

Extensions; higher dimensions. The following question is related to the complexity of levels in arrangements of lines: Let Γ be a set of n lines in the plane. Let Π be an x-monotone polygonal path whose vertices (the points at which Π bends) are a subset of the vertices of $\mathcal{A}(\Gamma)$ and whose edges are contained in the lines of Γ. What is the maximum number of vertices in Π? Matoušek [286] proved that there exists a set Γ of n lines in the plane whose arrangement contains an x-monotone path with $\Omega(n^{5/3})$ vertices. This has been considerably strengthened recently by Balogh *et al.* [73], who present such a construction with $\Omega(n^2/2^{c\sqrt{\log n}})$ vertices, for some constant $c > 0$.

Agarwal *et al.* [9] proved a nontrivial upper bound on the complexity of the k-level in an arrangement of segments. Combining their argument with that of Dey's, one can prove that the maximum complexity of the k-level in a planar arrangement of n segments is $O(n(k+1)^{1/3}\alpha(n/(k+1)))$. Recent significant progress has been made on the analysis of the complexity of a single level in an arrangement of n arcs in the plane. Tamaki and Tokuyama [371] proved that the complexity of any level in an arrangement of parabolas, each with a vertical axis, is $O(n^{23/12})$. (Their bound actually applies to *pseudo-parabolas*, i.e., graphs of continuous, totally defined, univariate functions, each pair of which intersect at most twice.) This has later been improved by Agarwal *et al.* [25], and further by Chan [113, 114]. The best bound [114] is $\tilde{O}(n^{3/2})$ (where $\tilde{O}(\cdot)$ hides polylogarithmic factors). Chan's analysis yields many other nontrivial bounds. In particular, the complexity of a single level in an arrangement of arcs, each pair of which intersects at most s times, is $O(n^{2-\frac{1}{2s}})$, for $s \geq 3$ odd, and $O(n^{2-\frac{1}{2(s-1)}})$, for $s \geq 4$ even. In spite of this impressive progress, none of these bounds is known (or suspected) to be tight.

Problem 2.6.6. *(i) What is the maximum complexity of a level in an arrangement of n lines in the plane?*

(ii) What is the maximum complexity of a level in an arrangement of n x-monotone Jordan arcs, each pair of which intersect in at most s points, for some constant $s > 1$?

Bárány *et al.* [74] proved an $O(n^{3-\gamma})$ bound on the complexity of the k-level in arrangements of n planes in $\mathbb{R}^3$, for any k, for some absolute constant $\gamma > 0$. The bound was improved by Aronov *et al.* [54] and Eppstein [186] to $O(n^{8/3}\text{polylog}n)$, and then by Dey and Edelsbrunner [151] to $O(n^{8/3})$. The best bound known, due to Sharir *et al.* [361], is $O(n(k+1)^{3/2})$. Agarwal *et al.* [9] established a weaker bound on the complexity of the k-level for arrangements of triangles in $\mathbb{R}^3$. A nontrivial bound on the complexity of the k-level in an arrangement of n hyperplanes in $d > 3$ dimensions, of the form $O(n^{d-\varepsilon_d})$, for some constant ε_d that decreases exponentially with d, was obtained in [44, 378]. This has later been slightly improved to $O(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil - \varepsilon_d})$ in [9]. Recently, Matoušek *et al.* [299] have improved the bound in $\mathbb{R}^4$ to $O(n^{4-2/45})$, with an elementary proof (the proof of the general bounds uses tools from algebraic topology). Table 2.1 summarizes the known upper bounds on k-levels.

Objects	Bound	Source
Lines in $\mathbb{R}^2$	$O(n(k+1)^{1/3})$	[150]
Segments in $\mathbb{R}^2$	$O(n(k+1)^{1/3}\alpha(n/(k+1)))$	[9, 150]
Planes in $\mathbb{R}^3$	$O(n(k+1)^{3/2})$	[361]
Triangles in $\mathbb{R}^3$	$O(n^2(k+1)^{5/6}\alpha(n/(k+1)))$	[9]
Hyperplanes in $\mathbb{R}^4$	$O(n^{4-2/45}$	[299]
Hyperplanes in $\mathbb{R}^d$	$O(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil - \varepsilon_d})$	[378]
Pseudo-parabolas in $\mathbb{R}^2$	$\tilde{O}(n^{3/2})$	[114]
s-intersecting curve segments in $\mathbb{R}^2$, $s \geq 3$ odd	$O(n^{1-\frac{1}{2s}})$	[114]
s-intersecting curve segments in $\mathbb{R}^2$, $s \geq 4$ even	$O(n^{1-\frac{1}{2(s-1)}})$	[114]

Table 2.1: Upper bounds on k-levels.

2.7 Many Cells and Related Problems

In the previous two sections we bounded the complexity of families of d-dimensional cells in $\mathcal{A}(\Gamma)$ that satisfied certain conditions (e.g., cells intersected by a surface, the cells of level at most k). We can ask a more general question: *What is the complexity of any m distinct cells in $\mathcal{A}(\Gamma)$?* A single cell in an arrangement of lines in the plane can have n edges, but can the total complexity of m cells in an arrangement of lines be $\Omega(mn)$? This is certainly false for $m = \Omega(n^2)$.

We can also formulate the above problem as follows: Let P be a set of m points and Γ a set of n surfaces in $\mathbb{R}^d$ satisfying assumptions (A1) and (A2). Define $\mathcal{C}(P, \Gamma)$ to be the set of cells in $\mathcal{A}(\Gamma)$ that contain at least one point of P. Define $\mu(P, \Gamma) = \sum_{C \in \mathcal{C}(P,\Gamma)} |C|$ and $\mu(m, n, \mathbf{G}) = \max \mu(P, \Gamma)$, where the maximum is taken over all sets P of m points and over all sets Γ of n surfaces in a given class $\mathbf{G}$.

Let $\mathbf{L}$ be the set of all lines in the plane. Canham [102] proved that $\mu(m,n,\mathbf{L}) = O(m^2+n)$, from which it easily follows that $\mu(m,n,\mathbf{L}) = O(m\sqrt{n}+n)$. Although this bound is optimal for $m \leq \sqrt{n}$, it is weak for larger values of m. Clarkson *et al.* [133] proved that $\mu(m,n,\mathbf{L}) = \Theta(m^{2/3}n^{2/3}+n)$. Their technique, based on random sampling, is general and constructive. It has led to several important combinatorial and algorithmic results on arrangements [133, 219, 220]. For example, following a similar, but considerably more involved, approach, Aronov *et al.* [55] proved that $\mu(m,n,\mathbf{E}) = O(m^{2/3}n^{2/3} + m\log n + n\alpha(n))$, where $\mathbf{E}$ is the set of all line segments in the plane. An improved bound can be attained if the number of vertices in the arrangement of segments is small. Hershberger and Snoeyink [248] proved an $O(m^{2/3}n^{2/3}+n)$ upper bound on the complexity of m distinct cells in the arrangements of n segments in the plane where the segments satisfy certain additional conditions.

The old bound of Clarkson *et al.* [133] on the complexity of m distinct cells in an arrangement of n circles (see Table 2.2 below) has recently been improved by Agarwal *et al.* [13] to $O(m^{2/3}n^{2/3} + m^{6/11}n^{9/11}\text{polylog}n + n\log n)$ (this is slightly better than the bound in [13, Theorem 4.4]). Still, when m is relatively small, no tight bounds are known.

Problem 2.7.1. *What is the maximum complexity of m distinct cells in an arrangement of n circles in the plane?*

Objects	Complexity	Source
Lines in $\mathbb{R}^2$	$\Theta(m^{2/3}n^{2/3}+n)$	[133]
Segments in $\mathbb{R}^2$	$O(m^{2/3}n^{2/3}+n\alpha(n)+n\log m)$	[55]
Unit circles in $\mathbb{R}^2$	$O(m^{2/3}n^{2/3}\alpha^{1/3}(n)+n)$	[133]
Circles in $\mathbb{R}^2$	$O(m^{2/3}n^{2/3}+m^{6/11}n^{9/11}\text{polylog}n+n\log n)$	[13]
Arcs in $\mathbb{R}^2$	$O(\sqrt{m}\lambda_q(n))$	[165]
Planes in $\mathbb{R}^3$	$\Theta(m^{2/3}n+n^2)$	[8]
Hyperplanes in $\mathbb{R}^d$, $d \geq 4$	$O(m^{1/2}n^{d/2}\log^\beta n)$ $\beta = (\lfloor d/2\rfloor - 2)/2$	[65]

Table 2.2: Complexity of many cells.

Complexity of many cells in hyperplane arrangements in higher dimensions was first studied by Edelsbrunner and Haussler [169]. Let $\mathbf{H}$ be the set of all hyperplanes in $\mathbb{R}^d$. They proved that the maximum number of $(d-1)$-dimensional faces in m distinct cells in an arrangement of n hyperplanes in $\mathbb{R}^d$ is $O(m^{1/2}n^{d/2}+n^{d-1})$. Refining an argument by Edelsbrunner *et al.* [168], Agarwal and Aronov [8] improved this bound to $O(m^{2/3}n^{d/3}+n^{d-1})$. By a result of Edelsbrunner and Haussler [169], this bound is tight in the worst case. Aronov and Sharir [65], slightly improving upon an earlier result in [58], proved that $\mu(m,n,\mathbf{H}) = O(m^{1/2}n^{d/2}\log^\beta n)$, where $\beta = (\lfloor d/2\rfloor - 2)/2$. Aronov *et al.* [58] also proved several lower bounds on $\mu(m,n,\mathbf{H})$: For odd values of d and $m \leq n$, $\mu(m,n,\mathbf{H}) = \Theta(mn^{\lfloor d/2\rfloor})$; for m of the form $\Theta(n^{d-2k})$ where $0 \leq k \leq \lfloor d/2\rfloor$ is an integer, $\mu(m,n,\mathbf{H}) = \Omega(m^{1/2}n^{\lfloor d/2\rfloor})$; and for arbitrary values of m, $\mu(m,n,\mathbf{H}) = \Omega(m^{1/2}n^{d/2-1/4})$. Agarwal [4], Guibas *et al.* [216], and Halperin and Sharir [234] obtained bounds on "special" subsets of cells in hyperplane arrangements.

A problem closely related to, but somewhat simpler than, the many-cells problem is the *incidence problem*. Here is a simple instance of this problem: Let Γ be a set of n lines and P a set of m points in the plane. Define $\mathcal{I}(P,\Gamma) = \sum_{\ell\in\Gamma} |P\cap\ell|$; set $\mathcal{I}(m,n) = \max \mathcal{I}(P,\Gamma)$, where the maximum is taken over all sets P of m distinct points and over all sets Γ of n distinct lines in the plane.

Of course, this problem is interesting only when the lines in Γ are in highly degenerate position. If $n = m^2 + m + 1$, then a finite projective plane of order m has n points and n lines and each line contains $m + 1 = \Omega(n^{1/2})$ points, so the number of incidences between n points and n lines is $\Omega(n^{3/2})$. Szemerédi and Trotter [366] proved that such a construction is impossible in $\mathbb{R}^2$. In a subsequent paper, Szemerédi and Trotter [367] proved that $\mathcal{I}(m, n) = O(m^{2/3}n^{2/3} + m + n)$. We will devote a full chapter (Chapter 4) to the topic of incidences and of many related problems and results.

2.8 Generalized Voronoi Diagrams

An interesting application of the new bounds on the complexity of lower envelopes is to generalized Voronoi diagrams in higher dimensions. Let S be a set of n pairwise-disjoint convex objects in $\mathbb{R}^d$, each of constant description complexity, and let ρ be some metric. For a point $\mathbf{x} \in \mathbb{R}^d$, let $\Phi(\mathbf{x})$ denote the set of objects nearest to $\mathbf{x}$, i.e.,

$$\Phi(\mathbf{x}) = \{s \in S \mid \rho(\mathbf{x}, s) \leq \rho(\mathbf{x}, s') \; \forall s' \in S\}.$$

The *Voronoi diagram* $\mathrm{Vor}_\rho(S)$ of S under the metric ρ (sometimes also simply denoted as $\mathrm{Vor}(S)$) is a partition of $\mathbb{R}^d$ into maximal connected regions C of various dimensions, so that the set $\Phi(\mathbf{x})$ is the same for all $\mathbf{x} \in C$. Let γ_i be the graph of the function $x_{d+1} = \rho(\mathbf{x}, s_i)$. Set $\Gamma = \{\gamma_i \mid 1 \leq i \leq n\}$. Edelsbrunner and Seidel [172] observed that $\mathrm{Vor}_\rho(S)$ is the minimization diagram of Γ.

In the classical case, in which ρ is the Euclidean metric and the objects in S are singletons (points), the graphs of these distance functions can be replaced by a collection of n hyperplanes in $\mathbb{R}^{d+1}$, using the linearization technique, without affecting the minimization diagram. Hence the maximum possible complexity of $\mathrm{Vor}(S)$ is $O(n^{\lceil d/2 \rceil})$, which actually can be achieved (see, e.g., [266, 351]). In more general settings, though, this reduction is not possible. Nevertheless, the bounds on the complexity of lower envelopes imply that, under reasonable assumption on ρ and on the objects in S, the complexity of the diagram is $O(n^{d+\varepsilon})$, for any $\varepsilon > 0$. While this bound is nontrivial, it is conjectured to be too weak. For example, this bound is near-quadratic for planar Voronoi diagrams, but the complexity of almost every planar Voronoi diagram is only $O(n)$, although there are certain distance functions for which the corresponding planar Voronoi diagram can have quadratic complexity [68].

In three dimensions, the above-mentioned bound for point sites and Euclidean metric is $\Theta(n^2)$. It has been a long-standing open problem to determine whether a similar quadratic or near-quadratic bound holds in $\mathbb{R}^3$ for more general objects and metrics (here the new results on lower envelopes only give an upper bound of $O(n^{3+\varepsilon})$). The problem stated above calls for improving this bound by roughly another factor of n. Since we are aiming for a bound that is two orders of magnitude better than the complexity of $\mathcal{A}(\Gamma)$, it appears to be a considerably more difficult problem than that of lower envelopes. The only hope of making progress here is to exploit the special structure of the distance functions $\rho(x, s)$.

Fortunately, some progress on this problem was made recently. It was shown by Chew *et al.* [132] that the complexity of the Voronoi diagram is $O(n^2\alpha(n)\log n)$ for the case where the objects of S are *lines* in $\mathbb{R}^3$ and the metric ρ is a *convex distance function* induced by a convex polytope with a constant number of facets (see [132] for more details). Note that such a distance function is not necessarily a metric, because it will fail to be symmetric if the defining polytope is not centrally symmetric. The L_1 and L_∞ metrics are special cases of such distance functions. The best known

lower bound for the complexity of the diagram in this special case is $\Omega(n^2\alpha(n))$. Recently, Koltun and Sharir [271] have extended this result to an arbitrary collection of pairwise disjoint *line segments* or *triangles*; the bound for the case of line segments is $O(n^2\alpha(n)\log n)$, and, for the case of triangles, is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$.

Dwyer [156] has shown that the expected complexity of the Voronoi diagram of a set of n random lines in $\mathbb{R}^3$ is $O(n^{3/2})$. Boissonnat *et al.* [93] have shown that the maximum complexity of the L_1-Voronoi diagram of a set of n points in $\mathbb{R}^3$ is $\Theta(n^2)$. Finally, it is shown in [368] that the complexity of the three-dimensional Voronoi diagram of point sites under a general polyhedral convex distance function (induced by a polytope with $O(1)$ facets) is $O(n^2\log n)$.

In spite of all this progress, the case of lines in $\mathbb{R}^3$ under the Euclidean metric is still open:

Problem 2.8.1. *Is the complexity of the Voronoi diagram of a set S of n lines under the Euclidean metric in $\mathbb{R}^3$ close to n^2?*

A partial affirmative answer has recently ben given by Koltun and Sharir [269], for the special case where the lines have a constant number of distinct orientations.

An interesting special case of these problems involves *dynamic Voronoi diagrams* for moving points in the plane. Let S be a set of n points in the plane, each moving along some line at some fixed velocity. The goal is to bound the number of combinatorial changes of the Euclidean $\mathrm{Vor}(S)$ over time. This dynamic Voronoi diagram can easily be transformed into a three-dimensional Voronoi diagram, by adding the time t as a third coordinate. The points become lines in $\mathbb{R}^3$, and the metric is a distance function induced by a horizontal disk (that is, the distance from a point $p(x_0, y_0, t_0)$ to a line ℓ is the Euclidean distance from p to the point of intersection of ℓ with the horizontal plane $t = t_0$). Here too the open problem is to derive a near-quadratic bound on the complexity of the diagram. Cubic or near-cubic bounds are known for this problem, even under more general settings [199, 218, 357], but subcubic bounds are known only in some very special cases [131].

Problem 2.8.2. *Is the complexity of the dynamic Euclidean Voronoi diagram of a set S of n moving points in the plane close to n^2? Is this true at least in the special case where the points have linear trajectories with constant common speed?*

Next, consider the problem of bounding the complexity of generalized Voronoi diagrams in higher dimensions. As mentioned above, when the objects in S are n points in $\mathbb{R}^d$ and the metric is Euclidean, the complexity of $\mathrm{Vor}(S)$ is $O(n^{\lceil d/2\rceil})$. As d increases, this becomes significantly smaller than the naive $O(n^{d+1})$ bound or the improved bound, $O(n^{d+\varepsilon})$, obtained by viewing the Voronoi diagram as a lower envelope in $\mathbb{R}^{d+1}$. The same bound of $O(n^{\lceil d/2\rceil})$ has been obtained in [93] for the complexity of the L_∞-diagram of n points in $\mathbb{R}^d$ (it was also shown that this bound is tight in the worst case). It is thus tempting to conjecture that the maximum complexity of generalized Voronoi diagrams in higher dimensions is close to this bound. Unfortunately, this was shown by Aronov to be false [52], by presenting a lower bound of $\Omega(n^{d-1})$. The sites used in this construction are lower-dimensional flats, and the distance is either Euclidean or a polyhedral convex distance function. (It is interesting that the lower bound in Aronov's construction depends on the affine dimension $0 \le k \le d-2$ of the sites: It is $\Omega(\max\{n^{k+1}, n^{\lceil (d-k)/2\rceil}\})$.) Thus, for $d = 3$, this lower bound does not contradict the conjecture made above, that the complexity of generalized Voronoi diagrams should be at most near-quadratic in this case. Also, in higher dimensions, the conjecture mentioned above is still not refuted when the sites are singleton points. Finally, for the general case,

the construction by Aronov still leaves a gap of roughly a factor of n between the known upper and lower bounds.

Another special case of generalized Voronoi diagrams are *power diagrams*. Here we have a set $\mathcal{B}$ of n balls in $\mathbb{R}^d$, and the distance from a point $x \in \mathbb{R}^d$ to a ball $B \in \mathcal{B}$ is power(x, B) = length of a tangent from x to B; that is, if B is centered at p and has radius r, then power$(x, B) = d^2(x, p) - r^2$. It is an easy exercise to see that power diagrams in $\mathbb{R}^d$ are minimization diagrams of lower envelopes of hyperplanes in $\mathbb{R}^{d+1}$; see, e.g., Boissonnat and Yvinec [94]. Hence their complexity has the same asymptotic bound $O(n^{\lceil d/2 \rceil})$ as standard Voronoi diagrams.

Another special case are *additive weight Voronoi diagrams*: Let P be a set of n points in $\mathbb{R}^d$, such that each $p \in P$ is associated with a weight $w_p \in \mathbb{R}$. The additively weighted distance from a point $x \in \mathbb{R}^d$ to $p \in P$ is $d_w(x, p) := d(x, p) + w_p$, and the additive weight Voronoi diagram is the partition of d-space into Voronoi cells, where the cell $V_w(p)$ of a point $p \in P$ is

$$V_w(p) = \{x \in \mathbb{R}^d \mid d_w(x, p) \leq d_w(x, q) \text{ for } q \in P\}.$$

Putting $\max_{p \in P} w_p$, we can interpret this diagram as the standard Euclidean Voronoi diagram of the n (not necessarily disjoint) balls $B(p, a - w_p)$, for $p \in P$. An elegant construction (see [94, Theorem 18.3.1]) shows that an additive weight Voronoi diagram in $\mathbb{R}^d$ can be embedded as a substructure of a power diagram in $\mathbb{R}^{d+1}$. Hence the complexity of an additive weight Voronoi diagram of n points (or, rather, the standard diagram of balls) in $\mathbb{R}^d$ is $O(n^{\lfloor d/2 \rfloor + 1})$.

2.9 Union of Geometric Objects

Let $\mathcal{K}$ be a family of n bodies in $\mathbb{R}^d$, whose boundaries satisfy assumptions (A1)–(A2), and let $\mathcal{U}$ denote the union of $\mathcal{K}$. In general, we can think of the arrangement $\mathcal{A}(\mathcal{K})$ of the boundaries of the objects of $\mathcal{K}$, and observe that the boundary of $\mathcal{U}$ is a substructure of $\mathcal{A}(\mathcal{K})$—each face of $\partial\mathcal{U}$ is a face of $\mathcal{A}(\mathcal{K})$. The combinatorial complexity of $\mathcal{U}$ is the number of these boundary faces, of all dimensions. The goal is to look for special properties of $\mathcal{K}$ that would ensure that the complexity of $\mathcal{U}$ is small. In the worst case, that complexity might be proportional to that of the entire arrangement $\mathcal{A}(\mathcal{K})$ and thus might be $\Theta(n^d)$. A simple example is a collection of long and thin triangles in $\mathbb{R}^2$, where it is easy to arrange them so that the complexity of their union is $\Theta(n^2)$, as illustrated in Figure 2.9.

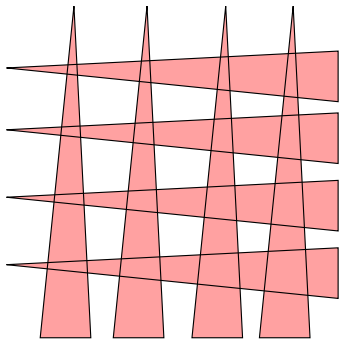

Figure 2.9: Union of Jordan regions.

However, in many favorable cases, the complexity of $\mathcal{U}$ is considerably smaller. Here is an easy example. Let $\mathcal{K}$ be a collection of n halfspaces in $\mathbb{R}^d$, each bounded by a hyperplane. Then the union of $\mathcal{K}$ is the complement of a convex polyhedron with at most n facets. By the celebrated Upper Bound Theorem [300], this complexity is only $O(n^{\lfloor d/2 \rfloor})$. In this section we will see many additional examples of this phenomenon.

Pseudodisks in the plane. A family $\mathcal{K}$ of simply connected regions in the plane is said to be a family of pseudodisks if the boundaries of each pair of objects in $\mathcal{K}$ intersect at most twice. Disks are obvious examples of pseudodisks, but there are many other examples. An interesting case is that of *Minkowski sums* of a fixed convex object B with n pairwise disjoint convex sets $A_1, A_2, \ldots, A_n$. Recall that the Minkowski sum $A \oplus B$ of two sets A, B is the set of pointwise sums of their elements:

$$A \oplus B = \{x + y \mid x \in A,\ y \in B\}.$$

Put $K_i = A_i \oplus B$. If ∂K_i and ∂K_j intersected at four or more points, then they would have to have at least four distinct common outer tangents. As is well known, if a line at orientation θ supports $A \oplus B$ at a point z, then there are points $x \in \partial A$, $y \in \partial B$, such that $z = x + y$, and the line with orientation θ that supports A (resp., B) passes through x (resp., y). It follows that each common outer tangent to ∂K_i and ∂K_j can be translated to become a common outer tangent to A_i and A_j. However, since these sets are disjoint, they have only two common outer tangents, showing that K_i and K_j are indeed pseudodisks.

This situation arises in the motion planning problem for a convex rigid robot B that is free to translate in the plane amid a collection of convex pairwise disjoint obstacles $A_1, \ldots, A_n$, which it must avoid. Assume that B is initially given at a placement that contains the origin o, and represent any translated position of B by the point $z \in \mathbb{R}^2$ to which o is translated (that is, this placement is $B + z$). Then $B + z$ intersects an obstacle A_i if there are points $x \in A_i$, $y \in B$, such that $x = y + z$, or $z = x - y$. Hence, denoting by $-B$ the reflection $\{-y \mid y \in B\}$ of B, it follows that $B + z$ collides with A_i if and only if $z \in A_i \oplus (-B)$. In other words, the *free configuration space* $\mathcal{F}$ of B, namely, the locus of all translations of B at which it does not collide with any obstacle, is the complement of the union $\mathcal{U} = \bigcup_i (A_i \oplus (-B))$.

Hence, to solve this translational motion planning problem, we need to compute $\mathcal{U}$. Before doing so, we need to determine its combinaorial complexity: How many vertices and edges of the arrangement $\mathcal{A}(\mathcal{K})$, where $\mathcal{K} = \{A_i \oplus (-B)\}_{i=1}^n$, appear on $\partial\mathcal{U}$? The surprising answer is that this complexity is linear in n; it was shown by Kedem *et al.* [259], one of the oldest results concerning unions of geometric objects:

Theorem 2.9.1 (Kedem et al. [259]). *The number of vertices of the boundary of the union of n pseudodisks in the plane is at most $6n - 12$, for $n \geq 3$. This bound is sharp.*

Proof: Let $\mathcal{K}$ be a family of n pseudodisks, and let $\mathcal{U}$ denote its union. The proof is an application of a beautiful result of hanani and Tutte [130, 374], stating that a graph drawn in the plane, so that each pair of its edges that do not have a common endpoint cross an even number of times, is planar. We construct such a graph G: We may assume that the boundary of each $K \in \mathcal{K}$ appears on $\partial\mathcal{U}$; indeed, by removing all other sets from $\mathcal{K}$, we do not lose any feature of $\partial\mathcal{U}$. For each $K \in \mathcal{K}$, choose an arbitrary fixed point $p_K \in \partial K \cap \partial\mathcal{U}$, and make these points the vertices of G. If $K, K' \in \mathcal{K}$ are such that their boundaries intersect at a point q that is a vertex of $\mathcal{U}$, draw an edge that connects p_K to $p_{K'}$ by following ∂K from p_K to q and then following $\partial K'$ from q to $p_{K'}$. See Figure 2.10.

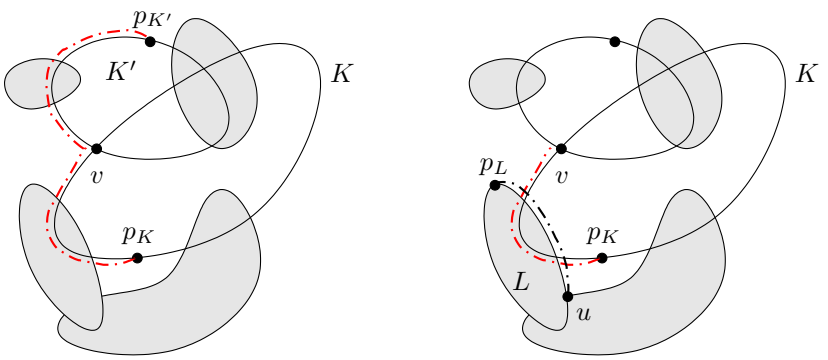

Figure 2.10: The union of pseudodisks via a planarity argument.

(As stated, edges of G overlap each other, so we perturb them slightly to make sure this does not happen.)

We claim that each pair of edges of G intersect an even number of times. In fact, each pair of "half-edges", one connecting some point p_K to an intersection point u, and the other connecting another point p_L to another intersection point v, are either disjoint or intersect exactly twice. This is easy to show, because each of these half-edges starts and ends on the boundary of the union, and the boundaries of the two corresponding pseudodisks intersect at most twice; see Figure 2.10. Hence, by the Hanani-Tutte's theorem, G is planar, and thus has at most $3n - 6$ edges, implying that $\partial\mathcal{U}$ can have at most twice that many vertices. □

3-Intersecting curves. A natural question that arises is what happens if any two boundaries intersect in at most three points. Notice that in general this question is meaningless, since any two closed curves must intersect in an even number of points (assuming nondegenerate configurations). To make the problem interesting, let Γ be a collection of n Jordan arcs, such that both endpoints of each arc $\gamma_i \in \Gamma$ lie on the x-axis, and such that K_i is the region between γ_i and the x-axis. Edelsbrunner *et al.* [164] have shown that the maximum combinatorial complexity of the union K is $\Theta(n\alpha(n))$. The upper bound requires a rather sophisticated analysis of the topological structure of K, and the lower bound follows from the construction by Wiernik and Sharir for lower envelopes of segments [384].

Fat objects in the plane. Define a convex planar body K to be *α-fat*, if there exist two disks D, D', such that $D \subseteq K \subseteq D'$, and the ratio between the radii of D', D is at most α. Let $\mathcal{K}$ be a family of n convex α-fat objects in the plane, whose boundaries satisfy assumptions (A1)–(A2). Intuitively, we should expect their union $\mathcal{U}$ to have small complexity, because the simple examples where the union has quadratic complexity involve long and skinny objects.

This has been shown in a series of papers. The simplest example of fat objects are fat triangles, where fatness can also be defined in terms of their angles, so that a triangle is α-fat (with a slight "abuse" of the parameter α) if each of its angles is at least α. Matoušek *et al.* [296] have shown

that the union of n α-fat triangles has only $O(n)$ holes (connected components of its complement) and its complexity is $O(n \log\log n)$ (and can be $\Omega(n\alpha(n))$ in the worst case). The constants of proportionality depend on α, where the best bound, due to Pach and Tardos [327], is $O(\frac{1}{\alpha}\log\frac{1}{\alpha})$. Earlier works by Alt *et al.* [47] and by Efrat *et al.* [181] consider the case of *fat wedges* (whose angle is at least some $\alpha > 0$), and show that the complexity of the union of such wedges is linear.

For arbitrary convex fat objects, Efrat and Sharir [183] have shown that the complexity of the union is $O(n^{1+\varepsilon})$, for any $\varepsilon > 0$, where the constant of proportionality depends on α, ε, and the degree b in (A2).

Several refinements of this result have also been established. Call a (not necessarily convex) object K κ-curved if for each point $p \in \partial K$ there exists a disk D that passes through p, is contained in K, and its radius is κ times the diameter of K. Informally, κ-round objects cannot have convex corners, nor can they have very thin bottlenecks. Efrat and Katz [180] have shown that the complexity of the union of n κ-round objects is is $O(\lambda_s(n)\log n)$ complexity, where s is a constant that depends on the description complexity of the input objects. This has further been extended by Efrat [179], to so-called (α,β)-*covered objects*, which are objects K with the property that for each point $p \in \partial K$ there exists an α-fat triangle T that has p as a vertex, is contained in K, and each of its edges is at least β times the diameter of K. Efrat has shown that if $\mathcal{K}$ is a collection of n (α,β)-covered objects, each pair of whose boundaries intersect in at most $s = O(1)$ points, then the complexity of their union is $O(\lambda_{s+2}(n)\log^2 n \log\log n)$, where s is the maximum number of intersections between the boundaries of any pair of the given objects.

Regular vertices and related problems. The analysis of the complexity bounds just cited requires as an initial but important substep an analysis of the number of *regular vertices* of the union: these are vertices of the union that are incident to two boundaries that intersect exactly twice. In fact, the analysis can only handle directly the irregular vertices of the union. Motivated by this problem, Pach and Sharir [326] have shown that, for an arbitrary collection of n convex regions, each pair of whose boundaries cross in a constant number of points, one has $R \le 2I + 6n - 12$, where R (resp. I) is the number of regular (resp. irregular) vertices on the boundary of the union. This result has been used in [183] and the other works to obtain near-linear bounds.

Nevertheless, regular vertices are interesting in their own right, and some additional results concerning them have been obtained by Aronov *et al.* [56]. First, if there are only regular vertices (i.e., every pair of boundaries intersect at most twice), then the inequality obtained by [326] implies that the complexity of the union in this case is at most $6n - 12$, so the result by Pach and Sharir extends the older results of Kedem *et al.* [259]. In general, though, I can be quadratic, so the above inequality only yields a quadratic upper bound on the number of regular vertices of the union. However, it was shown in [56] that in many cases R is subquadratic. This is the case when the given regions are such that every pair of their boundaries cross at most a constant number of times. If in addition all the regions are convex, the upper bound is close to $O(n^{3/2})$. recent work in progress aims at improving this bound.

As a final, somewhat unrelated result, we mention the work of Aronov and Sharir [63], who proved that the complexity of the union of n convex polygons in $\mathbb{R}^2$ with a total of s vertices is $O(n^2 + s\alpha(n))$.

Union of objects in higher dimensions. Compared with the extensive research on the union of objects in $\mathbb{R}^2$, the situation in three (and higher) dimensions leaves something (i.e., a lot of open prob-

lems) to be desired. Of course, the union of n thin plate-like objects that form a three-dimensional grid can have complexity $\Omega(n^3)$. However, as in the plane, it is important, and motivated by a line of practical applications, to consider realistic input models, and in particular to study the complexity of the union of *fat* objects. A prevailing conjecture is that the complexity of the union of fat objects (say, under the definition of α-fatness, requiring the ratio between the radii of the smallest enclosing ball and the largest inscribed ball of each object to be at most α) is near-quadratic. Such a bound has however proved quite elusive to obtain for general fat objects, and this has been recognized as one of the major open problems in computational geometry [149, Problem 4].

Let us first consider the case of convex polyhedra. Aronov *et al.* [66] (see also [61]) proved that the complexity of the union of n convex polyhedra in $\mathbb{R}^3$ with a total of s faces is $O(n^3 + sn \log s)$. This bound, which is nearly tight in the worst case, is interesting, because it is cubic only in the number of polyhedra, and not in the overall number of their facets.

The goal is of course is to obtain subcubic (ideally nearly-quadratic) bounds on the complexity of the union in certain special cases. Such bounds exist, though not in abundance. The complexity of the union of axis-parallel cubes is $O(n^2)$, and it drops to $O(n)$ if the cubes have the same size [93]. If the cubes are not axis-parallel but have equal (or "almost equal") size, the complexity of their union is $O(n^{2+\varepsilon})$ for any $\varepsilon > 0$, as shown recently by Pach *et al.* [324]. A general sub-cubic bound on the complexity of the union of cubes in three dimensions is currently not known. Even the case of (congruent or not) regular simplices is still open.

Minkowski sums in 3-space. Unions of objects also arise as subproblems in the study of generalized Voronoi diagrams, as follows. Let S and ρ be as in the previous section (say, for the 3-dimensional case). Let K denote the region consisting of all points $x \in \mathbb{R}^3$ whose smallest distance from a site in S is at most r, for some fixed parameter $r > 0$. Then $K = \bigcup_{s \in S} B(s, r)$, where $B(s, r) = \{x \in \mathbb{R}^3 \mid \rho(x, s) \leq r\}$. We thus face the problem of bounding the combinatorial complexity of the union of n objects in $\mathbb{R}^3$ (of some special type). For example, if S is a set of lines and ρ is the Euclidean distance, the objects are n congruent infinite cylinders in $\mathbb{R}^3$. In general, if the metric ρ is a distance function induced by some convex body P, the resulting objects are the *Minkowski sums* $s \oplus (-rP)$, for $s \in S$, where $A \oplus B = \{x + y \mid x \in A,\ y \in B\}$. Of course, this problem can also be stated in any higher dimension.

Since it has been conjectured that the complexity of the whole Voronoi diagram in $\mathbb{R}^3$ should be near-quadratic, the same conjecture should apply to the (simpler) structure K (whose boundary can be regarded as a *level curve* of the diagram at *height* r; it does indeed correspond to the cross-section at height r of the lower envelope in $\mathbb{R}^4$ that represents the diagram). This conjecture was confirmed by Aronov and Sharir in [64], in the special case where both P and the objects of S are convex polyhedra. They specialized their analysis of the union of convex polytopes to obtain an improved bound in the special case in which the polyhedra in question are Minkowski sums of the form $R_i \oplus P$, where the R_i's are n pairwise-disjoint convex polyhedra, P is a convex polyhedron, and the total number of faces of these Minkowski sums is s. The improved bounds are $O(ns \log n)$ and $\Omega(ns\alpha(n))$. If P is a cube, then the complexity of the Minkowski sum is $O(n^2\alpha(n))$ [240].

Agarwal and Sharir [33] have shown that the union of Minkowski sums of disjoint polyhedra of overall complexity n with a ball has complexity $O(n^{2+\varepsilon})$ for any $\varepsilon > 0$. As a special case (in which the given polyhedra are lines), the complexity of the union of n congruent infinite cylinders in $\mathbb{R}^3$ is nearly quadratic.

Other bounds in three and higher dimensions. The first rather general result on the complexity of the union of fat objects in 3-space stems from the analysis technique of Agarwal and Sharir [33] and appears in their paper: The complexity of the union $\mathcal{U}$ is $O(n^{2+\varepsilon})$ for any $\varepsilon > 0$ if $\mathcal{K}$ consists of n convex objects of near-equal size, with C^2-continuous boundaries, bounded mean curvature, and constant description complexity (that is, satisfying assumptions (A1)–(A2)).

In $d \geq 4$ dimensions, the results become even more scarce. As already mentioned, the complexity of the union of n halfspaces (each bounded by a hyperplane) in $\mathbb{R}^d$ is $O(n^{\lfloor d/2 \rfloor})$, as follows from the Upper Bound Theorem [300]. The complexity of the union of n balls in d-space is $O(n^{\lceil d/2 \rceil})$, as follows by lifting them to hyperplanes in $\mathbb{R}^{d+1}$. Boissonnat *et al.* [93] provide an upper bound of $O(n^{\lceil d/2 \rceil})$ for the union of n axis-parallel cubes in $\mathbb{R}^d$, which improves to $O(n^{\lfloor d/2 \rfloor})$ when the cubes have equal (or nearly equal) size. The union complexity of n convex bodies in $\mathbb{R}^d$ satisfying (A1)–(A2) *with a common interior point* o is $O(n^{d-1+\varepsilon})$ for any $\varepsilon > 0$, which follows from the observation that the boundary of their union can be interpreted as the upper envelope of $(d-1)$-variate functions (in spherical coordinates around o). See also a refined bound for polyhedra in $\mathbb{R}^3$ in [252]. Koltun and Sharir [270] extend the above-mentioned result of [33] to four dimensions, and proved that the complexity of the union of n convex objects of near-equal size, with C^2-continuous boundaries, bounded mean curvature, and constant description complexity in $\mathbb{R}^4$ is $O(n^{3+\varepsilon})$ for any $\varepsilon > 0$.

A more general result has recently been established in three and four dimensions. In complete analogy with the definition of κ-curved objects in two dimensions, a compact body c is *κ-round* (for a fixed $\kappa > 0$) if for every point $p \in \partial c$ there exists a closed ball $B(p, c, \kappa)$ of radius $\kappa \cdot \mathrm{diam}c$, which contains p and is contained in c. (If c is convex, $B(p, c, \kappa)$ is unique. The definition, though, allows c to be non-convex and to have *reflex* edges and vertices, although c cannot have any *convex* edge or vertex.) See Figure 2.11. Aronov *et al.* [57] have shown that the complexity of the union of a collection $\mathcal{K}$ of n κ-round (not necessarily convex) bodies in $\mathbb{R}^3$ or in $\mathbb{R}^4$, satisfying assumptions (A1)–(A2), is $O(n^{2+\varepsilon})$ in $\mathbb{R}^3$ and $O(n^{3+\varepsilon})$ in $\mathbb{R}^4$, for any $\varepsilon > 0$, where the constant of proportionality depends on ε, κ and the maximum degree b in (A2). In principle, the analysis of [57] can be applied

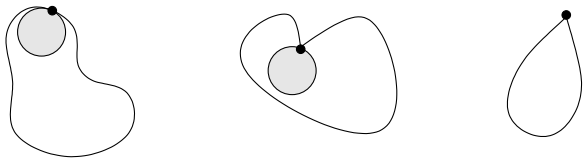

Figure 2.11: Examples of (planar analogues of) κ-round and non-κ-round bodies.

in any dimension $d \geq 3$, except for its last step, which reduces the problem to that of bounding the number of vertices of the *sandwich region* between the upper envelope of a collection of $(d-1)$-variate functions and the lower envelope of another such collection. As already mentioned, sharp bounds on the number of such vertices are known only for $d = 3$ and $d = 4$, which is the only reason for the present inability to extend the above bounds to $d > 4$.

We summarize some of the open problems involving unions in three and higher dimensions:

Problem 2.9.2. *(i) What is the complexity of the union of n cubes (arbitrarily aligned and of arbitrary sizes) in $\mathbb{R}^3$?*

(ii) What is the complexity of the union of n (say, congruent) arbitrarily aligned regular sim-

plices in $\mathbb{R}^3$?

(iii) What is the complexity of the union of n infinite cylinders of arbitrary radii in $\mathbb{R}^3$?

(iv) What is the complexity of the union of n κ-round objects (satisfying (A1)–(A2)) in $\mathbb{R}^d$, for $d \geq 5$?

(v) What is the complexity of the union of n α-fat objects (satisfying (A1)–(A2)) in $\mathbb{R}^d$, for $d \geq 3$?

2.10 Decomposition of Arrangements

Many applications call for decomposing each cell of the arrangement into subcells of constant complexity; see Sections 2.12 and 2.13 for a sample of such applications. In this section we describe a few general schemes that have been proposed for decomposition of arrangements.

2.10.1 Triangulating hyperplane arrangements

Each k-dimensional cell in an arrangement of hyperplanes is a convex polyhedron, so we can triangulate it into k-simplices. If the cell is unbounded, some of the simplices in the triangulation will be unbounded. A commonly used scheme to triangulate a convex polytope $\mathcal{P}$ is the so-called *bottom-vertex triangulation*, denoted $\mathcal{P}^\nabla$. It recursively triangulates every face of $\mathcal{P}$ as follows. An edge is a one-dimensional simplex, so there is nothing to do. Suppose we have triangulated all j-dimensional cells of $\mathcal{P}$ for $j < k$. We now triangulate a k-dimensional cell C as follows. Let v be the vertex of C with the minimum x_d-coordinate. For each $(k-1)$-dimensional simplex Δ lying on the boundary of C but not containing v (Δ was constructed while triangulating a $(k-1)$-dimensional cell incident to C), we extend Δ to a k-dimensional simplex by taking the convex hull of Δ and v; see Figure 2.12(i). (Unbounded cells require some care in this definition; see [135]). The number of simplices in $\mathcal{P}^\nabla$ is proportional to the number of vertices in $\mathcal{P}$.

If we want to triangulate the entire arrangement or more than one of its cells, we compute the bottom-vertex triangulation f^∇ for each face f in the increasing order of their dimension. Let $\mathcal{A}^\nabla(\Gamma)$ denote the bottom-vertex triangulation of $\mathcal{A}(\Gamma)$. A useful property of $A^\nabla(\Gamma)$ is that each simplex $\Delta \in \mathcal{A}^\nabla(\Gamma)$ is defined by a set $D(\Delta)$ of at most $d(d+3)/2$ hyperplanes of Γ, in the sense that $\Delta \in \mathcal{A}^\nabla(D(\Delta))$. Moreover, if $\mathcal{K}(\Delta) \subseteq \Gamma$ is the subset of hyperplanes intersecting Δ, then $\Delta \in \mathcal{A}^\nabla(R)$, for a subset $R \subseteq \Gamma$, if and only if $D(\Delta) \subseteq R$ and $\mathcal{K}(\Delta) \cap R = \emptyset$. A disadvantage of bottom-vertex triangulation is that some vertices may have large degree. Methods for obtaining low-degree triangulations have been proposed in two and three dimensions [153].

2.10.2 Vertical decomposition

Unfortunately, the bottom-vertex triangulation scheme does not work for arrangements of surfaces. Collins [139] described a general decomposition scheme, called *cylindrical algebraic decomposition*, that decomposes $\mathcal{A}(\Gamma)$ into $(bn)^{2^{O(d)}}$ cells, each semialgebraic of constant description complexity (however, the maximum algebraic degree involved in defining a cell grows exponentially with d) and homeomorphic to a ball of the appropriate dimension. Moreover, his algorithm produces a cell complex, i.e., closures of any two cells are either disjoint or their intersection is the closure of another

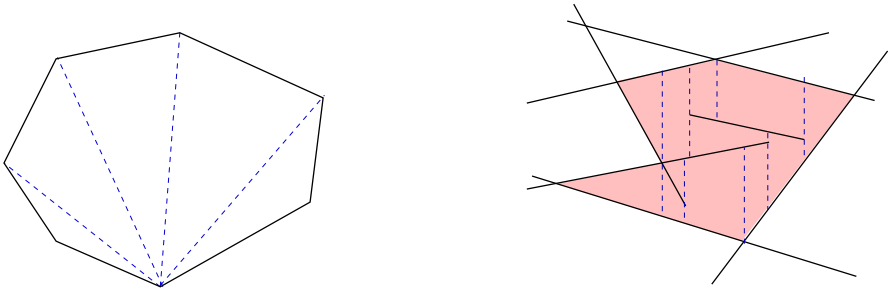

Figure 2.12: (i) Bottom vertex triangulation of a convex polygon; (ii) vertical decomposition of a cell in an arrangement of segments.

lower-dimensional cell of the decomposition. This bound is quite far from the known trivial lower bound of $\Omega(n^d)$, which is a lower bound on the size of the arrangement. A significantly better scheme for decomposing arrangements of general surfaces is their *vertical decomposition*. Although vertical decompositions of polygons in the plane have been in use for a long time, it was extended to higher dimensions only in the late 1980s. We describe this method briefly.

Let C be a d-dimensional cell in $\mathcal{A}(\Gamma)$. The vertical decomposition, $C^{\shortparallel}$, is computed as follows. We erect a vertical "wall" up and down (in the x_d-direction) within C from each $(d-2)$-dimensional face of C and from points of vertical tangencies (i.e., the points at which the tangent planes are parallel to the x_d-direction), and extend these walls until they hit another surface (or, failing this, all the way to $\pm\infty$). This results in a decomposition of C into subcells so that each subcell has a unique top facet and a unique bottom facet, and every vertical line cuts the subcell in a connected (possibly empty) interval. We next project each resulting subcell τ on the hyperplane $x_d = 0$. Let C_τ be the projected cell. We apply recursively the same technique to C_τ and compute its vertical decomposition $C_\tau^{\shortparallel}$. (We continue the recursion in this manner until we reach $d = 1$.) We then "lift" $C_\tau^{\shortparallel}$ back into $\mathbb{R}^d$, by extending each subcell $c \in C_\tau^{\shortparallel}$ into the vertical cylinder $c\times\mathbb{R}$, and by clipping the cylinder within τ. Using a standard argument, it can be shown that each cell of $C^{\shortparallel}$ is semialgebraic set of constant description complexity. In fact, they have the same structure as the Collins cells, but the number of subcells in $C^{\shortparallel}$ is much smaller than that in the Collins decomposition of C. Applying the above step to each cell of $\mathcal{A}(\Gamma)$, we obtain the vertical decomposition $\mathcal{A}^{\shortparallel}(\Gamma)$ of $\mathcal{A}(\Gamma)$. Note that $\mathcal{A}^{\shortparallel}(\Gamma)$ is not a cell complex.

It is easily seen that the complexity of the vertical decomposition of a cell in the plane is proportional to the number of edges in the cell. However, this is no longer the case in higher dimensions: Already for the case of a convex polytope with n facets in $\mathbb{R}^3$, the vertical decomposition may have complexity $\Omega(n^2)$. Nevertheless, for the entire arrangement, in dimensions $d = 2, 3, 4$, the vertical decomposition $\mathcal{A}^{\shortparallel}(\Gamma)$ behaves well: For $d = 2$, the complexity of $\mathcal{A}^{\shortparallel}(\Gamma)$ is $O(n^2)$ (trivial). For $d = 3$, it is $O(n^2\lambda_q(n))$, for some q that depends on the maximum degree b in (A2); see Chazelle *et al.* [119, 120]. A recent remarkable result of Koltun [267] shows that in $d = 4$ dimensions, $\mathcal{A}^{\shortparallel}(\Gamma)$ consists of $O(n^{4+\varepsilon})$, for any $\varepsilon > 0$, where the constant of proportionality depends on ε and b. In general, we have:

Theorem 2.10.1. *The number of cells in the vertical decomposition $\mathcal{A}^{\shortparallel}(\Gamma)$ of the arrangement $\mathcal{A}(\Gamma)$, for a set Γ of n surface patches in $\mathbb{R}^d$ satisfying (A1)–(A2), is $O(n^2)$ for $d = 2$, $O(n^2\lambda_q(n))$ for*

$d = 3$, where q is a constant depending on b, and $O(n^{2d-4+\varepsilon})$, for any $\varepsilon > 0$.

The only known lower bound on the size of $\mathcal{A}^{||}(\Gamma)$ is the trivial $\Omega(n^d)$, so there is a considerable gap here, for $d > 4$; for $d \leq 4$ the two bounds (nearly) coincide.

Problem 2.10.2. *What is the complexity of the vertical decomposition of the arrangement of n surfaces in $\mathbb{R}^5$ (or in higher dimensions) satisfying assumptions (A1)–(A2)?*

The bound stated above applies to the vertical decomposition of an entire arrangement of surfaces. In many applications, however, one is interested in the vertical decomposition of only a portion of the arrangement, e.g., a single cell, the lower envelope, the sandwich region between two envelopes, the zone of some surface, a specific collection of cells of the arrangement, etc. Since, in general, the complexity of such a portion is known to be smaller than the complexity of the entire arrangement, one would like to conjecture that a similar phenomenon applies to vertical decompositions. For example, it was shown by Schwarzkopf and Sharir [349] that the complexity of the vertical decomposition of a single cell in an arrangement of n surface patches in $\mathbb{R}^3$, as above, is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$. A similar near-quadratic bound has been obtained by Agarwal *et al.* [10] for the vertical decomposition of the region enclosed between the envelope and the upper envelope of two sets of bivariate surface patches. Another result by Agarwal *et al.* [16] gives a bound on the complexity of the vertical decomposition of $\mathcal{A}_{\leq k}(\Gamma)$ for a set Γ of surfaces in $\mathbb{R}^3$, which is only slightly larger that the worst-case complexity of $\mathcal{A}_{\leq k}(\Gamma)$.

Problem 2.10.3. *What is the complexity of the vertical decomposition of the minimization diagram of n surfaces in $\mathbb{R}^4$ satisfying assumptions (A1)–(A2)?*

Agarwal and Sharir [30] proved a near-cubic upper bound on the complexity of the vertical decomposition in the special case when the surfaces are graphs of trivariate polynomials and the intersection surface of any pair of surfaces is xy-monotone. In fact, their bound holds for a more general setting; see the original paper for details.

An interesting special case of vertical decomposition is that of hyperplanes. For such arrangements, the vertical decomposition is a too cumbersome construct, because, as described above, one can use the bottom-vertex triangulation (or any other triangulation) to decompose the arrangement into $\Theta(n^d)$ simplices. Still, it is probably a useful exercise to understand the complexity of the vertical decomposition of an arrangement of n hyperplanes in $\mathbb{R}^d$. Guibas *et al.* [217] give an almost tight bound of $O(n^4 \log n)$ for this quantity in $\mathbb{R}^4$, which has recently been tightened to $\Theta(n^4)$ by Koltun [268] (who has also obtained the sharp bound $O(n^4 \alpha(n) \log^2 n)$ on the complexity of the vertical decomposition of n 3-simplices in $\mathbb{R}^4$), but nothing significantly better than the general bound is known for $d > 4$. Another interesting special case is that of triangles in 3-space. This has been studied by [145, 369], where almost tight bounds were obtained for the case of a single cell ($O(n^2 \log^2 n)$), and for the entire arrangement ($O(n^2 \alpha(n) \log n + K)$, where K is the complexity of the undecomposed arrangement). The first bound is slightly better than the general bound of [349] mentioned above. Tagansky [369] also derives sharp complexity bounds for the vertical decomposition of many cells in an arrangement of simplices, including the case of all nonconvex cells.

2.10.3 Other decomposition schemes

Sometimes other decomposition schemes can be devised. For example, cells in generalized Voronoi diagrams in $\mathbb{R}^3$ can be decomposed into subcells of constant complexity by first decomposing the

Objects	Bound	Source
Surfaces in $\mathbb{R}^2$	$O(n^2)$	easy
Surfaces in $\mathbb{R}^3$	$O(n^2\lambda_q(n))$	[119, 359]
Surfaces in $\mathbb{R}^d$, $d \geq 4$	$O(n^{2d-4+\varepsilon})$	[267]
Triangles in $\mathbb{R}^3$	$O(n^2\alpha(n)\log n + K)$	[145, 369]
Surfaces in $\mathbb{R}^3$, single cell	$O(n^{2+\varepsilon})$	[349]
Triangles in $\mathbb{R}^3$, zone w.r.t. an algebraic surface	$\Theta(n^2 \log^2 n)$	[369]
Surfaces in $\mathbb{R}^3$, ($\leq k$)-level	$O(n^{2+\varepsilon}k)$	[16]
Hyperplanes in $\mathbb{R}^4$	$\Theta(n^4)$	[268]
3-Simplices in $\mathbb{R}^4$	$O(n^4\alpha(n)\log^2 n)$	[268]

Table 2.3: Combinatorial bounds on the maximum complexity of the vertical decomposition of an arrangement of n surfaces. In the fourth row, K is the combinatorial complexity of the arrangement.

boundary of each cell $V(s)$, using, say, a modified variant of planar vertical decomposition, and then extend each resulting cell τ to a 3-dimensional cell τ^*, which is the union of all segments that connect points $x \in \tau$ to their closest points on s. Thus, the overall number of resulting cells is proportional to the complexity of the entire Voronoi diagram. This also applies to the union of Minkowski sums, which, as discussed above, can be regarded as a substructure of an appropriate generalized Voronoi diagram. For example, this scheme yields a decomposition of the union of n congruent cylinders in $\mathbb{R}^3$ into $O(n^{2+\varepsilon})$ cells of constant complexity each.

It is not known how to extend these schemes to higher dimensions. Somewhat surprisingly, the scheme fails (or at least is not known to exist) for the *complement* of the union of Minkowski sums, already for complements of unions of cylinders:

Problem 2.10.4. *(i) What is the smallest asymptotic complexity of a decomposition of the cells of a generalized Voronoi diagram in $d \geq 4$ dimensions?*

(ii) What is the smallest asymptotic complexity of a decomposition of the complement of the union of n congruent cylinders in $\mathbb{R}^3$?

Linearization, defined in Section 2.3, can also be used to decompose the cells of the arrangement $\mathcal{A}(\Gamma)$ into cells of constant description complexity as follows. Suppose Γ admits a linearization of dimension k, i.e., there is a transformation $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}^k$ that maps each point $\mathbf{x} \in \mathbb{R}^d$ to a point $\varphi(\mathbf{x}) \in \mathbb{R}^k$, each surface $\gamma_i \in \Gamma$ to a hyperplane $h_i \subset \mathbb{R}^k$, and $\mathbb{R}^d$ to a d-dimensional surface $\Sigma \subseteq \mathbb{R}^k$. Let $H = \{h_i \mid 1 \leq i \leq n\}$. We compute the bottom-vertex triangulation $\mathcal{A}^{\nabla}(H)$ of $\mathcal{A}(H)$. For each simplex $\Delta \in \mathcal{A}^{\nabla}(H)$, let $\overline{\Delta} = \Delta \cap \Sigma$, and let $\Delta^* = \varphi^{-1}(\overline{\Delta})$ be the back projection of $\overline{\Delta}$ onto $\mathbb{R}^d$; Δ^* is a semialgebraic cell of constant description complexity. Set $\Xi = \{\Delta^* \mid \Delta \in \mathcal{A}^{\nabla}(H)\}$. Ξ is a decomposition of $\mathcal{A}(\Gamma)$ into cells of constant description complexity. If a simplex $\Delta \in \mathcal{A}^{\nabla}(H)$ intersects Σ, then Δ lies in the triangulation of a cell in $zone(\Sigma; H)$. Therefore, by Theorem 2.5.3, $|\Xi| = O(n^{\lfloor (d+k)/2 \rfloor} \log^{\gamma} n)$, where $\gamma = (d+k) \pmod 2$. Hence, we can conclude the following.

Theorem 2.10.5. *Let Γ be a set of hypersurfaces in $\mathbb{R}^d$ of degree at most b. If Γ admits a linearization of dimension k, then $\mathcal{A}(\Gamma)$ can be decomposed into $O(n^{\lfloor (d+k)/2 \rfloor} \log^{\gamma} n)$ cells of constant description complexity, where $\gamma = d + k \pmod 2$.*

As shown in Section 2.3, spheres in $\mathbb{R}^d$ admit a linearization of dimension $d+1$; therefore, the

arrangement of n spheres in $\mathbb{R}^d$ can be decomposed into $O(n^d \log n)$ cells of constant description complexity.

Aronov and Sharir [60] proposed another scheme for decomposing arrangements of triangles in $\mathbb{R}^3$ by combining vertical decomposition and triangulation. They first decompose each three-dimensional cell of the arrangement into convex polyhedron, using an incremental procedure, and then they compute a bottom-vertex triangulation of each polyhedron. Other specialized decomposition schemes in $\mathbb{R}^3$ have been proposed in [238, 297].

2.10.4 Cuttings

All the decomposition schemes described in this section decompose $\mathbb{R}^d$ into cells of constant description complexity, so that each cell lies entirely in a single face of $\mathcal{A}(\Gamma)$. In many applications, however, it suffices to decompose $\mathbb{R}^d$ into cells of constant description complexity so that each cell intersects only a few surfaces of Γ. Such a decomposition lies at the heart of divide-and-conquer algorithms for numerous geometric problems.

Let Γ be a set of n surfaces in $\mathbb{R}^d$ satisfying assumptions (A1)–(A2). For a parameter $r \leq n$, a family $\Xi = \{\Delta_1, \ldots, \Delta_s\}$ of cells of constant description complexity with pairwise disjoint interiors is called a $(1/r)$-*cutting* of $\mathcal{A}(\Gamma)$ if the interior of each cell in Ξ is crossed by at most n/r surfaces of Γ and Ξ covers $\mathbb{R}^d$. If Γ is a set of hyperplanes, then Ξ is typically a set of simplices. Cuttings have led to efficient algorithms for a wide range of geometric problems and to improved bounds for several combinatorial problems. For example, the proof by Clarkson *et al.* [133] on the complexity of m distinct cells in arrangements of lines uses cuttings; see the survey papers [3, 290] for a sample of applications of cuttings.

Clarkson [134] proved that a $(1/r)$-cutting of size $O(r^d \log^d r)$ exists for a set of hyperplanes in $\mathbb{R}^d$. The bound was improved by Chazelle and Friedman [124] to $O(r^d)$; see also [1, 284, 288, 295]. An easy counting argument shows that this bound is optimal for any nondegenerate arrangement. There has been considerable work on computing optimal $(1/r)$-cuttings efficiently [1, 116, 241, 284, 288]. Chazelle [116] showed that a $(1/r)$-cutting for a set of n hyperplanes in $\mathbb{R}^d$ can be computed in time $O(nr^{d-1})$.

Using Haussler and Welzl's result on ε-nets [243], one can show that if, for any subset $R \subseteq \Gamma$, there exists a canonical decomposition of $\mathcal{A}(R)$ into at most $g(|R|)$ cells of constant description complexity, then there exists a $(1/r)$-cutting of $\mathcal{A}(\Gamma)$ of size $O(g(r \log r))$. By the results of Koltun [267] and Chazelle *et al.* [119] on the vertical decomposition of $\mathcal{A}(\Gamma)$, there exists a $(1/r)$-cutting of size $O((r \log r)^{2d-4+\varepsilon})$ of $\mathcal{A}(\Gamma)$ in $d \geq 4$ dimensions, with corresponding bounds for $d = 2, 3$ (see above). On the other hand, if Γ admits a linearization of dimension k, then there exists a $(1/r)$-cutting of size $O((r \log r)^{\lfloor (d+k)/2 \rfloor} \log r)$.

2.11 Representation of Arrangements

Before we begin to present algorithms for computing arrangements and their substructures, we need to describe how we represent arrangements and their substructures. Planar arrangements of lines can be represented using any standard data structure for representing planar graphs such as quad-edge, winged-edge, and half-edge data structures [222, 262, 380]. However, representation of arrangements in higher dimensions is challenging because the topology of cells may be rather complex. Exactly

how an arrangement is represented largely depends on the specific application for which we need to compute it. For example, representations may range from simply computing a representative point within each cell, or the vertices of the arrangement, to storing various spatial relationships between cells. We first review representations of hyperplane arrangements and then discuss surface arrangements.

Hyperplane arrangements. A simple way to represent a hyperplane arrangement $\mathcal{A}(\Gamma)$ is by storing its 1-*skeleton* [157]. That is, we construct a graph (V, E) whose nodes are the vertices of the arrangement. There is an edge between two nodes v_i, v_j if they are endpoints of an edge of the arrangement. Using the 1-skeleton of $\mathcal{A}(\Gamma)$, we can traverse the entire arrangement in a systematic way. The incidence relationship of various cells in $\mathcal{A}(\Gamma)$ can be represented using a data structure called *incidence graph*. A k-dimensional cell C is called a *subcell* of a $(k+1)$-dimensional cell C' if C lies on the boundary of C'; C' is called the *supercell* of C. We assume that the empty set is a (-1)-dimensional cell of $\mathcal{A}(\Gamma)$, which is a subcell of all vertices of $\mathcal{A}(\Gamma)$; and $\mathbb{R}^d$ is a $(d+1)$-dimensional cell, which is the supercell of all d-dimensional cells of $\mathcal{A}(\Gamma)$. The incidence graph of $\mathcal{A}(\Gamma)$ has a node for each cell of $\mathcal{A}(\Gamma)$, including the (-1)-dimensional and $(d+1)$-dimensional cells. There is a (directed) arc from a node C to another node C' if C is a subcell of C'; see Figure 2.13. Note that the incidence graph forms a lattice. Many algorithms for computing the arrangement construct the incidence graph of the arrangement.

A disadvantage of 1-skeletons and incidence graphs is that they do not encode ordering information of cells. For examples, in planar arrangements of lines or segments, there is a natural ordering of edges incident to a vertex or of the edges incident to a two-dimensional face. The quad-edge data structure encodes this information for planar arrangements. Dobkin and Laszlo [154] extended the quad-edge data structure to $\mathbb{R}^3$, which was later extended to higher dimensions [96, 280, 281]. Dobkin *et al.* [152] described an algorithm for representing a simple polygon as a short Boolean formula, which can be used to store faces of segment arrangements to answer various queries efficiently.

Surface arrangements. Representing arrangements of surface patches is considerably more challenging than representing hyperplane arrangements because of the complex topology that cells in such an arrangement can have. A very simple representation of $\mathcal{A}(\Gamma)$ is to store a representative point from each cell of $\mathcal{A}(\Gamma)$ or to store the vertices of $\mathcal{A}(\Gamma)$. An even coarser representation of arrangements of graphs of polynomials is to store all realizable sign sequences. It turns out that this simple representation is sufficient for some applications [43, 89]. The notion of 1-skeleton can be generalized to arrangements of surfaces. However, all the connectivity information cannot be encoded by simply storing vertices and edges of the arrangement. Instead we need a finer one-dimensional structure, known as the *roadmap*. Roadmaps were originally introduced by Canny [103, 105] to determine whether two points lie in the same connected component of a semialgebraic set; see also [210, 212, 246]. They were subsequently used for computing a semialgebraic description of connected components of a semialgebraic set [82, 107, 247]. We can extend the notion of roadmaps to entire arrangements. Roughly speaking, a roadmap $\mathcal{R}(\Gamma)$ of $\mathcal{A}(\Gamma)$ is a one-dimensional semialgebraic set that satisfies the following two conditions.

(R1) For every cell C in $\mathcal{A}(\Gamma)$, $C \cap \mathcal{R}(\Gamma)$ is nonempty and connected.

(R2) Let C_w be the cross-section of a cell $C \in \mathcal{A}(\Gamma)$ at the hyperplane $x_1 = w$. For any $w \in \mathbb{R}$ and for cell $C \in \mathcal{A}(\Gamma)$, $C_w \neq \emptyset$ implies that every connected component of C_w intersects $\mathcal{R}(\Gamma)$.

We can also define the roadmap of various substructures of arrangements. See [79, 81, 103] for details on roadmaps.

A roadmap does not represent "ordering" of cells in the arrangement or adjacency relationship among various cells. If we want to encode the adjacency relationship among higher dimensional cells of $\mathcal{A}(\Gamma)$, we can compute the vertical decomposition or the cylindrical algebraic decomposition of $\mathcal{A}(\Gamma)$ and compute the adjacency relationship of cells in the decomposition [51, 346]. Brisson [96] describes the *cell-tuple* data structure that encodes topological structures, ordering among cells, the boundaries of cells, and other information for cells of surface arrangements.

Many query-type applications (e.g., point location, ray shooting) call for preprocessing $\mathcal{A}(\Gamma)$ into a data structure so that various queries can be answered efficiently. In these cases, instead of storing various cells of an arrangement explicitly, we can store the arrangement implicitly, e.g., using cuttings. Chazelle *et al.* [120] have described how to preprocess arrangements of surfaces for point-location queries; Agarwal *et al.* [10] have described data structures for storing lower envelopes in $\mathbb{R}^4$ for point-location queries.

2.12 Computing Arrangements

We now review algorithms to compute the arrangement $\mathcal{A}(\Gamma)$ of a set Γ of n surface patches satisfying assumptions (A1)–(A2). We need to assume here an appropriate model of computation in which various primitive operations on a constant number of surfaces can be performed in constant time. We will assume an infinite-precision real arithmetic model in which the roots of any polynomial of constant degree can be computed exactly in constant time.

Constructing arrangements of hyperplanes and simplices. Edelsbrunner *et al.* [171] describe an incremental algorithm that computes in time $O(n^d)$ the incidence graph of $\mathcal{A}(\Gamma)$, for a set Γ of n hyperplanes in $\mathbb{R}^d$. Roughly speaking, their algorithm adds the hyperplanes of Γ one by one and maintains the incidence graph of the arrangement of the hyperplanes added so far. Let Γ_i be the set of hyperplanes added in the first i stages, and let γ_{i+1} be the next hyperplane to be added. In the $(i+1)$st stage, the algorithm traces γ_{i+1} through $\mathcal{A}(\Gamma_i)$. If a k-face f of $\mathcal{A}(\Gamma_i)$ does not intersect γ_i, then f remains a face of $\mathcal{A}(\Gamma_{i+1})$. If f intersects γ_{i+1}, then $f \in zone(\gamma_{i+1}; \Gamma_i)$ and f is split into two k-faces f^+, f^-, lying in the two open halfspaces bounded by γ_{i+1}, and a $(k-1)$-face $f' = f \cap \gamma_{i+1}$. The algorithm therefore checks the faces of $zone(\gamma_{i+1}; \Gamma_i)$ whether they intersect γ_{i+1}. For each such intersecting face, it adds corresponding nodes in the incidence graph and updates the edges of the incidence graph. The $(i+1)$st stage can be completed in time proportional to the complexity of $zone(\gamma_{i+1}; \Gamma_i)$, which is $O(i^{d-1})$; see [158, 171]. Hence, the overall running time of the algorithm is $O(n^d)$.

A drawback of the algorithm just described is that it requires $O(n^d)$ "working" storage because it has to maintain the entire arrangement constructed so far in order to determine which of the cells intersect the new hyperplane. An interesting question is whether $\mathcal{A}(\Gamma)$ can be computed using only $O(n)$ working storage. Edelsbrunner and Guibas [163] proposed the *topological sweep* algorithm that can construct the arrangement of n lines in $O(n^2)$ time using $O(n)$ working storage. Their algorithm, which is a generalization of the sweep-line algorithm of Bentley and Ottmann [83], sweeps the plane by a pseudo-line. The algorithm by Edelsbrunner and Guibas can be extended to enumerate all vertices in an arrangement of n hyperplanes in $\mathbb{R}^d$ in $O(n^d)$ time using $O(n)$ space. See [49, 67, 174] for

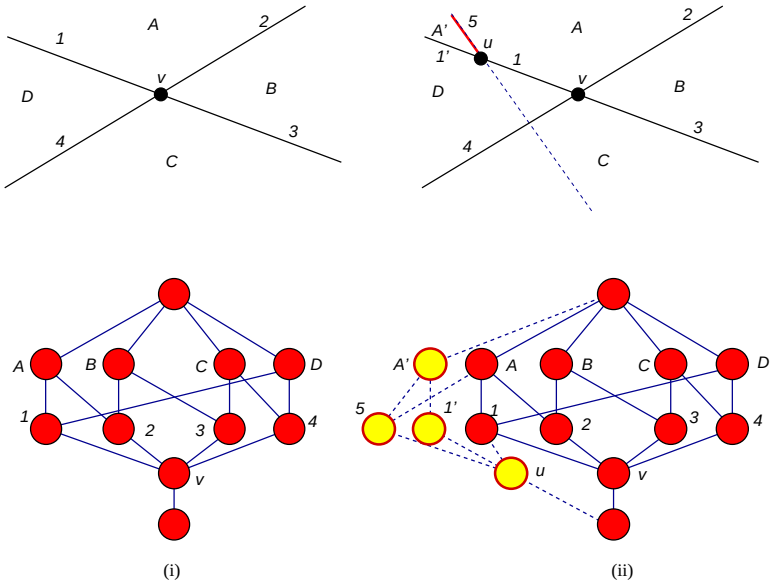

Figure 2.13: (i) Incidence graph of the arrangement of 2 lines. (ii) Adding a new line; incremental changes in the incidence graph as the vertex v, the edge 5, and the face A' are added.

other topological-sweep algorithms. Avis and Fukuda [70] developed an algorithm that can enumerate in $O(n^2k)$ time, using $O(n)$ space, all k vertices of the arrangement of a set Γ of n hyperplanes in $\mathbb{R}^d$ in which every vertex is incident to d hyperplanes. Their algorithm is useful when there are many parallel hyperplanes in Γ. See also [71, 201] for some related results.

Using the random-sampling technique, Clarkson and Shor [136] developed an $O(n \log n + k)$ expected time algorithm for constructing the arrangement of a set Γ of n line segments in the plane; here k is the number of vertices in $\mathcal{A}(\Gamma)$; see also [311, 312]. Chazelle and Edelsbrunner [118] developed a deterministic algorithm that can construct $\mathcal{A}(\Gamma)$ in time $O(n \log n + k)$, using $O(n + k)$ storage. The space complexity was improved to $O(n)$, without affecting the asymptotic running time, by Balaban [72]. If Γ is a set of n triangles in $\mathbb{R}^3$, $\mathcal{A}^{||}(\Gamma)$ can be constructed in $O(n^2 \log n + k)$ expected time using a randomized incremental algorithm [122, 359]. de Berg *et al.* [145] proposed a deterministic algorithm with $O(n^2\alpha(n) \log n + k \log n)$ running time for computing $\mathcal{A}^{||}(\Gamma)$.

Chazelle and Friedman [125] described an algorithm that can preprocess a set Γ of n hyperplanes into a data structure of size $O(n^d/\log^d n)$ so that a point-location query can be answered in $O(\log n)$ time. Their algorithm was later simplified by Matoušek [292] and Chazelle [116]. Mulmuley and Sen [314] developed a randomized dynamic data structure of size $O(n^d)$ for point location in arrangements of hyperplanes that can answer a point-location query in $O(\log n)$ expected time and can insert or delete a hyperplane in $O(n^{d-1} \log n)$ expected time. Hagerup *et al.* [225] described a randomized parallel algorithm for constructing the arrangement of hyperplanes under the CRCW model. Their algorithm runs in $O(\log n)$ time using $O(n^d/\log n)$ expected number of processors. A deterministic algorithm under the CREW model with the same worst-case performance was proposed by Goodrich [207].

Constructing arrangements of surfaces. The algorithm by Edelsbrunner *et al.* [171] for computing hyperplane arrangements can be extended to computing the vertical decomposition $\mathcal{A}^{||}(\Gamma)$ for a set Γ of n arcs in the plane. In the $(i+1)$st step, the algorithm traces γ_{i+1} through $zone(\gamma_{i+1}; \Gamma_i)$ and updates the trapezoids of $\mathcal{A}^{||}(\Gamma_i)$ that intersect γ_{i+1}. The running time of the $(i+1)$st stage is $O(\lambda_{s+2}(i))$, where s is the maximum number of intersection points between a pair of arcs in Γ. Hence, the overall running time of the algorithm is $O(n\lambda_{s+2}(n))$ [165]. Suppose Γ is a set of arcs in the plane in general position. If the arcs in Γ are added in a random order and a "history dag" (see Chapter 3) is used to efficiently find the trapezoids of $\mathcal{A}^{||}(\Gamma_i)$ that intersect γ_{i+1}, the expected running time of the algorithm can be improved to $O(n \log n + k)$, where k is the number of vertices in $\mathcal{A}(\Gamma)$ [122, 359].

Very little is known about computing the arrangement of a set Γ of surfaces in higher dimensions. Chazelle *et al.* [119] gave a randomized algorithm for constructing $\mathcal{A}^{||}(\Gamma)$ in $d \geq 4$ dimensions, using the random-sampling technique. Combined with Koltun's improved bounds [267], the expected running time of the algorithm is $O(n^{2d-4+\varepsilon})$, for any $\varepsilon > 0$. The algorithm can be made deterministic without increasing its asymptotic running time, but the deterministic algorithm is considerably more complex.

Computing arrangements robustly. Transforming the data structures and algorithms for computing with arrangements into effective computer programs is a difficult task. The typical assumptions of (i) the real RAM model of computation and (ii) general position, are not realistic in practice. This is not only a problem for implementing software for arrangements but rather a general problem in computational geometry. However, it is especially acute in the case of arrangements since here one needs to compute *intersection points* of curves and surfaces and use the computed values in further operations (to distinguish from say convex hull algorithms that only select a subset of the input points).

Recent surveys that cover the general area of robust geometric computations are [194, 228, 344, 387].

Exact computing. A general paradigm to overcome robustness problems is to compute exactly. For arrangements of linear objects, namely, arrangements of hyperplanes or of simplices, there is a fairly straightforward solution: using arbitrary precision rational arithmetic. This is regularly done by keeping arbitrary long integers for the enumerator and denominator of each number. Of course the basic numerical operations now become costly, and a method was devised to reduce the cost of rational arithmetic predicates through the use of *floating point filters*, which turn out to be very effective in practice, especially when the input is nondegenerate.

So far filtering has been applied to predicates but not to constructions. Notice that, if one has to produce the *exact* coordinates of an intersection point in an arrangement, there are no shortcuts and exact arithmetic needs to be used.

Matters are more complicated when the objects are not linear, namely when we deal with higher-degree curves and surfaces. First, there is the issue of representation, since intersection points of algebraic curves or surfaces will in general b irrational, so we cannot have a simple numerical representation of the vertices of the arrangement. An elegant solution to this problem is provided by special number types (so-called *algebraic number types*. Two software libraries support such number types (called *real* in both): LEDA [304] and Core [254]. The ideas behind the solution proposed by

both are similar and rely on separation bounds. In terms of arrangements, the power that these number types provide is that we can determine the exact topology of the arrangement in all cases including degenerate cases.

There has been considerable theoretical work on developing tools in computational real algebraic geometry for computing arrangements under the precise rational arithmetic model [81]. Canny [106] had described an $(nb)^{O(d)}$-time algorithm for computing a sample point from each cell of the arrangement of a set of n hypersurfaces in $\mathbb{R}^d$, each of degree at most b. The running time was improved by Basu *et al.* [80] to $n^{d+1}b^{O(d)}$. Basu *et al.* [79] described an $n^{d+1}b^{O(d^2)}$-time algorithm for computing the roadmap of a semialgebraic set defined by n polynomials, each of degree at most b. Although their goal is to develop the roadmap of a semialgebraic set, their algorithm first constructs the road map of the entire arrangement of the surfaces defining the semialgebraic set and then outputs the appropriate portion of the map.

While exact computing may seem to be the solution to all problems, the situation is far from being satisfactory for several reasons: (i) The existing number types considerably slow down the computation compared with standard machine arithmetic. (ii) It is difficult to implement the full fledged number types required for arrangements of curves and surfaces. The state-of-the-art libraries offer the necessary types for arrangements of circles but not even for arrangements of conic arcs. Recently, an alternative approach has been taken to enable the exact construction of arrangements of conic arcs. It is based on using the GCD of the defining polynomials of arrangement vertices [84], [381]. (iii) It still leaves open the question of handling degeneracies (see below).

The high cost of exact predicates has led researchers to look for alternative algorithmic solutions (for problems where good solutions, in the standard measures of computational geometry, have been known), solutions that use less costly predicates; see, e.g., [92].

Rounding. In rounding we transform an arbitrary precision arrangement into fixed precision representation. The most intensively studied case is that of planar arrangements of segments. A solution proposed independently by Hobby [249] and by Greene (improving on an earlier method in [211]), snaps vertices of the arrangement to centers of pixels in a prespecified grid. The method preserves several topological properties of the original arrangement and indeed expresses the vertices of the arrangement with limited precision numbers (say bounded bit-length integers). A dynamic algorithm is described in [215], and an improved algorithm for the case where there are many intersections within a pixel is given in [208]. Snap rounding has several drawbacks though: a line is substituted by a polyline possibly with many links, and a vertex of the arrangement can become very close to a non-incident edge (the latter problem has been overcome in an alternative scheme *iterated snap rounding* which guarantees a large separation between such features of the arrangement but pays in the quality of approximation [233]). Furthermore, a pair of input segments may intersect an arbitrarily large number of times in the rounded arrangements. Finally, the 3D version seems to produce a huge number of extra features [196]: a polyhedral subdivision of complexity n turns into a snapped subdivision of complexity $O(n^4)$.

Effective and consistent rounding of arrangements remains an important and largely open problem. The importance of rounding arrangements stems not only from its being a means to overcome robustness issues, but, not less significantly, from being a way to express the arrangement numerically with reasonable bit-size numbers. Even if highly efficient exact number types are developed, there will still remain the question of numerical representation of the output.

Approximate arithmetic in predicate evaluation. The behavior of fundamental algorithms for computing line arrangements (both sweep line and incremental), while using limited precision arithmetic, is studied in [197]. It is shown that the two algorithms can be implemented such that for n lines the maximum error of the coordinates of vertices is $O(n\varepsilon)$ where ε is the relative error of the approximate arithmetic used (e.g., floating point). An algorithm for constructing curve arrangements with rounded arithmetic is presented in [306].

Perturbation. An arrangement of lines is considered degenerate if it is not simple (has vertices that are incident to more than two lines). Intuitively this is a degeneracy since moving the lines slightly will result in a topologically different arrangement. Degeneracies in arrangements pose difficulties for two reasons. First and foremost they incredibly complicate programming (similar reasons led to the general position assumption—developing a theoretical algorithm that handles all possible degenerate cases is also a technically cumbersome and error-prone process). Although it has been proposed that handling degeneracies could be the solution in practice to relax the general position assumption [101], in three and higher dimensions handling all degeneracies in arrangements seems an extremely difficult task. The second difficulty posed by degeneracies is that the numerical computation at or near degeneracies typically requires higher precision and will for example cause floating point filters to fail and resort to exact computing resulting in longer running time.

To overcome the first difficulty, symbolic perturbation schemes have been proposed. They enable a consistent perturbation of the input objects so that all degeneracies are removed. These schemes modify the objects only symbolically and a limiting process is used to define the perturbed objects (corresponding to infinitesimal perturbations) such that all predicates will have non-zero results. They require the usage of exact arithmetic, and a postprocessing stage to determine the structure of the output. For a unifying view of these schemes and a discussion of their properties, see [354].

An alternative approach is to *actually* perturb the objects from their original placement. One would like to perturb the input objects as little as possible so that precision problems are resolved. This approach is viable in situations where the exact placement of the input can be compromised, as is the case in many engineering and scientific applications where the input is inexact due to measurement or modeling errors. An efficient such scheme for arrangements of spheres that model molecules is described in [239]; it has been adapted and extended to arrangements of polyhedral surfaces in [335]. It is referred to as *controlled* perturbation since it guarantees that the final arrangement is degeneracy free (and predicates can be safely computed with limited precision arithmetic), to distinguish from heuristic perturbation methods. An in-depth study of the parameters that govern the scheme in the case of planar arrangements of circles is given in [231].

Problem 2.12.1. *Devise efficient and consistent rounding schemes for arrangements of curves in the plane and for arrangements in three and higher dimensions.*

2.13 Computing Substructures in Arrangements

2.13.1 Lower envelopes

Let Γ be a set of surface patches satisfying assumptions (A1)–(A2). We want to compute the minimization diagram $\mathcal{M}(\Gamma)$ of Γ. We will present in Chapter 3 algorithms for computing the minimization diagram of a set of arcs in the plane. In this chapter we will focus on minimization diagrams

of sets of surface patches in higher dimensions. There are again several choices, depending on the application, as to what exactly we want to compute. The simplest choice is to compute the vertices or the 1-skeleton of $\mathcal{M}(\Gamma)$. A more difficult task is to compute all the faces of $\mathcal{M}(\Gamma)$ and represent them using any of the mechanisms described in the previous section. Another challenging task, which is required in many applications, is to store Γ into a data structure so that $L_\Gamma(\mathbf{x})$, for any point $\mathbf{x} \in \mathbb{R}^{d-1}$, can be computed efficiently.

For collections Γ of surface patches in $\mathbb{R}^3$, the minimization diagram $\mathcal{M}(\Gamma)$ is a planar subdivision. In this case, the latter two tasks are not significantly harder than the first one, because we can preprocess $\mathcal{M}(\Gamma)$ using any optimal planar point-location algorithm [147]. Several algorithms have been developed for computing the minimization diagram of bivariate (partial) surface patches [26, 90, 91, 143, 357, 359]. Some of these techniques use randomized algorithms, and their expected running time is $O(n^{2+\varepsilon})$, which is comparable with the maximum complexity of the minimization diagram of bivariate surface patches. The simplest algorithm is probably the deterministic divide-and-conquer algorithm presented by Agarwal *et al.* [26]. It partitions Γ into two subsets Γ_1, Γ_2 of roughly equal size, and computes recursively the minimization diagrams $\mathcal{M}_1$, $\mathcal{M}_2$ of Γ_1 and Γ_2, respectively. It then computes the overlay $\mathcal{M}^*$ of $\mathcal{M}_1$ and $\mathcal{M}_2$. Over each face f of $\mathcal{M}^*$ there are only (at most) two surface patches that can attain the final envelope (the one attaining $L(\Gamma_1)$ over f and the one attaining $L(\Gamma_2)$ over f), so we compute the minimization diagram of these two surface patches over f, replace f by this refined diagram, and repeat this step for all faces of $\mathcal{M}^*$. We finally merge any two adjacent faces f, f' of the resulting subdivision if the same surface patches attain $L(\Gamma)$ over both f and f'. The cost of this step is proportional to the number of faces of $\mathcal{M}^*$. By the result of Agarwal *et al.* [26], $\mathcal{M}^*$ has $O(n^{2+\varepsilon})$ faces. This implies that the complexity of the above divide-and-conquer algorithm is $O(n^{2+\varepsilon})$. If Γ is a set of triangles in $\mathbb{R}^3$, the running time of the algorithm is $O(n^2\alpha(n))$ [166]. This divide-and-conquer algorithm can also be used to compute $S(\Gamma, \Gamma')$, the region lying above all surface patches of one collection Γ' and below all surface patches of another collection Γ, in time $O(n^{2+\varepsilon})$, where $n = |\Gamma| + |\Gamma'|$ [26].

A more difficult problem is to devise *output-sensitive* algorithms for computing $\mathcal{M}(\Gamma)$, whose complexity depends on the actual combinatorial complexity of the envelope. A rather complex algorithm is presented by de Berg [142] for the case of triangles in $\mathbb{R}^3$, whose running time is $O(n^{4/3+\varepsilon} + n^{4/5+\varepsilon}k^{4/5})$, where k is the number of vertices in $\mathcal{M}(\Gamma)$. If the triangles in Γ are pairwise disjoint, the running time can be improved to $O(n^{1+\varepsilon} + n^{2/3+\varepsilon}k^{2/3})$ [21, 142].

The algorithm by Edelsbrunner *et al.* [166] can be extended to compute in $O(n^{d-1}\alpha(n))$ time all faces of the minimization diagram of $(d-1)$-simplices in $\mathbb{R}^d$ for $d \geq 4$. However, little is known about computing the minimization diagram of more general surface patches in $d \geq 4$ dimensions. Let Γ be a set of surface patches in $\mathbb{R}^d$ satisfying assumptions (A1)–(A2). Agarwal *et al.* [10] showed that all vertices, edges, and 2-faces of $\mathcal{M}(\Gamma)$ can be computed in randomized expected time $O(n^{d-1+\varepsilon})$. We sketch their algorithm below.

Assume that Γ satisfies assumptions (A1)–(A5). Fix a $(d-2)$-tuple of surface patches, say $\gamma_1, \ldots, \gamma_{d-2}$, and decompose their common intersection $\bigcap_{i=1}^{d-2} \gamma_i$ into smooth, x_1x_2-monotone, connected patches, using a stratification algorithm. Let Π be one such piece. Each surface γ_i, for $i \geq d-1$, intersects Π at a curve ξ_i, which partitions Π into two regions. If we regard each γ_i as the graph of a partially defined $(d-1)$-variate function, then we can define $K_i \subseteq \Pi$ to be the region whose projection on the hyperplane $H: x_d = 0$ consists of points $\mathbf{x}$ at which $\gamma_i(\mathbf{x}) \geq \gamma_1(\mathbf{x}) = \cdots = \gamma_{d-2}(\mathbf{x})$. The intersection $Q = \bigcap_{i \geq d-1} K_i$ is equal to the portion of Π that appears along the lower envelope $L(\Gamma)$. We repeat this procedure for all patches of the intersection $\bigcap_{i=1}^{d-2} \gamma_i$ and for all $(d-2)$-tuples of surface patches. This will give all the vertices, edges and 2-faces of $L(\Gamma)$.

Since Π is x_1x_2-monotone 2-manifold, computing Q is essentially the same as computing the intersection of $n-d+2$ planar regions. Q can thus be computed using an appropriate variant of the randomized incremental approach [122, 143]. It adds $\xi_i = \gamma_i \cap \Pi$ one by one in a random order (ξ may consist of $O(1)$ arcs), and maintains the intersection of the regions K_i for the arcs added so far. Let Q_r denote this intersection after r arcs have been added. We maintain the "vertical decomposition" of Q_r (within Π), and represent Q_r as a collection of *pseudo-trapezoids*. We maintain additional data structures, including a *history dag* and a union-find structure, and proceed exactly as in [122, 143] (see Chapter 3). We omit here the details.

We define the *weight* of a pseudo-trapezoid τ to be the number of surface patches γ_i, for $i \geq d-1$, whose graphs either cross τ or hide τ completely from the lower envelope (excluding the up to four function graphs whose intersections with Π *define* τ). The cost of the above procedure, summed over all $(d-2)$-tuples of Γ, is proportional to the number of pseudo-trapezoids that are created during the execution of the algorithm, plus the sum of their weights, plus an overhead term of $O(n^{d-1})$ needed to prepare the collections of arcs ξ_i over all two-dimensional patches Π. Modifying the analysis in the papers cited above, Agarwal *et al.* prove the following.

Theorem 2.13.1 (Agarwal et al. [10]). *Let Γ be a set of n surface patches in $\mathbb{R}^d$ satisfying assumptions (A1)–(A2). The vertices, edges, and 2-faces of $\mathcal{M}(\Gamma)$ can be computed in randomized expected time $O(n^{d-1+\varepsilon})$, for any $\varepsilon > 0$.*

For $d = 4$, the above algorithm can be extended to compute the incidence graph (or cell-tuple structure) of $\mathcal{M}(\Gamma)$. Their approach, however, falls short of computing such representations for $d > 4$. Agarwal *et al.* also show that the three-dimensional point-location algorithm by Preparata and Tamassia [334] can be extended to preprocess a set of trivariate surface patches in time $O(n^{3+\varepsilon})$ into a data structure of size $O(n^{3+\varepsilon})$ so that $L_\Gamma(\mathbf{x})$, for any point $\mathbf{x} \in \mathbb{R}^3$, can be computed in $O(\log^2 n)$ time.

Problem 2.13.2. *Let Γ be a set of n surface patches in $\mathbb{R}^d$, for $d > 4$, satisfying assumptions (A1)–(A3). How fast can Γ be preprocessed, so that $L_\Gamma(\mathbf{x})$, for a query point $\mathbf{x} \in \mathbb{R}^{d-1}$, can be computed efficiently?*

2.13.2 Single cells

Computing a single cell in an arrangement of n hyperplanes in $\mathbb{R}^d$ is equivalent, by duality, to computing the convex hull of a set of n points in $\mathbb{R}^d$ and is a widely studied problem; see, e.g., [158, 353] for a summary of known results. For $d \geq 4$, an $O(n^{\lfloor d/2 \rfloor})$ expected-time algorithm for this problem was proposed by Clarkson and Shor [136] (see also [352]), which is optimal in the worst case. By derandomizing this algorithm, Chazelle [117] developed an $O(n^{\lfloor d/2 \rfloor})$-time deterministic algorithm. A somewhat simpler algorithm with the same running time was later proposed by Brönnimann *et al.* [98]. These results imply that the Euclidean Voronoi diagram of a set of n points in $\mathbb{R}^d$ can be computed in time $O(n^{\lceil d/2 \rceil})$.

Since the complexity of a cell may vary between $O(1)$ and $O(n^{\lfloor d/2 \rfloor})$, it pays off to seek output-sensitive algorithms for computing a single cell in hyperplane arrangements; see [127, 264, 350]. For $d \leq 3$, Clarkson and Shor [136] gave randomized algorithms with expected time $O(n \log h)$, where h is the complexity of the cell, provided that the planes are in general position. Simple deterministic algorithms with the same worst-case bound were developed by Chan [108]. Seidel [350] proposed an algorithm whose running time is $O(n^2 + h \log n)$; the first term can be improved to

$O(n^{2-2/(\lfloor d/2\rfloor+1)}\log^c n)$ [291] or to $O((nh)^{1-1/(\lfloor d/2\rfloor+1)}\log^c n)$ [109]. Chan *et al.* [115] described another output-sensitive algorithm whose running time is $O((n+(nf)^{1-1/\lceil d/2\rceil}+fn^{1-2/\lceil d/2\rceil})\log^c n)$. Avis *et al.* [69] described an algorithm that can compute in $O(nf)$ time, using $O(n)$ space, all f vertices of a cell in an arrangement of n hyperplanes in $\mathbb{R}^d$; see also [95, 200]. All these output-sensitive bounds hold only for simple arrangements. Although many of these algorithms can be extended to nonsimple arrangements, the running time increases.

As will be mentioned in Chapter 3, Guibas *et al.* [221] developed an $O(\lambda_{s+2}(n)\log^2 n)$-time algorithm for computing a single face in an arrangement of n arcs, each pair of which intersect in at most s points. Later a randomized algorithm with expected time $O(\lambda_{s+2}(n)\log n)$ was developed by Chazelle *et al.* [122]. Since the complexity of the vertical decomposition of a single cell in an arrangement of n surface patches in $\mathbb{R}^3$ is $O(n^{2+\varepsilon})$ [349], an application of the random-sampling technique yields an algorithm for computing a single cell in time $O(n^{2+\varepsilon})$ in an arrangement of n surface patches in $\mathbb{R}^3$ [349]. If Γ is a set of triangles, the running time can be improved to $O(n^2\log^3 n)$ [143]. Halperin [226, 227] developed faster algorithms for computing a single cell in arrangements of "special" classes of bivariate surfaces that arise in motion-planning applications.

2.13.3 Levels

Constructing the $\leq k$-level. Let Γ be a set of n arcs in the plane, each pair of which intersect in at most s points. $\mathcal{A}_{\leq k}(\Gamma)$ can be computed by a simple divide-and-conquer algorithm as follows [355]. Partition Γ into two subsets Γ_1, Γ_2, each of size at most $\lceil n/2\rceil$, compute recursively $\mathcal{A}_{\leq k}(\Gamma_1), \mathcal{A}_{\leq k}(\Gamma_2)$, and then use a sweep-line algorithm to compute $\mathcal{A}_{\leq k}(\Gamma)$ from $\mathcal{A}_{\leq k}(\Gamma_1)$ and $\mathcal{A}_{\leq k}(\Gamma_2)$. The time spent in the merge step is proportional to the number of vertices in $\mathcal{A}_{\leq k}(\Gamma_1), \mathcal{A}_{\leq k}(\Gamma_2)$ and the number of intersections points between the edges of two subdivisions, each of which is a vertex of $\mathcal{A}(\Gamma)$ whose level is at most $2k$. Using Theorem 2.6.1, the total time spent in the merge step is $O(\lambda_{s+2}(n)k\log n)$. Hence, the overall running time of the algorithm is $O(\lambda_{s+2}(n)k\log^2 n)$. If we use a randomized incremental algorithm that adds arcs one by one in a random order and maintains $\mathcal{A}_{\leq k}(\Gamma_i)$, where Γ_i is the set of arcs added so far, the expected running time of the algorithm is $O(\lambda_{s+2}(n)k\log(n/k))$; see, e.g., [313]. Everett *et al.* [189] showed that if Γ is a set of n lines, the expected running time can be improved to $O(n\log n + nk)$. Agarwal *et al.* [15] gave another randomized incremental algorithm that can compute $\mathcal{A}_{\leq k}(\Gamma)$ in expected time $O(\lambda_{s+2}(n)(k+\log n))$.

In higher dimensions, little is known about computing $\mathcal{A}_{\leq k}(\Gamma)$, for collections Γ of surface patches. For $d=3$, Mulmuley [313] gave a randomized incremental algorithm for computing the $\leq k$-level in an arrangement of n planes whose expected running time is $O(nk^2\log(n/k))$. The expected running time can be improved to $O(n\log^3 n + nk^2)$ using the algorithm by Agarwal *et al.* [15]. There are, however, several technical difficulties in extending this approach to computing levels in arrangements of surface patches. Using the random-sampling technique, Agarwal *et al.* [16] developed an $O(n^{2+\varepsilon}k)$ expected-time algorithm for computing $\mathcal{A}_{\leq k}(\Gamma)$, for a collection Γ of n surface patches in $\mathbb{R}^3$. Their algorithm can be derandomized without affecting the asymptotic running time. For $d\geq 4$, Agarwal *et al.*'s and Mulmuley's algorithm can compute the $\leq k$-level in arrangements of n hyperplanes in expected time $O(n^{\lfloor d/2\rfloor}k^{\lceil d/2\rceil})$. These algorithms do not extend to computing the $\leq k$-level in surface arrangements because no nontrivial bound is known for the complexity of a triangulation of $\mathcal{A}_{\leq k}(\Gamma)$ in four and higher dimensions.

Constructing a single level. Edelsbrunner and Welzl [177] gave an $O(n \log n + b \log^2 n)$-time algorithm to construct the k-level in an arrangement of n lines in the plane, where b is the number of vertices of the k-level. This bound was slightly improved by Cole *et al.* [138] to $O(n \log n + b \log^2 k)$. However, these algorithms do not extend to computing the k-level in arrangements of curves. The approach by Agarwal *et al.* [15] can compute the k-level in an arrangement of lines in randomized expected time $O(n \log^2 n + nk^{1/3} \log^{2/3} n)$, and it extends to arrangements of curves and to arrangements of hyperplanes. In an unpublished note, Chan [111] slightly improves these bounds; in particular, he computes a single level in a line arrangement in randomized expected time $O(n \log n + nk^{1/3})$. Agarwal and Matoušek [23] describe an output-sensitive algorithm for computing the k-level in an arrangement of planes. The running time of their algorithm, after a slight improvement by Chan [109], is $O(n \log b + b^{1+\varepsilon})$, where b is the number of vertices of the k-level. Their algorithm can compute the k-level in an arrangement of hyperplanes in $\mathbb{R}^d$ in time $O(n \log b + (nb)^{1-1/(\lfloor d/2 \rfloor + 1)+\varepsilon} + bn^{1-2/(\lfloor d/2 \rfloor + 1)+\varepsilon})$. As in the case of single cells, all the output-sensitive algorithms assume that the hyperplanes are in general position.

2.13.4 Marked cells

Let Γ be a set of n lines in the plane and S a set of m points in the plane. Edelsbrunner *et al.* [167] presented a randomized algorithm, based on the random-sampling technique, for computing $\mathcal{C}(S, \Gamma)$, the set of cells in $\mathcal{A}(\Gamma)$ that contain at least one point of S, whose expected running time is $O(m^{2/3-\varepsilon} n^{2/3+2\varepsilon} + m \log n + n \log n \log m)$, for any $\varepsilon > 0$. A deterministic algorithm with running time $O(m^{2/3} n^{2/3} \log^c n + n \log^3 n + m \log n)$ was developed by Agarwal [2]. However, both algorithms are rather complicated. A simple randomized divide-and-conquer algorithm, with $O((m\sqrt{n}+n) \log n)$ expected running time, was recently proposed by Agarwal *et al.* [24]. Using random sampling, they improved the expected running time to $O(m^{2/3} n^{2/3} \log^{2/3}(n/\sqrt{m}) + (m+n) \log n)$. If we are interested in computing the incidences between Γ and S, the best known algorithm is by Matoušek whose expected running time is $O(m^{2/3} n^{2/3} 2^{O(\log^*(m+n))} + (m+n) \log(m+n))$ [293]. An $\Omega(m^{2/3} n^{2/3} + (m+n) \log(m+n))$ lower bound for this problem is proved by Erickson [188] under a restricted model of computation. Matoušek's algorithm can be extended to higher dimensions to count the number of incidences between m points and n hyperplanes in $\mathbb{R}^d$ in time $O((mn)^{1-1/(d+1)} 2^{O(\log^*(m+n))} + (m+n) \log(m+n))$ [293].

The above algorithms can be modified to compute marked cells in arrangements of segments in the plane. The best known randomized algorithm is by Agarwal *et al.* [24] whose running time is $O(m^{2/3} n^{2/3} \log^2(n/\sqrt{m}) + (m + n \log m + n\alpha(n)) \log n)$. Little is known about computing marked cells in arrangements of arcs in the plane. Using a randomized incremental algorithm, $\mathcal{C}(S, \Gamma)$ can be computed in expected time $O(\lambda_{s+2}(n)\sqrt{m} \log n)$, where s is the maximum number of intersection points between a pair of arcs in Γ [359]. If Γ is a set of n unit-radius circles and S is a set of m points in the plane, the incidences between Γ and S can be computed using Matoušek's algorithm [293]. A recent result of Agarwal and Sharir [34] computes all incidences between m points and n circles in time close to $O(m^{2/3} n^{2/3} + m^{6/11} n^{9/11} + m + n)$. See also Chapter 4.

Randomized incremental algorithms can be used to construct marked cells in arrangements of hyperplanes in higher dimensions in time close to their worst-case complexity. For example, if Γ is a set of n planes in $\mathbb{R}^3$ and S is a set of m points in $\mathbb{R}^3$, then the incidence graph of cells in $\mathcal{C}(S, \Gamma)$ can be computed in expected time $O(nm^{2/3} \log n)$ [143]. For $d \geq 4$, the expected running time is $O(m^{1/2} n^{d/2} \log^\gamma n)$, where $\gamma = (\lfloor d/2 \rfloor - 1)/2$. de Berg *et al.* [148] describe an efficient point-location algorithm in the zone of a k-flat in an arrangement of hyperplanes in $\mathbb{R}^d$. Their algorithm can answer

a query in $O(\log n)$ time using $O(n^{\lfloor (d+k)/2 \rfloor} \log^{\gamma} n)$ space, where $\gamma = d + k \pmod 2$.

2.13.5 Union of objects

Let Γ be a set of n semialgebraic simply connected regions in the plane, each of constant description complexity. The union of Γ can be computed in $O(f(n) \log^2 n)$ time by a divide-and-conquer technique, similar to that described in Section 2.13.3 for computing $\mathcal{A}_{\leq k}(\Gamma)$. Here $f(m)$ is the maximum complexity of the union of a subset of Γ of size m. Alternatively, $\bigcup \Gamma$ can be computed in $O(f(n) \log n)$ expected time using the *lazy* randomized incremental algorithm by De Berg *et al.* [143]. As a consequence, the union of n convex fat objects, each of constant description complexity, can be computed in $O(n^{1+\varepsilon})$ time, for any $\varepsilon > 0$. The same holds for κ-round or (α, β)-covered objects; see Section 2.9.

While the union of triangles can have quadratic complexity in the worst case, there have been several attempts to compute the union efficiently in certain favorable situations; see [190, 191].

Aronov *et al.* [66] modified the approach by Agarwal *et al.* [10] so that the union of n convex polytopes in $\mathbb{R}^3$ with a total of s vertices can be computed in expected time $O(sn \log n \log s + n^3)$. The same approach can be used to compute the union of n congruent cylinders in time $O(n^{8/3+\varepsilon})$. (Again, consult Section 2.9 for the corresponding bounds on the complexity of the union.)

Many applications call for computing the volume or surface area of $\bigcup \Gamma$ instead of its combinatorial structure. Overmars and Yap [321] showed that the volume of the union of n axis-parallel boxes in $\mathbb{R}^d$ can be computed in $O(n^{d/2} \log n)$ time. Edelsbrunner [160] gave an elegant formula for the volume and the surface area of the union of n balls in $\mathbb{R}^d$, which can be used to compute the volume efficiently.

2.14 Applications

In this section we present a sample of applications of arrangements. We discuss a few specific problems that can be reduced to bounding the complexity of various substructures of arrangements of surfaces or to computing these substructures. We also mention a few general areas that have motivated several problems involving arrangements and in which arrangements have played an important role.

2.14.1 Range searching

A typical range-searching problem is defined as follows: *Preprocess a set S of n points in $\mathbb{R}^d$, so that all points of S lying in a query region can be reported (or counted) quickly.* A special case of range searching is halfspace range searching, in which the query region is a halfspace. Because of numerous applications, range searching has received much attention during the last twenty years. See [5, 18, 294] for surveys on range searching and its applications.

If we define the dual of a point $p = (a_1, \ldots, a_d)$ to be the hyperplane $p^* : x_d = -a_1 x_1 - \cdots - a_{d-1} x_{d-1} + a_d$, and the dual of a hyperplane $h : x_d = b_1 x_1 + \cdots + b_{d-1} x_{d-1} + b_d$ to be the point $h^* = (b_1, \ldots b_d)$, then p lies above (resp. below, on) h if and only if the hyperplane p^* lies above (resp. below, on) the point h^*. Hence, halfspace range searching has the following equivalent "dual" formulation: Preprocess a set Γ of n hyperplanes in $\mathbb{R}^d$ so that the hyperplanes of H lying below a query point

can be reported quickly, or the level of a query point can be computed quickly. Using the point-location data structure for hyperplane arrangements given in [116], the level of a query point can be computed in $O(\log n)$ time using $O(n^d/\log^d n)$ space. This data structure can be modified to report all t hyperplanes lying below a query point in time $O(\log n + t)$. Chazelle *et al.* [126] showed, using results on arrangements, that a two-dimensional halfspace range-reporting query can be answered in $O(\log n + t)$ time using $O(n)$ space. In higher dimensions, by constructing $(1/r)$-cuttings for $\mathcal{A}_{\leq k}(\Gamma)$, Matoušek [289] developed a data structure that can answer a halfspace range-reporting query in time $O(\log n + t)$ using $O(n^{\lfloor d/2 \rfloor} \log^c n)$ space, for some constant c. He also developed a data structure that can answer a query in time $O(n^{1-1/\lfloor d/2 \rfloor} \log^c n + t)$ using $O(n \log\log n)$ space [289]. See also [7, 128]. Using linearization, a semialgebraic range-searching query, where one wants to report all points of S lying inside a semialgebraic set of constant description complexity, can be answered efficiently using some of the halfspace range-searching data structures [22, 386].

Point location in hyperplane arrangements can be used for simplex range searching [129], ray shooting [21, 22, 297], and several other geometric searching problems [37].

2.14.2 Transversals

Let S be a set of n compact convex sets in $\mathbb{R}^d$. A hyperplane h is called a *transversal* of S if h intersects every member of S. Let $\mathcal{T}(S)$ denote the space of all hyperplane transversals of S. We wish to study the structure of $\mathcal{T}(S)$. To facilitate this study, we apply the dual transform described in Section 2.14.1. Let $h : x_d = a_1 x_1 + \cdots + a_{d-1} x_{d-1} + a_d$ be a hyperplane that intersects a set $s \in S$. Translate h up and down until it becomes tangent to s. Denote the resulting upper and lower tangent hyperplanes by

$$x_d = a_1 x_1 + \cdots + a_{d-1} x_{d-1} + U_s(a_1, \ldots, a_{d-1})$$

and

$$x_d = a_1 x_1 + \cdots + a_{d-1} x_{d-1} + L_s(a_1, \ldots, a_{d-1}),$$

respectively. Then we have

$$L_s(a_1, \ldots, a_{d-1}) \leq a_d \leq U_s(a_1, \ldots, a_{d-1}).$$

Now if h is a transversal of S, we must have

$$\max_{s \in S} L_s(a_1, \ldots, a_{d-1}) \leq a_d \leq \min_{s \in S} U_s(a_1, \ldots, a_{d-1}).$$

In other words, if we define $\Gamma = \{U_s \mid s \in S\}$ and $\Gamma' = \{L_s \mid s \in S\}$, then $\mathcal{T}(S)$ is $\mathcal{S}(\Gamma, \Gamma')$, the region lying below the lower envelope of Γ and above the upper envelope of Γ'. The results of Agarwal *et al.* [26] (resp., of Koltun and Sharir [270]) imply that if each set in S has constant description complexity, then the complexity of $\mathcal{T}(S)$ is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, in $\mathbb{R}^3$, and $O(n^{3+\varepsilon})$, for any $\varepsilon > 0$, in $\mathbb{R}^4$. The results in [26] concerning the complexity of the vertical decomposition of $S(\Gamma, \Gamma')$ imply that $\mathcal{T}(S)$ can be constructed in $O(n^{2+\varepsilon})$ time. In four dimensions, the results of [270] do not yield a near-cubic bound on the complexity of the vertical decomposition of the sandwich region. However, using the algorithm by Agarwal *et al.* [10] for point location in the minimization diagram of trivariate functions, we can preprocess S into a data structure of size $O(n^{3+\varepsilon})$ so that we can determine in $O(\log n)$ time whether a hyperplane h is a transversal of S. No sharp bounds are known on $\mathcal{T}(S)$ in higher dimensions.

The problem can be generalized by considering lower-dimensional transversals. For example, in $\mathbb{R}^3$ we can also consider the space of all line transversals of S (lines that meet every member of S), where a similar reduction to the sandwich region between two envelopes can be shown. This will be discussed in detail in Chapter 7.

2.14.3 Geometric optimization

In the past decade, many problems in geometric optimization have been successfully attacked by techniques that reduce the problem to constructing and searching in various substructures of surface arrangements. Hence, the area of geometric optimization is a natural extension, and a good application area, of the study of arrangements. See [27, 29, 310] for surveys on geometric optimization.

One of the basic techniques for geometric optimization is the *parametric searching* technique, originally proposed by Megiddo [302]. This technique reduces the optimization problem to a *decision problem*, where one needs to compare the optimal value to a given parameter. In most cases, the decision problem is easier to solve than the optimization problem. The parametric searching technique proceeds by a parallel simulation of a generic version of the decision procedure with the (unknown) optimum value as an input parameter. In most applications, careful implementation of this technique leads to a solution of the optimization problem whose running time is larger than that of the decision algorithm only by a polylogarithmic factor. See [29] for a more detailed survey of parametric searching and its applications.

Several alternatives to parametric searching have been developed during the past decade. They use randomization [30, 110, 287], expander graphs [256], and searching in monotone matrices [198]. Like parametric searching, all these techniques are based on the availability of an efficient procedure for the decision problem. When applicable, they lead to algorithms with running times that are similar to, and sometimes slightly better than, those yielded by parametric searching.

These methods have been used to solve a wide range of geometric optimization problems, many of which involve arrangements. We mention a sample of such results.

Slope selection. Given a set S of n points in $\mathbb{R}^2$ and an integer k, find the line with the kth smallest slope among the lines passing through pairs of points of S. If we dualize the points in S to a set Γ of lines in $\mathbb{R}^2$, the problem becomes that of computing the kth leftmost vertex of $\mathcal{A}(\Gamma)$. Cole *et al.* [137] developed a rather sophisticated $O(n \log n)$-time algorithm for this problem, which is based on parametric searching. (Here the decision problem is to determine whether at most k vertices of $\mathcal{A}(\Gamma)$ lie to the left of a given vertical line.) A considerably simpler algorithm, based on $(1/r)$-cuttings, was later proposed by Brönnimann and Chazelle [97]. See also [255, 287].

Distance selection. Given a set S of n points in $\mathbb{R}^2$ and a parameter $k \leq \binom{n}{2}$, find the k-th largest distance among the points of S [14, 256]. The corresponding decision problem reduces to point location in a set of congruent disks in $\mathbb{R}^2$. Specifically, given a set Γ of m congruent disks in the plane, we wish to count efficiently the number of containments between disks of Γ and points of S. This problem can be solved using parametric searching [14], expander graphs [256], or randomization [287]. The deterministic algorithm of Katz and Sharir [256], runs in $O(n^{4/3} \log^{3+\varepsilon} n)$ time.

Segment center. Given a set S of n points in $\mathbb{R}^2$ and a line segment e, find a placement of e that minimizes the largest distance from the points of S to e [17, 182]. The decision problem reduces to determining whether given two families Γ and Γ' of bivariate surfaces, $\mathcal{S}(\Gamma, \Gamma')$, the region lying between L_Γ and $U_{\Gamma'}$, is empty. Exploiting the special properties of Γ and Γ', Efrat and Sharir [182] show that the complexity of $\mathcal{S}(\Gamma, \Gamma')$ is $O(n \log n)$. They describe an $O(n^{1+\varepsilon})$-time algorithm to determine whether $S(\Gamma, \Gamma')$ is empty, which leads to an $O(n^{1+\varepsilon})$-time algorithm for the segment-center problem.

Extremal polygon placement. Given a convex m-gon P and a closed polygonal environment Q with n vertices, find the largest similar copy of P that is fully contained in Q [362]. Here the decision problem is to determine whether P, with a fixed scaling factor, can be placed inside Q; this is a variant of the corresponding motion-planning problem for P inside Q, and is solved by constructing an appropriate representation of the 3-dimensional free configuration space, as a collection of cells in a corresponding 3-dimensional arrangement of surfaces. The running time of the whole algorithm is only slightly larger than the time needed to solve the fixed-size placement problem. The best running time is $O(mn\lambda_6(mn) \log^3 mn \log^2 n)$ [12]; see also [261, 362]. If Q is a convex n-gon, the largest similar copy of P that can be placed inside Q can be computed in $O(mn^2 \log n)$ time [6].

Diameter in 3D. Given a set S of n points in $\mathbb{R}^3$, determine the maximum distance between a pair of points in S. The problem is reduced to determining whether S lies in the intersection of a given set Γ of n congruent balls. A randomized algorithm with $O(n \log n)$ expected time was proposed by Clarkson and Shor [136]. A series of papers [85, 121, 298, 337, 336] describe near-linear-time deterministic algorithms, culminating in an optimal $O(n \log n)$ algorithm by Ramos [338].

Width in 3D. Given a set S of n points in $\mathbb{R}^3$, determine the smallest distance between two parallel planes enclosing S between them. This problem has been studied in a series of papers [10, 30, 121], and the currently best known randomized algorithms computes the width in $O(n^{3/2+\varepsilon})$ expected time [30]. The technique used in attacking the decision problems for this and the two following problems reduce them to point location in the region above the lower envelope of a collection of trivariate functions in $\mathbb{R}^4$.

Biggest stick in a simple polygon. Compute the longest line segment that can fit inside a given simple polygon with n edges. The current best solution is $O(n^{3/2+\varepsilon})$ [30] (see also [10, 35]).

Minimum-width annulus. Compute the annulus of smallest width that encloses a given set of n points in the plane. This problem arises in fitting a circle through a set of points in the plane. Again, the current best solution is $O(n^{3/2+\varepsilon})$ [30] (see also [10, 35]).

Approximations. Since the last three solution are fairly complicated, it is desirable to seek simpler approximation algorithms. Several recent works address this issue, such as Chan's elegant algorithms [112], and the coreset approach of Agarwal *et al.* [19, 20].

Geometric matching. Consider the problem where we are given two sets S_1, S_2 of n points in the plane, and we wish to compute a minimum-weight matching in the complete bipartite graph $S_1 \times S_2$,

where the weight of an edge (p, q) is the Euclidean distance between p and q. One can also consider the analogous nonbipartite version of the problem, which involves just one set S of $2n$ points, and the complete graph on S. The goal is to explore the underlying geometric structure of these graphs, to obtain faster algorithms than those available for general abstract graphs. Vaidya [375] had shown that both the bipartite and the nonbipartite versions of the problem can be solved in time close to $O(n^{5/2})$. A fairly sophisticated application of vertical decomposition in three-dimensional arrangements, given in [16], has improved the running time for the bipartite case to $O(n^{2+\varepsilon})$. Varadarajan [376] gave an $O(n^{3/2} \log^c n)$-time algorithm for the nonbipartite case.

Center point. A *center point* of a set S of n points in the plane is a point $\pi \in \mathbb{R}^2$ so that each line ℓ passing through π has the property that at least $\lfloor n/3 \rfloor$ points lie in each halfplane bounded by ℓ. It is well known that such a center point always exists [158]. If we dualize S to a set Γ of n lines in the plane, then π^*, the line dual to π, lies between $\mathcal{A}_{\lfloor n/3 \rfloor}(\Gamma)$ and $\mathcal{A}_{\lceil 2n/3 \rceil}(\Gamma)$. Cole *et al.* [138] described an $O(n \log^3 n)$-time algorithm for computing a center point of S, using parametric searching. A simple $O(n)$-time algorithm is given by Jadhav and Mukhopadhyay [253]. The problem of computing the set of all center points reduces to computing the convex hull of $\mathcal{A}_k(\Gamma)$ for a given k. Matoušek [285] described an $O(n \log^4 n)$-time algorithm for computing the convex hull of $\mathcal{A}_k(\Gamma)$ for any $k \leq n$; recall, in contrast, that the best known upper bound for $\mathcal{A}_k(\Gamma)$ is $O(n(k+1)^{1/3})$. In three dimensions, a recent result of Agarwal *et al.* [36] shows that the complexity of the center region of a set of n points is $O(n^2)$, and that it can be computed in $O(n^{2+\varepsilon})$ time. A much earlier work by Naor and Sharir [315] gives an algorithm for finding a single center point in three dimensions in nearly quadratic time.

Ham sandwich cuts. Let $S_1, S_2, \ldots, S_d$ be d sets of points in $\mathbb{R}^d$, each containing n points. Suppose n is even. A *ham sandwich cut* is a hyperplane h so that each open halfspace bounded by h contains at most $n/2$ points of S_i, for $i = 1, \ldots, d$. It is known [158, 385] that such a cut always exists. Let Γ_i be the set of hyperplanes dual to S_i. Then the problem reduces to computing a vertex of the intersection of $\mathcal{A}_{n/2}(\Gamma_1)$ and $\mathcal{A}_{n/2}(\Gamma_2)$. Megiddo [303] developed a linear-time algorithm for computing a ham sandwich cut in the plane if S_1 and S_2 can be separated by a line. For arbitrary point sets in the plane, a linear-time algorithms was later developed by Lo *et al.* [282]. Lo *et al.* also described an algorithm for computing a ham sandwich cut in $\mathbb{R}^3$ whose running time is $O(\psi_{n/2}(n) \log^2 n)$, where $\psi_k(n)$ is the maximum complexity of the k-level in an arrangement of n lines in the plane. By Dey's result on k-levels [150], the running time of their algorithm is $O(n^{4/3} \log^2 n)$.

2.14.4 Robotics

As mentioned in the introduction, *motion planning* for a robot system has been a major motivation for the study of arrangements. Let B be a robot system with d degrees of freedom, which is allowed to move freely within a given two- or three-dimensional environment cluttered with obstacles. Given two placements I and F of B, determining whether there exists a collision-free path between these placements reduces to determining whether I and F lie in the same cell of the arrangement of the family Γ of "contact surfaces" in $\mathbb{R}^d$, regarded as the configuration space of B (see the introduction for more details). If I and F lie in the same cell, then a path between I and F in $\mathbb{R}^d$ that does not intersect any surface of Γ corresponds to a collision-free path of B in the physical environment from I to F. Schwartz and Sharir [346] developed an $n^{2^{O(d)}}$-time algorithm for this problem. If d is a part of the input, the problem was later proved to be PSPACE-complete [104, 340]. Canny [103, 105] gave an

$n^{O(d)}$-time algorithm to compute the roadmap of a single cell in an arrangement $\mathcal{A}(\Gamma)$ of a set Γ of n surfaces in $\mathbb{R}^d$ provided that the cells in $\mathcal{A}(\Gamma)$ form a Whitney regular stratification of $\mathbb{R}^d$ (see [209] for the definition of Whitney stratification). Using a perturbation argument, he showed that his approach can be extended to obtain a Monte Carlo algorithm to determine whether two points lie in the same cell of $\mathcal{A}(\Gamma)$. The algorithms was subsequently extended and improved by many researchers see [79, 210, 246]. The best known algorithm, due to Basu *et al.* [79], can compute the roadmap in time $n^{d+1}b^{O(d^2)}$. Much work has been done on developing efficient algorithms for robots with a small number of degrees of freedom, say, two or three [226, 238, 260]. The result by Schwarzkopf and Sharir [349] gives an efficient algorithm for computing a collision-free path between two given placements for fairly general robot systems with three degrees of freedom. See [229, 347, 358] for surveys on motion-planning algorithms.

It is impractical to compute the roadmap, or any other explicit representation, of a single cell in $\mathcal{A}(\Gamma)$ if d is large. A general Monte Carlo algorithm for computing a *probabilistic roadmap* of a cell in $\mathcal{A}(\Gamma)$ is described by Kavraki *et al.* [258]. This approach avoids computing the cell explicitly. Instead, it samples a large number of random points in the configuration space and only those configurations that lie in the free configuration space (FP) are retained (they are called *milestones*); we also add I and F as milestones. The algorithm then builds a "connectivity graph" whose nodes are these milestones, and whose edges connect pairs of milestones if the line segment joining them in configuration space lies in FP (or if they satisfy some other "local reachability" rule). Various strategies have been proposed for choosing random configurations [48, 76, 251, 257]. The algorithm returns a path from I to F if they lie in the same connected component of the resulting network. Note that this algorithm may fail to return a collision-free path from I to F even if there exists one. This technique nevertheless has been successful in several real-world applications.

Assembly planning is another area in which the theory of arrangements has led to efficient algorithms. An *assembly* is a collection of objects (called parts) placed rigidly in some specified relative positions so that no two objects overlap. A *subassembly* of an assembly A is a subset of objects in A in their relative placements in A. An assembly operation is a motion that merges some subassemblies of A into a new and larger subassembly. An *assembly sequence* for A is a sequence of assembly operations that starts with the individual parts separated from each other and ends up with the full assembly A. The goal of assembly planning is to compute an assembly sequence for a given assembly. A classical approach to assembly sequencing is *disassembly sequencing*, which separates an assembly into its individual parts [250]. The reverse order of a sequence of disassemblying operations yields an assembly sequence. Several kinds of motion have been considered in separating parts of an assembly, including translating a subassembly along a straight line, arbitrary translational motion, rigid motion, etc. A common approach to generate a disassembly sequence is the so-called *nondirectional blocking graph* approach. It partitions the space of all allowable motions of separation into a finite number of cells so that within each cell the set of "blocking relations" between all pairs of parts remains fixed. The problem is then reduced to computing representative points in cells of the arrangement of a family of surfaces. This approach has been successful in many instances, including polyhedral assembly with infinitesimal rigid motion [216]; see also [229, 230].

Other problems in robotics that have exploited arrangements include fixturing [339], MEMS (micro electronic mechanical systems) [89], path planning with uncertainty [144], and manufacturing [38].

2.14.5 Molecular modeling

In the introduction, we described the Van der Waals model, in which a molecule M is represented as a collection Γ of spheres in $\mathbb{R}^3$. (See [140, 161, 305] for other geometric models of molecules.) Let $\Sigma = \partial(\bigcup \Gamma)$. Σ is called the "surface" of M. Many problems in molecular biology, especially those that study the interaction of a protein with another molecule, involve computing the molecular surface, a portion of the surface (e.g., the so-called *active site* of a protein), or various features of the molecular surface [162, 232, 278, 377]. We briefly describe two problems in molecular modeling that can be formulated in terms of arrangements.

The chemical behavior of solute molecules in a solution is strongly dependent on the interactions between the solute and solvent molecules. These interactions are critically dependent on those molecular fragments that are accessible to the solvent molecules. Suppose we use the Van der Waals model for the solute molecule and model the solvent by a sphere S. By rolling S on the molecular surface Σ, we obtain a new surface Σ', traced by the center of the rolling sphere. If we enlarge each sphere of Γ by the radius of S, Σ' is the boundary of the union of the enlarged spheres.

As mentioned above, several methods have been proposed to model the surface of a molecule. The best choice of the model depends on the chemical problem the molecular surface is supposed to represent. For example, the Van der Waals model represents the space requirement of molecular conformations, while isodensity contours and molecular electrostatic potential contour surfaces [305] are useful in studying molecular interactions. An important problem in molecular modeling is to study the interrelations among various molecular surfaces of the same molecule. For example, let $\mathbf{\Sigma} = \{\Sigma_1, \ldots, \Sigma_m\}$ be a family of molecular surfaces of the same molecule. We may want to compute the arrangement $\mathcal{A}(\mathbf{\Sigma})$, or we may want to compute the subdivision of Σ_i induced by $\{\Sigma_j \cap \Sigma_i \mid 1 \leq j \neq i \leq m\}$.

Researchers have also been interested in computing "connectivity" of a molecule, e.g., computing voids, tunnels, and pockets of Σ. A *void* of Σ is a bounded component of $\mathbb{R}^3 \setminus (\bigcup \Gamma)$; a *tunnel* is a hole through $\bigcup \Gamma$ that is accessible from the outside, i.e., an "inner" part of a non-contractible loop in $\mathbb{R}^3 \setminus \bigcup \Gamma$; and a *pocket* is a depression or cavity on Σ. Pockets are not holes in the topological sense and are not well defined; see [141, 162] for some of the definitions proposed so far. Pockets and tunnels are interesting because they are good candidates to be *binding sites* for other molecules.

Efficient algorithms have been developed for computing Σ, connectivity of Σ, and the arrangement $\mathcal{A}(\Gamma)$ [161, 232, 377]. Halperin and Shelton [239] describe an efficient perturbation scheme to handle degeneracies while constructing $\mathcal{A}(\Gamma)$ or Σ. Some applications require computing the measure of different substructures of $\mathcal{A}(\Gamma)$, including the volume of Σ, the surface area of Σ, or the volume of a void of Σ. Edelsbrunner *et al.* [162] describe an efficient algorithm for computing these measures; see also [160, 161].

2.15 Conclusions

In this survey we reviewed a wide range of topics on arrangements of surfaces. We mentioned a few old results, but the emphasis of the survey was on the tremendous progress made in this area during the last fifteen years. We discussed combinatorial complexity of arrangements and their substructures, representation of arrangements, algorithms for computing arrangements and their substructures, and several geometric problems in which arrangements play pivotal roles. Although the survey covered

a broad spectrum of results, many topics on arrangements were either not included or very briefly touched upon. For example, we did not discuss arrangements of pseudo-lines and oriented matroids, we discussed algebraic and topological issues very briefly, and we mentioned a rather short list of applications that have exploited arrangements. There are numerous other sources where more details on arrangements and their applications can be found; see e.g. the books [86, 295, 323, 359] and the survey papers [32, 206, 228, 309, 322].

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Chapter 3

Davenport–Schinzel Sequences and Their Geometric Applications

3.1 Introduction

Davenport–Schinzel sequences, introduced by H. Davenport and A. Schinzel in the 1960s, are interesting and powerful combinatorial structures that arise in the analysis and construction of the lower (or upper) envelope of collections of univariate functions, and therefore have applications in a variety of geometric problems that can be reduced to computing such an envelope. In addition, Davenport–Schinzel sequences play a central role in many related geometric problems involving arrangements of curves and surfaces. For these reasons, they have become one of the major tools in the analysis of combinatorial and algorithmic problems in geometry.

Definition 3.1.1 Let n and s be two positive integers. A sequence $U = \langle u_1, \ldots, u_m \rangle$ of integers is an (n, s) *Davenport–Schinzel sequence* (a *$DS(n, s)$-sequence* for short) if it satisfies the following conditions:

(i) $1 \leq u_i \leq n$ for each $i \leq m$,

(ii) $u_i \neq u_{i+1}$ for each $i < m$, and

(iii) there do not exist $s + 2$ indices $1 \leq i_1 < i_2 < \cdots < i_{s+2} \leq m$ such that

$$u_{i_1} = u_{i_3} = u_{i_5} = \cdots = a, \quad u_{i_2} = u_{i_4} = u_{i_6} = \cdots = b,$$

and $a \neq b$.

In other words, the third condition forbids the presence of long alternations of any pair of distinct symbols in a Davenport–Schinzel sequence. We refer to s as the *order* of U, to n as the *number of symbols* composing U, and to $|U| = m$ as the *length* of the sequence U. Define

$$\lambda_s(n) = \max\{\, |U| \mid U \text{ is a } DS(n, s)\text{-sequence} \,\}.$$

Curiously, the original papers by Davenport and Schinzel [48, 49] were entitled *On a combinatorial problem connected with differential equations*, because they were motivated by a particular application that involved the pointwise maximum of a collection of independent solutions of a linear differential equation. This, however, is only a special case of more general lower or upper envelopes. Davenport and Schinzel did establish in [48, 49] the connection between envelopes and these sequences, and obtained several non-trivial bounds on the length of the sequences, which were later strengthened by Szemerédi [148]. The potential of *DS*-sequences to geometric problems, however, remained unnoticed until Atallah rediscovered and applied them to several problems in dynamic computational geometry [22]. It is easy to show that $\lambda_1(n) = n$ and $\lambda_2(n) = 2n - 1$ (see Theorem 3.3.1). Hart and Sharir [78] proved that $\lambda_3(n) = \Theta(n\alpha(n))$, where $\alpha(n)$ is the inverse Ackermann function (see below for details), and later Agarwal et al. [11] (see also Sharir [136, 137]) proved sharp bounds on $\lambda_s(n)$ for $s > 3$. These somewhat surprising bounds show that $\lambda_s(n)$ is nearly linear in n for any fixed s. Davenport–Schinzel sequences have become a useful and powerful tool for solving numerous problems in discrete and computational geometry, usually by showing that the geometric structure being analyzed has smaller combinatorial complexity than what more naive methods would have implied. Many such geometric applications have been obtained in the past twenty years, but we will skip them in these notes. we review some of these applications below. The book by Sharir and Agarwal [141] gives a more detailed description of the theory of *DS*-sequences and of their geometric applications. See also the survey of Agarwal and Sharir [10] and the recent book of Matoušek [103].

As noted above, and will be shown in more detail below, Davenport–Schinzel sequences provide a complete combinatorial characterization of the lower envelope of a collection of *univariate* functions. In many geometric problems, though, one faces the more difficult problem of calculating or analyzing the envelope of a collection of multivariate functions. Even for bivariate functions this problem appears to be considerably harder than the univariate case. Nevertheless, considerable progress has been made on the multivariate case, leading to almost-tight bounds on the complexity of envelopes in higher dimensions [75, 357]. Higher-dimensional lower envelopes and related combinatorial structures will be reviewed in the next chapter on arrangements.

The material reviewed in this chapter mostly covers the basic combinatorial analysis of Davenport–Schinzel sequences and some of their basic geometric applications, both combinatorial and algorithmic. Section 3.2 shows the connection between *DS*-sequences and lower envelopes. Sections 3.3–3.5 discuss the analysis of the maximum length of (n, s) Davenport–Schinzel sequences. Section 3.6 presents basic combinatorial geometric applications of Davenport–Schinzel sequences to two-dimensional arrangements of lines, segments, and arcs, and studies the role that these sequences play in various structures in such arrangements, including envelopes, individual faces, zones, and levels.

3.2 Davenport–Schinzel Sequences and Lower Envelopes

3.2.1 Lower envelopes of totally defined functions

Let $\mathcal{F} = \{f_1, \ldots, f_n\}$ be a collection of n real-valued, continuous totally defined functions so that the graphs of every pair of distinct functions intersect in at most s points (this is the case for polynomials of fixed degree, Chebychev systems, etc). The *lower envelope* of $\mathcal{F}$ is defined as

$$E_{\mathcal{F}}(x) = \min_{1 \le i \le n} f_i(x),$$

i.e., $E_{\mathcal{F}}$ is the *pointwise minimum* of the functions f_i; see Figure 3.1. Let $I_1, \ldots, I_m$ be the maximal connected intervals on the x-axis so that they cover the entire x-axis and, for each $k \leq m$, the same function f_{u_k} appears on $E_{\mathcal{F}}$ for all points in I_k (i.e., $E_{\mathcal{F}}(x) = f_{u_k}(x)$ for all $x \in I_k$). In other words, m is the number of (maximal) connected portions of the graphs of the f_i's that constitute the graph of $E_{\mathcal{F}}$. The endpoints of the intervals I_k are called the *breakpoints* of $E_{\mathcal{F}}$. Assuming that $I_1, \ldots, I_m$ are sorted from left to right, put

$$U(\mathcal{F}) = \langle u_1, \ldots, u_m \rangle.$$

$U(\mathcal{F})$ is called the *lower-envelope sequence* of $\mathcal{F}$; see Figure 3.1. The *minimization diagram* of $\mathcal{F}$, denoted by $M_{\mathcal{F}}$, is the partition of the x-axis induced by the intervals $I_1, \ldots, I_m$. The endpoints of these intervals are called the *breakpoints* of $M_{\mathcal{F}}$. For convenience, we add $-\infty, +\infty$ as the breakpoints of $M_{\mathcal{F}}$.

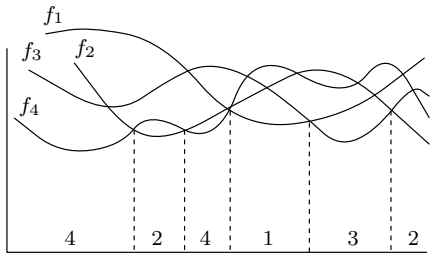


FIGURE 3.1. A lower-envelope sequence.

The *upper envelope* of $\mathcal{F}$ is defined, in a fully symmetric manner, to be

$$E^*_{\mathcal{F}}(x) = \max_{1 \leq i \leq n} f_i(x),$$

and the *maximization digram* $M^*_{\mathcal{F}}$ is defined as the corresponding partition of the real line, as in the case of lower envelopes. In this chapter we mostly consider lower envelopes. This choice is arbitrary, and all the results, of course, apply equally well to upper envelopes.

Theorem 3.2.1 ([22, 49]). *$U(\mathcal{F})$ is a $DS(n, s)$-sequence. Conversely, for any given $DS(n, s)$-sequence U, one can construct a set $\mathcal{F} = \{f_1, \ldots, f_n\}$ of continuous, totally defined, univariate functions, each pair of whose graphs intersect in at most s points, such that $U(\mathcal{F}) = U$.*

Proof (Sketch): For the first part, note that, by definition, the lower-envelope sequence $U = U(\mathcal{F})$ does not contain a pair of adjacent equal elements. For simplicity, assume that the graphs of functions in $\mathcal{F}$ intersect transversally at each intersection point. The proof can easily be extended to the case when the graphs of two functions touch each other. Suppose U contains $s+2$ indices $i_1 < i_2 < \cdots < i_{s+2}$ so that $u_{i_1} = u_{i_3} = \cdots = a$ and $u_{i_2} = u_{i_4} = \cdots = b$ for $a \neq b$. By definition of the lower-envelope sequence, we must have $f_a(x) < f_b(x)$ for $x \in (int(I_{i_1}) \cup int(I_{i_3}) \cup \cdots)$ and $f_a(x) > f_b(x)$ for $x \in (int(I_{i_2}) \cup int(I_{i_4}) \cup \cdots)$, where $int(J)$ denotes the interior of the interval J. Since f_a and f_b are continuous, there must exist $s+1$ distinct points $x_1, \ldots, x_{s+1}$ so that x_r lies between the intervals I_{i_r} and $I_{i_{r+1}}$ and $f_a(x_r) = f_b(x_r)$, for $r = 1, \ldots, s+1$. This, however, contradicts the fact that the graphs of f_a and f_b intersect in at most s points.

For the converse statement, let $U = \langle u_1, \ldots, u_m \rangle$ be a given $DS(n, s)$-sequence. Without loss of generality, suppose the symbols $1, 2, \ldots, n$, of which U is composed, are ordered so that the leftmost

appearance of symbol i in U precedes the leftmost appearance of symbol j in U if and only if $i < j$. We now define the required collection of functions $\mathcal{F} = \{f_1, \ldots, f_n\}$ as follows. We choose $m-1$ distinct "transition points" $x_2 < x_3 < \ldots < x_m$ on the x-axis, and $n+m-1$ distinct horizontal "levels," say, at $y = 1, 2, \ldots, n, -1, -2, \ldots, -(m-1)$. For each symbol $1 \le a \le n$ the graph of the corresponding function f_a is always horizontal at one of these levels, except at short intervals near some of the transition points, where it can drop very steeply from one level to a lower one. At each transition point exactly one function changes its level. More specifically:

(i) Before x_2, the function f_a is at the level $y = a$, for $a = 1, \ldots, n$.

(ii) At the transition point x_i, let $a = u_i$; then f_a drops down from its current level to the highest still "unused" level. See Figure 3.2 for an illustration.

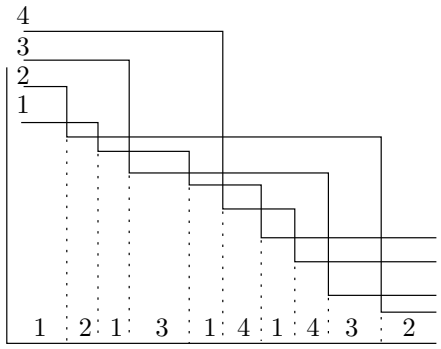


FIGURE 3.2. Realization of the $DS(4,3)$-sequence $\langle 1,2,1,3,1,4,1,4,3,2\rangle$.

It is clear from this construction that $U(\mathcal{F}) = U$, and it can be shown that each pair of functions intersect in at most s points. This completes the proof of the theorem. □

Corollary 3.2.2. *For any collection $\mathcal{F} = \{f_1, \ldots, f_n\}$ of n continuous, totally defined, univariate functions, each pair of whose graphs intersect in at most s points, the length of the lower-envelope sequence $U(\mathcal{F})$ is at most $\lambda_s(n)$, and this bound can be attained by such a collection $\mathcal{F}$.*

Corollary 3.2.3. *Let $\mathcal{F} = \{f_1, \ldots, f_n\}$ and $\mathcal{G} = \{g_1, \ldots, g_n\}$ be two collections of n continuous, totally defined, univariate functions, such that the graphs of any pair of functions of $\mathcal{F}$, or of any pair of functions of $\mathcal{G}$, intersect in at most s points, and the graphs of any pair of functions in $\mathcal{F} \times \mathcal{G}$ intersect in a (possibly larger) constant number of points. Then the number of intersection points of graphs of functions in $\mathcal{F} \cup \mathcal{G}$ that lie on the boundary of the region lying between the upper envelope of $\mathcal{G}$ and the lower envelope of $\mathcal{F}$ (see Figure 3.3), i.e., the region*

$$\Pi_{\mathcal{F},\mathcal{G}} = \{(x,y) \mid E^*_{\mathcal{G}}(x) \le y \le E_{\mathcal{F}}(x)\},$$

is $O(\lambda_s(n))$.

Proof: Let $L = (b_1, \ldots, b_t)$ be the sequence of breakpoints of $M_{\mathcal{F}}$ and $M^*_{\mathcal{G}}$, sorted from left to right. By definition, $t \le 2\lambda_s(n)$. In each interval (b_i, b_{i+1}), the envelopes $E_{\mathcal{F}}$, $E^*_{\mathcal{G}}$ are attained by a unique pair of functions $f^{(i)} \in \mathcal{F}$, $g^{(i)} \in \mathcal{G}$. Hence, there are $O(1)$ intersection points on the boundary of $\Pi_{\mathcal{F},\mathcal{G}}$ whose x-coordinates lie in (b_i, b_{i+1}). This completes the proof of the corollary. □

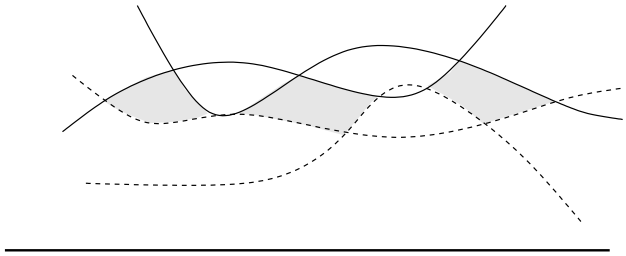


FIGURE 3.3. The region between $E^*_{\mathcal{G}}$ and $E_{\mathcal{F}}$ is shown shaded; the graphs of the functions in $\mathcal{F}$ (resp. in $\mathcal{G}$) are drawn solid (resp. dashed).

3.2.2 Lower envelopes of partially defined functions

It is useful to note that a similar equivalence exists between Davenport–Schinzel sequences and lower envelopes of *partially defined* functions. Specifically, let $f_1, \ldots, f_n$ be a collection of partially defined and continuous functions, so that the domain of definition of each function f_i is an interval I_i, and suppose further that the graphs of each pair of these functions intersect in at most s points. The lower envelope of $\mathcal{F}$ is now defined as

$$E_{\mathcal{F}}(x) = \min f_i(x),$$

where the minimum is taken over those functions that are defined at x. One can then define the minimization diagram $M_{\mathcal{F}}$ and the lower-envelope sequence $U(\mathcal{F})$ in much the same way as for totally defined functions; see Figure 3.4. In this case the following theorem holds.

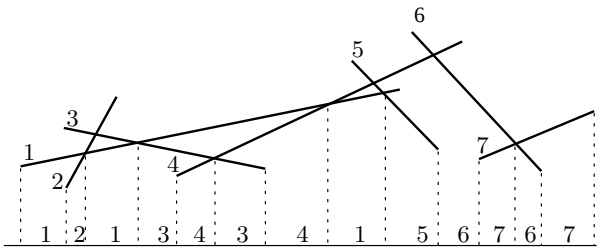


FIGURE 3.4. The lower envelope of a collection of (nonvertical) segments.

Theorem 3.2.4 ([78]). *The lower-envelope sequence $U(\mathcal{F})$ is a $DS(n, s+2)$-sequence. Conversely, for any $DS(n, s+2)$-sequence U one can construct a collection $\mathcal{F} = \{f_1, \ldots, f_n\}$ of partially-defined, continuous functions, each defined over an interval, and each pair of which intersect in at most s points, such that $U(\mathcal{F}) = U$.*

Hence, we can conclude

Theorem 3.2.5 ([78]). *Let $\mathcal{F}$ be a collection of n partially-defined, continuous, univariate functions, with at most s intersection points between the graphs of any pair. Then the length of the lower-envelope sequence $U(\mathcal{F})$ is at most $\lambda_{s+2}(n)$.*

The functions constructed in Theorems 3.2.1 and 3.2.4, to realize arbitrary $DS(n,s)$-sequences, have fairly irregular structure (cf. Figure 3.2). A problem that arises naturally in this context is whether any $DS(n,s)$-sequence can be realized as the lower envelope sequence of a collection of n partial or total functions of some canonical form. For example, can any $(n,3)$-Davenport–Schinzel sequence be realized as the lower envelope sequence of a collection of n line segments (see Figure 3.4)? Some partially affirmative results on geometric relatization of $DS(n,s)$-sequences will be mentioned below, although the problem is still wide open.

3.2.3 Constructing lower envelopes

We conclude this section by presenting a simple, efficient divide-and-conquer algorithm for computing the minimization diagram of a set $\mathcal{F}$ of n continuous, totally defined, univariate functions, each pair of whose graphs intersect at most s times, for some constant parameter s. Here we assume a model of computation that allows us to compute the intersections between any pair of functions in $\mathcal{F}$ in $O(1)$ time.

We partition $\mathcal{F}$ into two subsets $\mathcal{F}_1, \mathcal{F}_2$, each of size at most $\lceil n/2 \rceil$, compute the minimization diagrams $M_{\mathcal{F}_1}, M_{\mathcal{F}_2}$ recursively, and merge the two diagrams to obtain $M_{\mathcal{F}}$. We merge the lists of breakpoints of $M_{\mathcal{F}_1}$ and of $M_{\mathcal{F}_2}$ into a single list $V = (v_1 = -\infty, v_2, \ldots, v_t = +\infty)$, sorted from left to right. Notice that, for any $1 \le i < t$, there is a unique pair of functions $f_i^{(1)} \in \mathcal{F}_1$, $f_i^{(2)} \in \mathcal{F}_2$, that attain the respective envelopes $E_{\mathcal{F}_1}$, $E_{\mathcal{F}_2}$ over (v_i, v_{i+1}). We compute the real roots $r_1, \ldots, r_k$ $(k \le s)$ of the function $f_i^{(1)} - f_i^{(2)}$ that lie in the interval (v_i, v_{i+1}), and add them to the list V. Let $V' = (v'_1, \ldots, v'_{t'})$ denote the new list of points. It is clear that, for each i, a unique function $f_i \in \mathcal{F}$ attains $E_{\mathcal{F}}$ over the interval (v'_i, v'_{i+1}). We associate f_i with this interval. If the same function is associated with two adjacent intervals (v'_{i-1}, v'_i) and (v'_i, v'_{i+1}), we delete the breakpoint v'_i from V'. The resulting list represents the minimization diagram $M_{\mathcal{F}}$ of $\mathcal{F}$. The total time spent in the merge step is

$$O(|V'|) = O(|V|) = O(|M_{\mathcal{F}_1}| + |M_{\mathcal{F}_2}|) = O(\lambda_s(n)).$$

Hence, the overall running time of the algorithm is $O(\lambda_s(n) \log n)$.

If the functions in $\mathcal{F}$ are partially defined, an easy modification of the above algorithm constructs $M_{\mathcal{F}}$ in time $O(\lambda_{s+2}(n) \log n)$. In this case, however, $M_{\mathcal{F}}$ can be computed in time $O(\lambda_{s+1}(n) \log n)$, using a more clever algorithm due to Hershberger [79].

Theorem 3.2.6 ([22, 79]). *The lower envelope of a set $\mathcal{F}$ of n continuous, totally defined, univariate functions, each pair of whose graphs intersect in at most s points, can be constructed, in an appropriate model of computation, in $O(\lambda_s(n) \log n)$ time. If the functions in $\mathcal{F}$ are partially defined, then $E_{\mathcal{F}}$ can be computed in $O(\lambda_{s+1}(n) \log n)$ time. In particular, the lower envelope of a set of n segments in the plane can be computed in optimal $O(n \log n)$ time.*

3.3 Simple Bounds and Variants

One of the main goals in the analysis of DS-sequences is to estimate the value of $\lambda_s(n)$. In this section we review some of the earlier results that established nontrivial bounds on $\lambda_s(n)$. These bounds are somewhat weaker than the best known bounds, but have simpler proofs. We begin our analysis by disposing of the two simple cases $s = 1$ and $s = 2$.

Theorem 3.3.1 ([49]). (a) $\lambda_1(n) = n$. (b) $\lambda_2(n) = 2n - 1$.

Proof (Sketch): (a) Let U be a $DS(n, 1)$-sequence. U cannot contain any subsequence of the form $\langle a \cdots b \cdots a \rangle$, for $a \neq b$, and any two adjacent elements of U are distinct, therefore all elements of U are distinct, which implies that $|U| \leq n$. The bound is tight, because $U = \langle 1\ 2\ 3 \cdots n \rangle$ is a $DS(n, 1)$-sequence.

(b) The proof proceeds by induction on n. The case $n = 1$ is obvious. Suppose the claim holds for $n - 1$, and let U be any $DS(n, 2)$-sequence. Without loss of generality, we can assume that the leftmost occurrence of i in U is before the leftmost occurrence of j if and only if $i < j$. It can then be shown that there is only one occurrence of n in U, or else a forbidden subsequence of the form $\langle x \cdots n \cdots x \cdots n \rangle$ would arise. Remove this single appearance of n from U, and if the two symbols adjacent to n are equal, remove also one of them from U. The resulting sequence is clearly a $DS(n - 1, 2)$-sequence, and is one or two elements shorter than U. The induction hypothesis then implies $|U| \leq 2n - 3 + 2 = 2n - 1$. Since the sequence $\langle 1\ 2\ 3 \cdots n - 1\ n\ n - 1 \cdots 3\ 2\ 1 \rangle$ is clearly a $DS(n, 2)$-sequence of length $2n - 1$, the bound is tight. □

A cyclic sequence U is called a $DS(n, 2)$-*cycle* if no two adjacent symbols are equal and if U does not contain a subcycle of the form $\langle a \cdots b \cdots a \cdots b \rangle$, for any $a \neq b$. Notice that the maximum length of a $DS(2, 2)$-cycle is 2. The same argument as in Theorem 3.3.1(b) can be used to prove the following.

Corollary 3.3.2. *The maximum length of a $DS(n, 2)$-cycle consisting of n symbols is $2n - 2$.*

As we will see later, obtaining a sharp bounds on the maximum length of a $DS(n, s)$-sequence, for $s \geq 3$, is not as simple. Let us first give a simple proof of the following bound:

Theorem 3.3.3 ([49]). $\lambda_3(n) = O(n \log n)$.

Proof (Sketch): Let U be a $DS(n, 3)$-sequence of length $\lambda_3(n)$. There must exist a symbol x that appears in U at most $\lambda_3(n)/n$ times. For any appearance of x which is neither the leftmost nor the rightmost, the symbols immediately preceding and succeeding x must be different, or else we would have obtained a forbidden subsequence of the form $\langle x \cdots yxy \cdots x \rangle$. Hence, if we erase from U all appearances of x, and, if necessary, at most two other elements, near the first and last appearances of x, we obtain a $DS(n - 1, 3)$-sequence, so this analysis implies the recurrence

$$\lambda_3(n) \leq \lambda_3(n - 1) + \frac{\lambda_3(n)}{n} + 2 \, ,$$

or

$$\frac{\lambda_3(n)}{n} \leq \frac{\lambda_3(n - 1)}{n - 1} + \frac{2}{n - 1} \, ,$$

from which the claim follows easily. □

This bound was later improved by Davenport [48] to $O(n \log n / \log \log n)$. For any given n and s, a trivial upper bound on $\lambda_s(n)$ is $sn(n - 1)/2 + 1$ (use Corollary 3.2.2 and the observation that the total number of intersections between the functions in $\mathcal{F}$ is at most $s\binom{n}{2}$). Roselle and Stanton [130] proved that, for $s > n$, one has

$$\lambda_s(n) \geq sn(n - 1)/2 - cn^3,$$

where $c < 1$ is a constant. Davenport and Schinzel [49] proved that, for any fixed s, there is a constant C_s depending on s, such that $\lambda_s(n) \leq n \cdot 2^{C_s\sqrt{\log n}}$. The problem was also studied in several early papers [52, 107, 111, 121, 129, 131, 146, 147], but the next significant improvement on the bound of $\lambda_s(n)$ was made by Szemerédi [148], who proved that $\lambda_s(n) \leq A_s n \log^* n$, for each $s \geq 3$ and for appropriate positive constants A_s (doubly exponential in s). The currently best known bounds on $\lambda_s(n)$ for $s \geq 3$, stated below, are by Hart and Sharir [78] and Agarwal et al. [11].

$$\begin{aligned}
\lambda_3(n) &= \Theta(n\alpha(n)), \\
\lambda_4(n) &= \Theta(n \cdot 2^{\alpha(n)}), \\
\lambda_{2s+2}(n) &= n \cdot 2^{\Theta(\alpha^s(n))}, \qquad \text{for } s \geq 2, \\
\lambda_{2s+3}(n) &= n \cdot 2^{O(\alpha^s(n)\log\alpha(n))}, \quad \text{for } s \geq 1;
\end{aligned}$$

more precise forms of these bounds are given in Theorems 3.4.3, 3.4.5, 3.5.1, and 3.5.2 below.

We conclude this section by mentioning some generalizations of $DS(n, s)$-sequences. Let $U = \langle u_1, u_2, \ldots, u_m\rangle$ be a $DS(n, s)$-sequence. For $1 \leq j \leq m$, let $\mu(j)$ denote the number of symbols whose leftmost occurrences in U occur at an index $\leq j$ and whose rightmost occurrences occur at an index $> j$. We define the *depth* of U to be the maximum value of $\mu(j)$, for $j \leq m$. Define a $DS(n, s, t)$-sequence to be a $DS(n, s)$-sequence whose depth is at most t, and let $\lambda_{s,t}(n)$ denote the maximum length of a $DS(n, s, t)$-sequence. Huttenlocher et al. [81] proved that $\lambda_{s,t}(n) \leq \lceil n/t\rceil\lambda_s(2t)$ (see also Har-Peled [76]). This result has the following interesting consequence:

Corollary 3.3.4 ([81]). *Let $\mathcal{F} = \{f_1, \ldots, f_t\}$ be a collection of t continuous, real-valued, piecewise-linear functions (i.e., the graph of each f_i is an x-monotone polygonal chain). Let n be the total number of edges in the graphs of the functions of $\mathcal{F}$. Then the lower envelope of $\mathcal{F}$ has at most $\lambda_{3,t}(n) \leq \lceil n/t\rceil\lambda_3(2t) = O(n\alpha(t))$ breakpoints.*

Adamec et al. [2] have studied some generalizations of Davenport–Schinzel sequences. In particular, they bound the length of sequences not containing more general forbidden subsequences, for example, subsequences consisting of more than two symbols. They also showed that the maximum length of a sequence not containing any forbidden subsequence $\langle a^{i_1}b^{i_2}a^{i_3}b^{i_4}\rangle$, where i_1, i_2, i_3, i_4 are some fixed positive constants, is linear. See also [93, 94, 95, 96, 97] for related results.

3.4 Sharp Upper Bounds on $\lambda_s(n)$

In the previous section we mentioned some weak upper bounds on $\lambda_s(n)$. The problem of bounding $\lambda_s(n)$ lay dormant for about 10 years after Szemerédi's result [148], until Hart and Sharir [78] proved a tight bound of $\Theta(n\alpha(n))$ on $\lambda_3(n)$; here $\alpha(n)$ is the inverse Ackermann function, defined below. Later, Sharir [136] extended the analysis of Hart and Sharir to prove that $\lambda_s(n) = n \cdot \alpha(n)^{O(\alpha(n)^{s-3})}$, for $s > 3$. Applying a more careful analysis, Agarwal et al. [11] improved the bounds further, and obtained sharp, nearly tight bounds on $\lambda_s(n)$, for any fixed s. The best known upper bounds on $\lambda_s(n)$ are summarized in Theorem 3.4.3 (for $s = 3$) and Theorem 3.4.5 (for larger values of s). Since the proofs of these theorems are quite technical, we will sketch the proof of Theorem 3.4.3, and only briefly mention how the proof extends to the case $s > 3$.

3.4.1 Ackermann's function—A review

In this subsection we recall the definition of *Ackermann's function* and its functional inverse, which appears in the upper and lower bounds for $\lambda_s(n)$. Ackermann's function (also called "generalized exponentials") is an extremely fast growing function defined over the integers in the following recursive manner [1].

Let $\mathbb{N}$ denote the set of positive integers. Given a function g from a set into itself, denote by $g^{(s)}$ the composition $g \circ g \circ \ldots \circ g$ of g with itself s times, for $s \in \mathbb{N}$. Define inductively a sequence $\{A_k\}_{k=1}^{\infty}$ of functions from $\mathbb{N}$ into itself as follows:

$$\begin{array}{lcll} A_1(n) & = & 2n & n \geq 1, \\ A_k(1) & = & 2 & k \geq 2, \\ A_k(n) & = & A_{k-1}(A_k(n-1)) & n \geq 2, k \geq 2. \end{array}$$

Finally, define *Ackermann's function* itself as $A(n) = A_n(n)$. The function A grows very quickly; its first few values are: $A(1) = 2$, $A(2) = 4$, $A(3) = 16$, and $A(4)$ is an exponential "tower" of 65536 2s. See [91, 120, 125] for a discussion on Ackermann's and other rapidly growing functions.

Let α_k and α denote the functional inverses of A_k and A, respectively. That is,

$$\alpha_k(n) = \min\{s \geq 1 \mid A_k(s) \geq n\} \quad \text{and} \quad \alpha(n) = \min\{s \geq 1 \mid A(s) \geq n\}.$$

The functions α_k are easily seen to satisfy the following recursive formula:

$$\alpha_k(n) = \min\{s \geq 1 : \alpha_{k-1}^{(s)}(n) = 1\}; \tag{3.4.1}$$

that is, $\alpha_k(n)$ is the number of iterations of α_{k-1} needed to go from n to 1. In particular, (3.4.1) implies that, for $n \in \mathbb{N}$,

$$\alpha_1(n) = \lceil n/2 \rceil, \quad \alpha_2(n) = \lceil \log n \rceil, \quad \text{and} \quad \alpha_3(n) = \log^* n.$$

For each k, the function α_k is nondecreasing and unbounded. The same holds for α too, which grows more slowly than any of the α_k. Note that $\alpha(n) \leq 4$ for all $n \leq A(4)$, which is an exponential tower with 65536 2s, thus $\alpha(n) \leq 4$ for all practical values of n. We will need the following two easily established properties of $\alpha_k(n)$:

$$\alpha_{\alpha(n)}(n) = \alpha(n) \quad \text{and,} \quad \text{for } n > 4, \quad \alpha_{\alpha(n)+1}(n) \leq 4. \tag{3.4.2}$$

3.4.2 The upper bound for $\lambda_3(n)$

Let U be a $DS(n, 3)$-sequence. A *chain* in U is a contiguous subsequence in which each symbol appears at most once. One can show that any such U can be decomposed into at most $2n - 1$ pairwise disjoint chains, by splitting U just before the leftmost and rightmost appearances of each symbol. Let $\Psi(m, n)$ denote the maximum length of a $DS(n, 3)$-sequence that can be decomposed into at most m chains.

Lemma 3.4.1. *Let $m, n \geq 1$, and let $b > 1$ be a divisor of m. Then there exist integers $n^\star, n_1, n_2, \ldots, n_b \geq 0$ such that*

$$n^\star + \sum_{i=1}^{b} n_i = n,$$

and

$$\Psi(m,n) \leq 4m + 4n^\star + \Psi(b, n^\star) + \sum_{i=1}^{b} \Psi\left(\frac{m}{b}, n_i\right). \tag{3.4.3}$$

Proof: Let U be a $DS(n,3)$-sequence, consisting of at most m chains $c_1, \ldots, c_m$, of length $\Psi(m,n)$, and let $b > 1$ be a divisor of m. Partition the sequence U into b *blocks* (contiguous subsequences) $L_1, \ldots, L_b$, so that the block L_i consists of $p = m/b$ chains $c_{(i-1)p+1}, c_{(i-1)p+2}, \ldots, c_{ip}$. Call a symbol a *internal* to block L_i if all the occurrences of a in U are within L_i. A symbol is called *external* if it is not internal to any block. Suppose that there are n_i internal symbols in block L_i and $n^\star$ external symbols; thus $n^\star + \sum_{i=1}^{b} n_i = n$.

We estimate the total number of occurrences in U of symbols that are internal to L_i, as follows. Erase all external symbols from L_i. Next scan L_i from left to right and erase each element that has become equal to the element immediately preceding it. This leaves us with a sequence $L_i^\star$, which is clearly a $DS(n_i, 3)$-sequence consisting of at most m/b chains, and thus its length is at most $\Psi(m/b, n_i)$. Moreover, if two equal internal elements in L_i have become adjacent after erasing the external symbols, then these two elements must have belonged to two distinct chains, thus the total number of deletions of internal symbols is at most $(m/b) - 1$. Hence, summing over all blocks, we conclude that the total contribution of internal symbols to $|U|$ is at most

$$m - b + \sum_{i=1}^{b} \Psi\left(\frac{m}{b}, n_i\right).$$

Next, to estimate the contribution of external symbols to $|U|$, we argue as follows. For each L_i, call an external symbol a a *middle* symbol if none of its occurrences in L_i is the first or the last occurrence of a in U. Otherwise we call a a *non-middle* symbol. We will consider the contribution of middle and non-middle external symbols separately.

Consider first the middle symbols. To estimate their contribution to the length of L_i, we erase all internal and non-middle symbols from L_i, and also erase a middle symbol if it has become equal to the symbol immediately preceding it. As above, at most $(m/b) - 1$ deletions of external middle symbols will be performed. Let $L_i^\star$ be the resulting subsequence, and suppose that it is composed of p_i distinct symbols. It is easily seen that $L_i^\star$ is a $DS(p_i, 1)$-sequence, so its length is at most p_i. Hence, summing over all blocks, the total contribution of external middle symbols is at most $m - b + \sum_{i=1}^{b} p_i$. But $\sum_{i=1}^{b} p_i$ is the length of the sequence obtained by concatenating all the subsequences $L_i^\star$. This concatenation can contain at most b pairs of adjacent equal elements, and if we erase each element that is equal to its predecessor, we obtain a sequence $U^\star$ which is clearly a $DS(n^\star, 3)$-sequence composed of b chains (namely the subsequences $L_i^\star$). The length of $U^\star$ is thus at most $\Psi(b, n^\star)$. Hence, the contribution of middle external elements to the length of U is at most $m + \Psi(b, n^\star)$.

Consider next the contribution of non-middle symbols. A symbol is called *starting* (resp. ending) in block L_i if does not occur in any block before (resp. after) L_i. To estimate the contribution of starting symbols to the length of L_i we erase from L_i all symbols occurring there except for starting symbols, and, if necessary, also erase each occurrence of a *starting* symbol that has become equal to the element immediately preceding it. As above, at most $(m/b) - 1$ deletions of external starting symbols will be performed. Let $L_i^\#$ be the resulting subsequence, and suppose that it is composed of p_i distinct symbols.

Note first that each external symbol can appear as a starting symbol in exactly one block, thus $\sum_{i=1}^{b} p_i = n^\star$. It is easily seen that $L_i^\#$ is a $DS(p_i, 2)$-sequence, so the length of $L_i^\#$ is at most $2p_i - 1$,

and, summing over all blocks, we conclude that the contribution of all external starting symbols to the length of U is at most

$$m - b + \sum_{i=1}^{b}(2p_i - 1) = m - 2b + 2n^\star.$$

In a completely symmetric manner, the contribution of external *ending* symbols to the length of U is also at most $m - 2b + 2n^\star$. Summing up all these contributions we finally obtain the asserted inequality (3.4.3). □

Next, we solve the recurrence derived in the previous lemma.

Lemma 3.4.2. *For all $m, n \geq 1$, and for $k \geq 2$,*

$$\Psi(m, n) \leq (8k - 8)m\alpha_k(m) + (4k - 2)n. \tag{3.4.4}$$

Proof (Sketch): For the sake of simplicity, we will only show that for $n, s \geq 1$, $k \geq 2$, and m dividing $A_k(s)$,

$$\Psi(m, n) \leq (4k - 4)ms + (4k - 2)n. \tag{3.4.5}$$

If $m = A_k(s)$, then $s = \alpha_k(m)$, and (3.4.5) implies the assertion of the lemma for these values of m also. The case of an arbitrary m is then easy to handle; see [78, 141] for details.

We will use (3.4.3) repeatedly to obtain the series of upper bounds on Ψ, stated in (3.4.5) for $k = 2, 3, \ldots$. At each step we choose b in an appropriate manner, and estimate $\Psi(b, n^\star)$ using the bound obtained in the preceding step. This yields a new recurrence relation on Ψ, which we solve to obtain a better upper bound on Ψ.

Specifically, we proceed by double induction on k and s. For $k = 2$, m divides $A_2(s) = 2^s$, so m is a power of 2. Choose $b = 2$ in (3.4.3); it is easily checked that $\Psi(b, n^\star) = \Psi(2, n^\star) = 2n^\star$ for all $n^\star$, so (3.4.3) becomes

$$\Psi(m, n) \leq 4m + 6n^\star + \Psi\left(\frac{m}{2}, n_1\right) + \Psi\left(\frac{m}{2}, n_2\right).$$

The solution to this recurrence relation, for m a power of 2 and $n = n^\star + n_1 + n_2$ arbitrary, is easily verified to be

$$\Psi(m, n) \;\leq\; 4m \log m + 6n.$$

The case $k > 2$ and $s = 1$ is now a consequence of this bound (because m divides $A_k(1) = 2$ in this case).

Suppose next that $k > 2$ and $s > 1$, and that the induction hypothesis is true for all $k' < k$ and $s' \geq 1$, and for $k' = k$ and all $s' < s$. Let $m = A_k(s)$, and $t = A_k(s - 1)$, and choose $b = m/t$, which is an integer dividing $m = A_k(s) = A_{k-1}(t)$. Hence, by the induction hypothesis for $k - 1$ and t, we have

$$\Psi(b, n^\star) \;\leq\; (4k - 8)bt + (4k - 6)n^\star \;=\; (4k - 8)m + (4k - 6)n^\star.$$

Then (3.4.3) becomes

$$\Psi(m, n) \leq (4k - 8)m + (4k - 6)n^\star + 4m + 4n^\star + \sum_{i=1}^{b}\Psi(t, n_i).$$

Using the induction hypothesis once more (for k and $s-1$), we obtain

$$\begin{aligned}\Psi(m,n) &\leq (4k-4)m + (4k-2)n^\star + \sum_{i=1}^{b}((4k-4)t(s-1) + (4k-2)n_i) \\ &= (4k-4)ms + (4k-2)n,\end{aligned}$$

because $n^\star + \sum_{i=1}^{b} n_i = n$.

The case where m only divides $A_k(s)$ is handled by taking a concatenation of $p = A_k(s)/m$ copies of a sequence whose length is $\Psi(m,n)$, using pairwise-disjoint sets of symbols for the copies. The concatenated sequence is composed of pn symbols and has at most pm chains, so

$$p\Psi(m,n) \leq \Psi(pm,pn) \leq (4k-4)pms + (4k-2)pn\,,$$

from which (3.4.5) follows.

This completes the proof of the asserted bound. □

Theorem 3.4.3 ([78]). $\lambda_3(n) = O(n\alpha(n))$.

Proof: By putting $k = \alpha(m)+1$ in (3.4.4) and using (3.4.2), we obtain

$$\Psi(m,n) \leq 32m\alpha(m) + (4\alpha(m)+2)n.$$

As noted in the beginning of this subsection, $\lambda_3(n) \leq \Psi(2n-1,n)$. Since $\alpha(2n-1) \leq \alpha(n)+1$, the theorem follows. □

Applying a more careful analysis, Klazar [96] has shown that

$$\lambda_3(n) \leq 4n\alpha(n) + O(n\sqrt{\alpha(n)}),$$

provided that n is sufficiently large.

An immediate corollary of Theorem 3.4.3 is that the lower envelope of n segments in the plane has $O(n\alpha(n))$ breakpoints.

3.4.3 Upper bounds on $\lambda_s(n)$

We now briefly mention how the upper bounds on $\lambda_s(n)$, for $s > 3$, are derived in [11]. Let $\Psi_s^t(m,n)$ denote the maximum length of a $DS(n,s)$-sequence composed of at most m contiguous subsequences, each of which is a $DS(n,t)$-sequence. As above, Agarwal et al. [11] obtain a recurrence relation for $\Psi_s(m,n)$, the length of a $DS(n,s)$-sequences composed of at most m chains, but the recurrence is now written in terms of Ψ_s and Ψ_s^{s-2}. Let S be a given $DS(n,s)$-sequence composed of at most m chains. The analysis in [11] divides S into b blocks and counts the contributions of internal, middle, and non-middle symbols separately, in a manner similar to that given above. This leads to the following lemma.

Lemma 3.4.4. *Let m, $n \geq 1$ and $1 < b < m$ be integers. For any partitioning $m = \sum_{i=1}^{b} m_i$, with $m_1, \ldots, m_b \geq 1$, there exist integers $n^\star$, $n_1, n_2, \ldots, n_b \geq 0$ such that*

$$n^\star + \sum_{i=1}^{b} n_i = n$$

and

$$\Psi_s(m,n) \le \Psi_s^{s-2}(b,n^\star) + 2\Psi_{s-1}(m,n^\star) + 4m + \sum_{i=1}^{b} \Psi_s(m_i,n_i). \tag{3.4.6}$$

If we choose $b = 2$, the solution of the recurrence is $O(n \log^{s-2} n)$. However, extending the proof of Lemma 3.4.2, but using a rather involved analysis, one can obtain the following bounds on $\lambda_s(n)$.

Theorem 3.4.5 ([11]). (i) $\lambda_4(n) = O(n \cdot 2^{\alpha(n)})$.

(ii) *For $s > 1$, there exists a polynomial $C_s(q)$ of degree at most $s-1$, such that*

$$\begin{aligned} \lambda_{2s+1}(n) &\le n \cdot 2^{\alpha^{s-1}(n)\log\alpha(n) + C_s(\alpha(n))}, \\ \lambda_{2s+2}(n) &\le n \cdot 2^{\alpha^{s}(n) + C_s(\alpha(n))}. \end{aligned}$$

3.5 Lower Bounds on $\lambda_s(n)$

An even more surprising result in the theory of Davenport–Schinzel sequences is that the bounds stated in Theorems 3.4.3 and 3.4.5 are optimal for $s = 3$ and 4, and are very close to optimal for $s > 4$. The first superlinear bound on $\lambda_s(n)$ was obtained by Hart and Sharir [78], who proved that $\lambda_3(n) = \Omega(n\alpha(n))$. Their original proof transforms $DS(n,3)$-sequences into certain path compression schemes on rooted trees. A more direct proof for the lower bound on $\lambda_3(n)$ was given by Wiernik and Sharir [151] — they describe an explicit recursive scheme for constructing a $DS(n,3)$-sequence of length $\Omega(n\alpha(n))$. See also [98] for another proof of the same lower bound. We sketch Wiernik and Sharir's construction, omitting many details, which can be found in [141, 151].

Let $\{C_k(m)\}_{k\ge1}$ be a sequence of functions from $\mathbb{N}$ to itself, defined by

$$\begin{aligned} C_1(m) &= 1 && m \ge 1, \\ C_k(1) &= 2C_{k-1}(2) && k \ge 2, \\ C_k(m) &= C_k(m-1) \cdot C_{k-1}(C_k(m-1)) && k \ge 2, m \ge 2. \end{aligned}$$

It can be shown that, for all $k \ge 4, m \ge 1$,

$$A_{k-1}(m) \le C_k(m) \le A_k(m+3). \tag{3.5.7}$$

In what follows, let $\mu = C_k(m-1), \nu = C_{k-1}(C_k(m-1))$, and $\gamma = \mu \cdot \nu$.

For each $k, m \ge 1$, we construct a sequence $S_k(m)$ that satisfies the following two properties:

(P1) $S_k(m)$ is composed of $N_k(m) = m \cdot C_k(m)$ distinct symbols. These symbols are named (d,l), for $d = 1,\dots,m$, $l = 1,\dots,\gamma$, and are ordered in lexicographical order, so that $(d,l) < (d',l')$ if $l < l'$ or $l = l'$ and $d < d'$.

(P2) $S_k(m)$ contains γ *fans* of size m, where each fan is a contiguous subsequence of the form $\langle (1,l)\,(2,l) \cdots (m,l)\rangle$, for $l = 1,\dots,\gamma$.

Since fans are pairwise disjoint, by definition, the naming scheme of the symbols of $S_k(m)$ can be interpreted as assigning to each symbol the index l of the fan in which it appears, and its index d within that fan. The construction of $S_k(m)$ proceeds by double induction on k and m, as follows.

1. $k = 1$: The sequence is a single fan of size m: $S_1(m) = \langle (1,1)\,(2,1)\cdots(m,1)\rangle$. Properties (P1) and (P2) clearly hold here ($C_1(m) = 1$).

2. $k = 2$: The sequence contains a pair of disjoint fans of size m, with a block of elements following each of these fans. Specifically,

$$\begin{aligned} S_2(m) \;=\; & \langle (1,1)\,(2,1)\cdots(m-1,1)\,(m,1)\,(m-1,1)\cdots(1,1) \\ & (1,2)\,(2,2)\cdots(m-1,2)\,(m,2)\,(m-1,2)\cdots(1,2)\rangle. \end{aligned}$$

   Indeed, $S_2(m)$ contains $C_2(m) = 2$ fans and is composed of $2m$ dsitinct symbols.

3. $k \geq 3, m = 1$: The sequence is identical to the sequence for $k' = k-1$ and $m' = 2$, except for renaming of its symbols and fans: $S_{k-1}(2)$ contains $C_{k-1}(2) = \frac{1}{2}C_k(1)$ fans, each of which consists of two symbols; the symbol renaming in $S_k(1)$ causes each of these two elements to become a 1-element fan. Properties (P1) and (P2) clearly hold.

4. The general case $k \geq 3, m > 1$:

   (i) Generate inductively the sequence $S' = S_k(m-1)$; by induction, it contains μ fans of size $m-1$ each and is composed of $(m-1)\cdot\mu$ symbols.

   (ii) Create ν copies of S' whose sets of symbols are pairwise disjoint. For each $j \leq \nu$, rename the symbols in the jth copy S'_j of S' as (d,i,j) where $1 \leq d \leq m-1$ is the index of the symbol in the fan of S'_j containing it, and $1 \leq i \leq \mu$ is the index of this fan in S'_j.

   (iii) Generate inductively the sequence $S^\star = S_{k-1}(\mu)$ whose set of symbols is disjoint from that of any S'_j; by induction, it contains ν fans of size μ each. Rename the symbols of $S^\star$ as (m,i,j) (where i is the index of that symbol within its fan, and j is the index of that fan in $S^\star$). Duplicate the last element (m,μ,j) in each of the ν fans of $S^\star$.

   (iv) For each $1 \leq i \leq \mu$, $1 \leq j \leq \nu$, extend the ith fan of S'_j by duplicating its last element $(m-1,i,j)$, and by inserting the corresponding symbol (m,i,j) of $S^\star$ between these duplicated appearances of $(m-1,i,j)$. This process extends the $(m-1)$-fans of S'_j into m-fans and adds a new element after each extended fan.

   (v) Finally construct the desired sequence $S_k(m)$ by merging the ν copies S'_j of S' with the sequence $S^\star$. This is done by replacing, for each $1 \leq j \leq \nu$, the jth fan of $S^\star$ by the corresponding copy S'_j of S', as modified in (iv) above. Note that the duplicated copy of the last element in each fan of $S^\star$ (formed in step (iii) above) appears now after the copy S'_j that replaces this fan; see Figure 3.5 for an illustration of this process.

   It is easily checked that $S_k(m)$ consists of

$$N_k(m) = \nu(m-1)\mu + \mu C_{k-1}(\mu) = mC_k(m)$$

symbols, and it can also be shown that $S_k(m)$ is a $DS(N_k(m),3)$-sequence satisfying properties (P1) and (P2). If we let $\sigma_k(m)$ denote the length of $S_k(m)$, then

$$\begin{aligned} \sigma_1(m) &= m, \\ \sigma_2(m) &= 4m-2, \\ \sigma_k(1) &= \sigma_{k-1}(2), \\ \sigma_k(m) &= \nu\sigma_k(m-1) + \sigma_{k-1}(\mu) + \nu(\mu+1). \end{aligned}$$

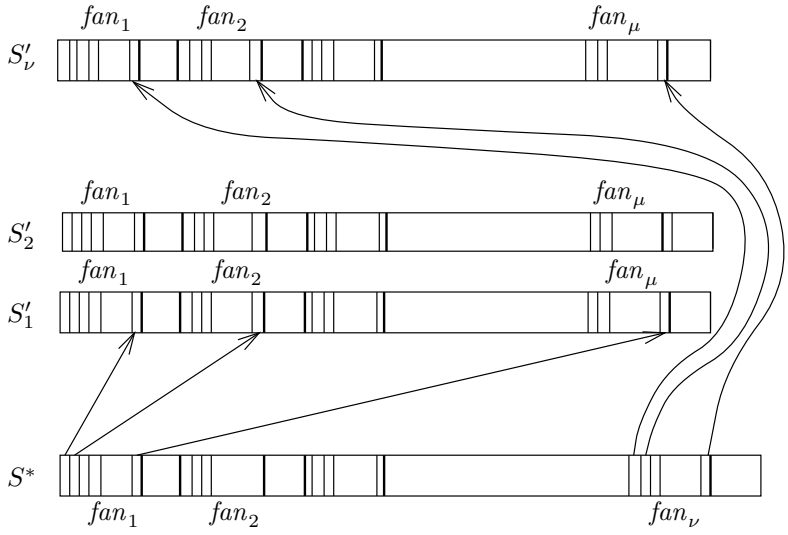


FIGURE 3.5. Lower bound construction: merging the subsequences.

The third term in the last equation is due to the duplication of the rightmost symbol of each fan of S^* and of each S'_j (see Steps 4 (iii)-(iv)). Using a double induction on k and m, one can prove that

$$\sigma_k(m) > (km - 2)C_k(m) + 1.$$

Theorem 3.5.1 ([78, 151]). $\lambda_3(n) = \Omega(n\alpha(n))$.

Proof: Choose $m_k = C_{k+1}(k-3)$. Then

$$n_k = N_k(m_k) = C_{k+1}(k-2) \leq A_{k+1}(k+1)$$

where the last inequality follows from (3.5.7). Therefore $\alpha(n_k) \leq k+1$, and hence

$$\lambda_3(n_k) \geq \sigma_k(m_k) \geq kn_k - 2C_k(m_k) \geq (k-2)n_k \geq n_k(\alpha(n_k) - 3).$$

As shown in [141], this bound can be extended to any integer n, to prove that $\lambda_3(n) = \Omega(n\alpha(n))$. □

Generalizing the above construction and using induction on s — basically replacing each chain of the sequence $S_k(m)$ by a $DS(n, s-2)$-sequence, which, in turn, is constructed recursively — Sharir [137] proved that $\lambda_{2s+1}(n) = \Omega(n\alpha(n)^s)$. Later Agarwal et al. [11] proved that the upper bounds stated in Theorem 3.4.5 are almost optimal. In particular, using a rather involved doubly-inductive scheme, they constructed a $DS(n, 4)$-sequence of length $\Omega(n2^{\alpha(n)})$. Then, by recursing on s, they generalized their construction of $DS(n, 4)$-sequences to higher-order sequences. The following theorem summarizes their result.

Theorem 3.5.2 ([11]). (i) $\lambda_4(n) = \Omega(n \cdot 2^{\alpha(n)})$.

(ii) *For $s > 1$, there exists a polynomial $Q_s(q)$ of degree at most $s-1$, such that*

$$\lambda_{2s+2}(n) \geq n \cdot 2^{\frac{\alpha^s(n)}{s!} + Q_s(\alpha(n))}.$$

Problem 3.5.3. *Obtain tights bounds on $\lambda_s(n)$ for $s > 4$, especially for odd values of s.*

Wiernik and Sharir [151] proved that the $DS(n, 3)$-sequence $S_k(m)$ constructed above can be realized as the lower envelope sequence of a set of n segments, which leads to the following fairly surprising result:

Theorem 3.5.4 ([151]). *The lower envelope of n segments can have $\Omega(n\alpha(n))$ breakpoints in the worst case.*

Shor [144] gave a simpler example of n segments whose lower envelope also has $\Omega(n\alpha(n))$ breakpoints. These results also yield an $\Omega(n\alpha(n))$ lower bound on many other unrelated problems, including searching in totally monotone matrices [92] and counting the number of distint edges in the convex hull of a planar point set as the points are being updated dynamically [150]. Shor has also shown that there exists a set of n degree-4 polynomials whose lower envelope has $\Omega(n\alpha(n))$ breakpoints [145] (which is somewhat weak, because the upper bound for this quantity is $\lambda_4(n) = O(n \cdot 2^{\alpha(n)})$). We conclude this section by mentioning another open problem, which we believe is one of the most challenging and interesting problems related to Davenport–Schinzel sequences.

Problem 3.5.5. *Is there a natural geometric realization of higher order sequences? For example, can the lower envelope of n conic sections have $\Omega(n2^{\alpha(n)})$ breakpoints?*

3.6 Davenport–Schinzel Sequences and Arrangements

In this section we consider certain geometric and topological structrues induced by a family of arcs in the plane, where Davenport–Schinzel sequences play a major role in their analysis.

This section somewhat overlaps the material given in Chapter 2, except that here the discussion is specialized to planar arrangements. Readers unfamiliar with this basic theory might want to read this section first.

Specifically, let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a collection of n Jordan arcs in the plane, each pair of which intersect in at most s points, for some fixed constant s.[1]

Definition 3.6.1 The *arrangement* $\mathcal{A}(\Gamma)$ of Γ is the planar subdivision induced by the arcs of Γ; that is, $\mathcal{A}(\Gamma)$ is a planar map whose *vertices* are the endpoints of the arcs of Γ and their pairwise intersection points, whose *edges* are maximal (relatively open) connected portions of the γ_i's that do not contain a vertex, and whose *faces* are the connected components of $\mathbb{R}^2 - \bigcup \Gamma$. The *combinatorial complexity* of a face is the number of vertices (or edges) on its boundary, and the *combinatorial complexity* of $\mathcal{A}(\Gamma)$ is the total complexity of all of its faces.

The maximum combinatorial complexity of $\mathcal{A}(\Gamma)$ is clearly $\Theta(sn^2) = \Theta(n^2)$, and $\mathcal{A}(\Gamma)$ can be computed in time $O(n^2 \log n)$, under an appropriate model of computation, using the sweep-line algorithm of Bentley and Ottmann [29]. A slightly faster algorithm, with running time $O(n\lambda_{s+2}(n))$, is mentioned in Section 3.6.3. Many applications, however, need to compute only a small portion of the arrangement, such as a single face, a few faces, or some other substructures that we will

[1] A *Jordan arc* is an image of the closed unit interval under a continuous bijective mapping. Similarly, a *closed Jordan curve* is an image of the unit circle under a similar mapping, and an *unbounded Jordan curve* is an image of the open unit interval (or of the entire real line) that separates the plane.

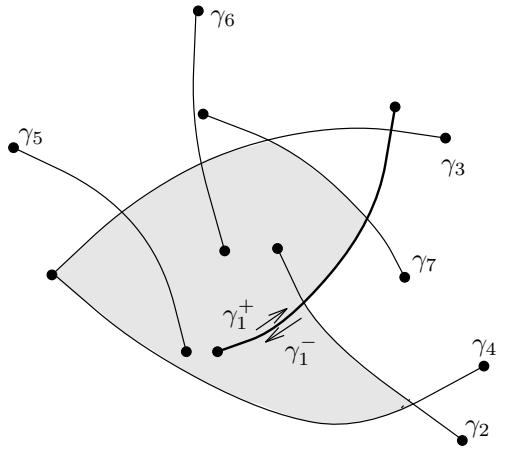
consider shortly. Using DS-sequences, one can show that the combinatorial complexity of these substructures is substantially smaller than that of the entire arrangement. This fact is then exploited in the design of efficient algorithms, whose running time is close to the bound on the complexity of the substructures that these algorithms aim to construct. In this section we review combinatorial and algorithmic results related to these substructures, in which DS-sequences play a crucial role.

3.6.1 Complexity of a single face

It is well known that the complexity of a single face in an arrangement of n lines is at most n [123], and a linear bound on the complexity of a face in an arrangement of rays is also known (see Alevizos et al. [14, 15]). The result of Wiernik and Sharir [151] on the lower envelopes of segments implies that the unbounded face in an arrangement of n line segments has $\Omega(n\alpha(n))$ vertices in the worst case. A matching upper bound was proved by Pollack et al. [122], which was later extended by Guibas et al. [73] to general Jordan arcs. The case of closed or unbounded Jordan curves was treated in [134].

Theorem 3.6.2 ([73, 134]). *Let Γ be a set of n Jordan arcs in the plane, each pair of which intersect in at most s points, for some fixed constant s. Then the combinatorial complexity of any single face in $\mathcal{A}(\Gamma)$ is $O(\lambda_{s+2}(n))$. If each arc in Γ is a Jordan curve (closed or unbounded), then the complexity of a single face is at most $\lambda_s(n)$.*

Proof (Sketch): We only consider the first part of the theorem; the proof of the second part is simpler, and can be found in [134, 141]. Let f be a given face in $\mathcal{A}(\Gamma)$, and let C be a connected component of its boundary. We can assume that C is the only connected component of ∂f. Otherwise, we repeat the following analysis for each connected component and sum their complexities. Since each arc appears in at most one connected component, the bound follows. For each arc γ_i, let u_i and v_i be its endpoints, and let γ_i^+ (respectively, γ_i^-) be the directed arc γ_i oriented from u_i to v_i (respectively, from v_i to u_i).



$$S = \langle \gamma_1^+ \gamma_2^- \gamma_2^+ \gamma_1^+ \gamma_7^- \gamma_3^- \gamma_6^+ \gamma_6^- \gamma_3^- \gamma_5^+ \gamma_5^- \gamma_3^- \gamma_4^+ \gamma_2^- \gamma_1^- \rangle$$

FIGURE 3.6. A single face and its associated boundary sequence; all arcs are positively oriented from left to right.

Without loss of generality, assume that C is the exterior boundary component of f. Traverse C in counterclockwise direction (so that f lies to our left) and let $S = \langle s_1, s_2, \ldots, s_t \rangle$ be the circular sequence of oriented arcs in Γ in the order in which they appear along C (if C is unbounded, S is a linear, rather than circular, sequence). More precisely, if during our traversal of C we encounter an arc γ_i and follow it in the direction from u_i to v_i (respectively, from v_i to u_i) then we add γ_i^+ (respectively, γ_i^-) to S. See Figure 3.6 for an illustration. Note that in this example *both* sides of an arc γ_i belong to the outer connected component.

Let $\xi_1, \ldots, \xi_{2n}$ denote the oriented arcs of Γ. For each ξ_i we denote by $|\xi_i|$ the nonoriented arc γ_j coinciding with ξ_i. For the purpose of the proof, we transform each arc γ_i into a very thin closed Jordan curve $\gamma_i^\star$ by taking two nonintersecting copies of γ_i lying very close to one another, and by joining them at their endpoints. This will perturb the face f slightly but can always be done in such a way that the combinatorial complexity of C does not decrease. Note that this transformation allows a natural identification of one of the two sides of $\gamma_i^\star$ with γ_i^+ and the other side with γ_i^-.

It can be shown (see [73, 141]) that the portions of each arc ξ_i appear in S in a circular order that is consistent with their order along the oriented ξ_i. In particular, there exists a starting point in S (which depends on ξ_i) so that if we read S in circular order starting from that point, we encounter these portions of ξ_i in their order along ξ_i. For each directed arc ξ_i, consider the linear sequence V_i of all appearances of ξ_i in S, arranged in the order they appear along ξ_i. Let μ_i and ν_i denote respectively the index in S of the first and of the last element of V_i. Consider $S = \langle s_1, \ldots, s_t \rangle$ as a linear, rather than a circular, sequence (this change is not needed if C is unbounded). For each arc ξ_i, if $\mu_i > \nu_i$ we split the symbol ξ_i into two distinct symbols ξ_{i1}, ξ_{i2}, and replace all appearances of ξ_i in S between the places μ_i and t (respectively, between 1 and ν_i) by ξ_{i1} (respectively, by ξ_{i2}). Note that the above claim implies that we can actually split the arc ξ_i into two connected subarcs, so that all appearances of ξ_{i1} in the resulting sequence represent portions of the first subarc, whereas all appearances of ξ_{i2} represent portions of the second subarc. This splitting produces a sequence $S^\star$, of the same length as S, composed of at most $4n$ symbols.

With all these modifications, one can then prove that S^* is a $DS(4n, s+2)$-sequence. This is done by showing that each quadruple of the form $\langle a \cdots b \cdots a \cdots b \rangle$ in S^* corresponds, in a unique manner, to an intersection point between the two arcs of Γ that a and b represent. See [73, 141] for more details. This completes the proof of the first part of the theorem. □

Theorem 3.6.2 has the following interesting consequence. Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a set of n closed Jordan curves, each pair of which intersects in at most s points. Let $K = conv(\Gamma)$ be the convex hull of the curves in Γ. Divide the boundary of K into a minimum number of subarcs, $\alpha_1, \alpha_2, \ldots, \alpha_m$, such that the relative interior of each α_i has a nonempty intersection with exactly one of the curves γ_j. Then the number m of such arcs is at most $\lambda_s(n)$; see [134] for a proof.

Arkin et al. [20] showed that the complexity of a single face in an arrangement of line segments with h distinct endpoints is only $O(h \log h)$ (even though the number of segments can be $\Theta(h^2)$). A matching lower bound is proved is by Matoušek and Valtr [105]. The upper bound by Arkin et al. does not extend to general arcs. Har-Peled [76] has also obtained improved bounds on the complexity of a single face in many special cases.

3.6.2 Computing a single face

Let Γ be a collection of n Jordan arcs, as above, and let x be a point that does not lie on any arc of Γ. We wish to compute the face of $\mathcal{A}(\Gamma)$ that contains x. We assume that each arc in Γ has at most a constant number of points of vertical tangency, so that we can break it into $O(1)$ x-monotone Jordan arcs.

We assume a model of computation allowing infinite-precision real arithmetic, in which certain primitive operations involving one or two arcs (e.g., computing the intersection points of a pair of arcs, the points of vertical tangency of an arc, the intersections of an arc with a vertical line, etc.) are assumed to take constant time.

If Γ is a set of n lines, or a set of n rays, then a single face can be computed in time $O(n \log n)$. In the case of lines, this is done by dualizing the lines to points and using any optimal convex hull algorithm [123]; the case of rays is somewhat more involved, and is described in [14, 15]. However, these techniques do not extend to arrangements of more general Jordan arcs. Pollack et al. [122] presented an $O(n\alpha(n)\log^2 n)$-time algorithm for computing the unbounded face in certain arrangements of line segments, but the first algorithm that works for general arcs was given by Guibas et al. [73]. Later, several other efficient algorithms—both randomized and deterministic—have been proposed. We first present randomized (Las Vegas) algorithms[2] for computing a single face, and then review the deterministic solution of [73], and mention some other related results. Randomized algorithms have recently been designed for many geometric problems; see, e.g., [45, 114, 135]. They are often much simpler than their deterministic counterparts, and are sometimes more efficient, as the present case will demonstrate. The efficiency of a Las Vegas randomized algorithm will be measured by its expected running time in the worst case, where the expectation is taken with respect to the internal randomizations performed by the algorithm.

Randomized algorithms. The randomized algorithms that we will describe actually compute the so-called *vertical decomposition* of f. This decomposition, which we denote by $f^{||}$, is obtained by drawing a vertical segment from each vertex and from each point of vertical tangency of the boundary of f in both directions, and extend it until it meets another edge of f, or else all the way to $\pm\infty$. The vertical decomposition partitions f into 'pseudo-trapezoidal' cells, each bounded by at most two arcs of Γ and at most two vertical segments. To simplify the presentation, we will refer to these cells simply as trapezoids; see Figure 3.7 for an illustration.

We first present a rather simple randomized divide-and-conquer algorithm due to Clarkson [42] (see also [141]). The basic idea of the algorithm is as follows: Randomly choose a subset $\Gamma_1 \subseteq \Gamma$ of $\lfloor n/2 \rfloor$ arcs. Recursively compute the vertical decompositions $f_1^{||}, f_2^{||}$ of the faces f_1, f_2 containing x in $\mathcal{A}(\Gamma_1)$ and in $\mathcal{A}(\Gamma \setminus \Gamma_1)$, respectively. Then merge $f_1^{||}$ and $f_2^{||}$ to compute the vertical decomposition of the face f of $\mathcal{A}(\Gamma)$ that contains x. The merge step essentially performs a simultaneous depth-first search over the trapezoids of $f_1^{||}$ and of $f_2^{||}$, in which it computes the *intersection cells* $\triangle_1 \cap \triangle_2$, for $\triangle_1 \in f_1^{||}$, $\triangle_2 \in f_2^{||}$, that lie in $f^{||}$. After having computed all such intersection cells, $f^{||}$ can be computed in additional $O(|f^{||}|)$ time; see [141] for details. Although the merge step is quite naive, and can take quadratic time in the worst case, one can nevertheless show that the randomization makes this step fast—its expected time is only $O(\lambda_{s+2}(n))$. Hence, the expected running time of the algorithm is $O(\lambda_{s+2}(n)\log n)$.

[2] A *Las Vegas* algorithm always terminates with the correct output, but its running time is a random variable (over the internal randomizations it performs).

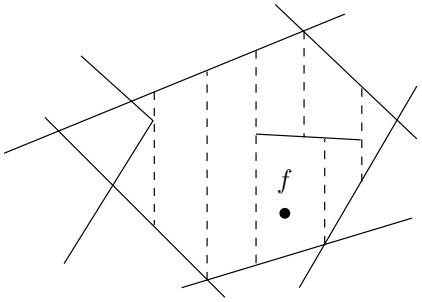


FIGURE 3.7. Vertical decomposition of a face in an arrangement of line segments; here each cell is indeed a trapezoid or a triangle.

The second randomized algorithm, due to Chazelle et al. [35], constructs the vertical decomposition $f^{||}$ of the face containing x incrementally, by adding the arcs of Γ one by one in a random order (the choice of the insertion order is the only randomized step in the algorithm), where each permutation of Γ is chosen with equal probability. While the worst-case running time of this algorithm is also quadratic, the expected running time is only $O(\lambda_{s+2}(n)\log n)$, as for the preceding algorithm.

The basic idea of this algorithm is as follows. Let $\langle \gamma_1, \gamma_2, \ldots, \gamma_n \rangle$ denote the insertion sequence, let $\Gamma_i = \{\gamma_1, \ldots, \gamma_i\}$, and let $f_i^{||}$ be the vertical decomposition of the face containing x in $\mathcal{A}(\Gamma_i)$, for $i = 1, \ldots, n$. When γ_{i+1} is inserted, it may chop off a part of f_i by separating it from the point x, so some of the trapezoids of $f_i^{||}$ may not appear in $f_{i+1}^{||}$, and some of them, which are crossed by γ_{i+1}, will have to be replaced by new trapezoids that have γ_{i+1} on their boundary. Thus, adding γ_{i+1} requires the following steps: Compute the set of trapezoids in $f_i^{||}$ that γ_{i+1} intersects; determine the set of new cells that appear in $f_{i+1}^{||}$, and find the portion of f_i that is chopped off by γ_{i+1}, if any; finally, discard the trapezoids of $f_i^{||}$ that do not appear in $f_{i+1}^{||}$.

To facilitate the execution of these steps, the algorithm stores $f_i^{||}$ as a vertical adjacency graph, whose edges connect pairs of trapezoids sharing a vertical edge (more precisely, having overlapping vertical edges); for each trapezoid in $f_i^{||}$, we store the list of trapezoids that are its neighbors in the vertical adjacency graph. The algorithm also maintains a directed acyclic graph (dag) G, referred to as the *history dag*. The nodes of G, after the ith insertion stage, correspond to the trapezoids that appeared in at least one $f_j^{||}$, for $j \le i$. The root of the dag corresponds to the entire plane. There is a directed edge from a node v to a node w if the corresponding trapezoids τ_v and τ_w intersect and if τ_v (resp. τ_w) appeared in $f_j^{||}$ (resp. $f_k^{||}$) for some $j < k$. If τ_v is a trapezoid of $f_i^{||}$, then v is an *active* leaf (in the version of G after the ith insertion), and if τ_v was a trapezoid of $f_{i+1}^{||}$ but is not in $f_i^{||}$, and γ_i does not cross τ_v, then v is an *inactive* leaf, in the sense that no successor of τ_v will ever be created. All other nodes of G are *inner* nodes, and represent trapezoids that existed in some $f_j^{||}$, but were crossed by some arc γ_k, for $j < k \le i$. The purpose of the dag G is to facilitate, through a top-down traversal of it, a simple and efficient technique for finding all active trapezoids that the newly inserted arc intersects.

How exactly the above steps are executed and how the data structures are updated is somewhat involved, and is described in detail in [35, 141]. As mentioned above, the expected running time

of the algorithm is $O(\lambda_{s+2}(n)\log n)$. Moreover, the expected size and depth of G are $O(\lambda_{s+2}(n))$ and $O(\log n)$, respectively, so we also obtain a point-location data structure that can determine, in $O(\log n)$ expected time, whether a query point lies in f. A somewhat simpler variant of the randomized incremental algorithm is given in [50].

Theorem 3.6.3 ([35, 42, 50]). *Given a collection Γ of n Jordan arcs, each pair of which intersect in at most s points, and a point x not lying on any arc, the face of $\mathcal{A}(\Gamma)$ containing x can be computed by a randomized algorithm in $O(\lambda_{s+2}(n)\log n)$ expected running time, in an appropriate model of computation.*

Deterministic algorithms. We now sketch a deterministic, divide-and-conquer algorithm, due to Guibas et al. [73], for computing f. The high-level description of the algorithm is quite simple, and is similar to the first randomized algorithm described above. We partition Γ into two subsets Γ_1, Γ_2, of roughly $n/2$ arcs each, recursively compute the faces, f_1, f_2, of $\mathcal{A}(\Gamma_1)$, $\mathcal{A}(\Gamma_2)$, respectively, that contain x, and then 'merge' these two faces to obtain the desired face f. Note that f is the connected component of $f_1 \cap f_2$ containing x. However, as already noted, it is generally too expensive to compute this intersection in its entirety, and then select the component containing x, because the boundaries of f_1 and f_2 might have $\Omega(n^2)$ points of intersection. We therefore need a more careful way of performing the merge.

The setup for the merge step is as follows. We are given two connected (but not necessarily simply connected) regions in the plane, which we denote, respectively, as the red region R and the blue region B. Both regions contain the point x in their interior, and our task is to calculate the connected component f of $R \cap B$ that contains x. The boundaries of R and B are composed of (maximal connected) portions of the given curves in Γ, each of which will be denoted in what follows as an *arc segment* (or 'subarc').

For technical reasons, we extend this task as follows. Let P be the set of points containing x and all endpoints of the arcs of Γ that lie on the boundary of either R or B. Clearly, $|P| \leq 2n+1$. For each $w \in P$, let f_w denote the connected component of $R \cap B$ that contains w (these components are not necessarily distinct, and some may be empty). Our task is now to calculate all these components (but produce each distinct component just once, even if it contains several points of P). We refer to this task as the *red–blue merge*. We call the resulting components f_w *purple regions*, as each of them is covered by both the red and the blue regions. An illustration of this merge is shown below in Figure 3.8.

The algorithm relies heavily on the following technical result, called the *combination lemma*, which is interesting in its own right. We first introduce a few notation. Let $R_1, \ldots, R_m$ be a collection of distinct faces in an arrangement of a set Γ_r of 'red' Jordan arcs, and let $B_1, \ldots, B_n$ be a similar collection of faces in an arrangement of a set Γ_b of 'blue' Jordan arcs (where, again, each pair of arcs from $\Gamma_r \cup \Gamma_b$ are assumed to intersect in at most s points). Let $P = \{p_1, \ldots, p_k\}$ be a collection of points, so that each $p_i \in P$ belongs to one red face R_{m_i} and to one blue face B_{n_i}. Let E_i be the connected component of $R_{m_i} \cap B_{n_i}$ containing p_i (i.e., E_i is the 'purple' face of the combined arrangement $\mathcal{A}(\Gamma_r \cup \Gamma_b)$ containing p_i). Then we have the following result.

Lemma 3.6.4 (Combination Lemma, [73]). *The total combinatorial complexity of all the regions E_i is at most $O(r+b+k)$, where r and b are the total number of arc segments composing the boundaries of the red faces and of the blue faces, respectively.*

Remark 3.6.5 A stronger combination lemma was obtained by Edelsbrunner et al. [57] for the

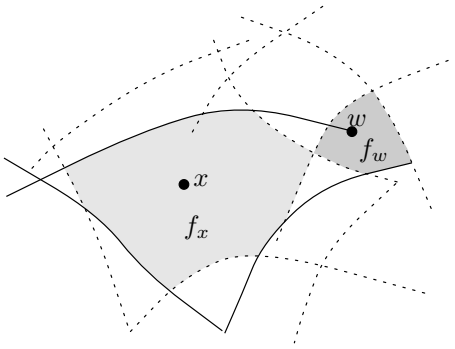


FIGURE 3.8. The red–blue merge; the solid arcs are the blue arcs, and the dashed arcs are red.

case of line segments. They proved that the total complexity of the purple regions E_i's is bounded by $r + b + O(u + v + k)$, where u (resp. v) is the total number of connected components of the boundaries of the red (resp. blue) faces. Recently, Har-Peled [76] generalized the combination lemma to the overlay of more than two arrangements.

The combination lemma implies that the complexity of all the 'purple' regions in the overlay of the faces f_1 and f_2 is $O(r + b) = O(\lambda_{s+2}(n))$. Exploiting this bound, Guibas et al. [73] describe a somewhat involved sweep-line algorithm that sweeps over f_1 and f_2, and computes the purple regions in time $O(\lambda_{s+2}(n) \log n)$. The main idea behind this sweep is that it is performed separately, but simultaneously, over the red, blue, and purple arrangements, in a manner that processes only a small number of red-blue intersections. See [73, 141] for more details. Hence, the overall running time of the algorithm is $O(\lambda_{s+2}(n) \log^2 n)$.

Recently, Amato et al. [18] have succeeded in derandomizing the algorithm by Chazelle et al. [35], described above, for a set of segments. The worst-case running time of their algorithm is $O(n\alpha^2(n) \log n)$. Hence, we can conclude the following.

Theorem 3.6.6 ([18, 73]). *Given a collection Γ of n Jordan arcs, each pair of which intersect in at most s points, and a point x not lying on any arc, the face of $\mathcal{A}(\Gamma)$ containing x can be computed by a deterministic algorithm in time $O(\lambda_{s+2}(n) \log^2 n)$, in an appropriate model of computation. The running time improves to $O(\lambda_s(n) \log^2 n)$ for collections of Jordan curves (closed or unbounded), and to $O(n\alpha^2(n) \log n)$ for collections of line segments.*

We conclude this subsection by mentioning two open problems.

Problem 3.6.7. *(i) Given a set Γ of n segments and a point p, can the face in $\mathcal{A}(\Gamma)$ containing p be computed in time $O(n \log h)$, where h is the number of edges in the face?*

(ii) Given a set Γ of n Jordan arcs, each pair of which intersects in at most s points, and a point p, can the face in $\mathcal{A}(\Gamma)$ containing p be computed in time $O(\lambda_{s+2} \log n)$?

3.6.3 Zones

The *zone* of a curve γ_0 in the arrangement $\mathcal{A}(\Gamma)$ of a collection Γ of n Jordan arcs is the set of all faces of $\mathcal{A}(\Gamma)$ that γ_0 intersects. The complexity of the zone is the sum of the complexities of all the faces in the zone.

Zones were initially studied for arrangements of lines and hyperplanes [53, 59, 60], but they are also easy to analyze in the context of general arcs. The following theorem demonstrates a close relationship between zones and faces in an arrangement.

Theorem 3.6.8 ([55]). *The complexity of the zone of a curve γ_0 in an arrangement $\mathcal{A}(\Gamma)$ of n Jordan arcs, each pair of which intersect in at most s points, is $O(\lambda_{s+2}(n))$, assuming that γ_0 intersects every arc of Γ in at most some constant number of points.*

Proof: Split every arc $\gamma \in \Gamma$ into two subarcs at each intersection point of γ and γ_0, and leave sufficiently small gaps between these pieces. In this manner all faces in the zone of γ_0 are merged into one face, at the cost of increasing the number of arcs from n to $O(n)$. Now we can apply Theorem 3.6.2 to conclude the proof. □

If Γ is a set of n lines and γ_0 is also a line, then after splitting each line of Γ at their intersection points with γ_0 we obtain a collection of $2n$ rays, and therefore the complexity of the unbounded face is $O(n)$. In fact, in this case one can show that the edges of the zone form a $DS(4n, 2)$ sequence, thereby obtaining an upper bound of $8n - 1$ on the complexity of the zone. Applying a more careful analysis, Bern et al. [31] proved the following theorem.

Theorem 3.6.9 ([31]). *The complexity of the zone of a line in an arrangement of n lines is at most $5.5n$, and this bound is tight within an additive constant term, in the worst case.*

See [14, 31, 37, 55] for other results and applications of zones of arcs.

An immediate consequence of Theorem 3.6.8 is an efficient algorithm for computing the arrangement $\mathcal{A}(\Gamma)$. Suppose we add the arcs of Γ one by one and maintain the arrangement of the arcs added so far. Let Γ_i be the set of arcs added in the first i stages, and let γ_{i+1} be the next arc to be added. Then in the $(i + 1)$st stage one has to update only those faces of $\mathcal{A}(\Gamma_i)$ which lie in the zone of γ_{i+1}, and this can easily be done in time proportional to the complexity of the zone; see Edelsbrunner et al. [55] for details. By Theorem 3.6.8, the total running time of the algorithm is $O(n\lambda_{s+2}(n))$, and, by Theorem 3.6.9, the arrangement of a set of n lines can be computed in $O(n^2)$ time. If the arcs of Γ are added in a *random* order, then the expected running time of the above algorithm is $O(n \log n + k)$, where k is the number of vertices in $\mathcal{A}(\Gamma)$ [35, 46, 114], which is at most quadratic in n. The latter time bound is worst-case optimal.

Theorem 3.6.8 can also be used to obtain an upper bound on the complexity of any m faces of $\mathcal{A}(\Gamma)$. Specifically, let $\{f_1, \ldots, f_m\}$ be a subset of m distinct faces in $\mathcal{A}(\Gamma)$, and let n_f denote the

number of vertices in a face f of $\mathcal{A}(\Gamma)$. Then, using the Cauchy-Schwarz inequality,

$$\begin{aligned}\sum_{i=1}^{m} n_{f_i} &\leq m^{1/2}\Big(\sum_i n_{f_i}^2\Big)^{1/2} \leq m^{1/2}\Big(\sum_{f\in\mathcal{A}(\Gamma)} n_f^2\Big)^{1/2} \\ &= O\left[m^{1/2}\Big(\sum_{f\in\mathcal{A}(\Gamma)} n_f\lambda_{s+2}(k_f)\Big)^{1/2}\right] \\ &= O\left[m^{1/2}\left(\frac{\lambda_{s+2}(n)}{n}\right)^{1/2}\Big(\sum_{f\in\mathcal{A}(\Gamma)} n_f k_f\Big)^{1/2}\right],\end{aligned}$$

where k_f is the number of arcs in Γ that appear along the boundary of f. It is easily verified that

$$\sum_{f\in\mathcal{A}(\Gamma)} n_f k_f \leq \sum_{\gamma\in\Gamma}\sum_{f\in zone(\gamma,\Gamma\setminus\{\gamma\})} n_f = O(n\lambda_{s+2}(n))\,.$$

Hence, we obtain the following result.

Theorem 3.6.10 ([55, 76]). *Let Γ be a set of n arcs satisfying the conditions stated earlier. The maximum number of edges bounding any m distinct faces of $\mathcal{A}(\Gamma)$ is $O(m^{1/2}\lambda_{s+2}(n))$.*

It should be noted that Theorem 3.6.10 is weaker than the best bounds known for the complexity of m distinct faces in arrangements of several special types of arcs, such as lines, segments, and circles (see [21, 33, 43]), but it applies to arrangements of more general arcs.

3.6.4 Levels in arrangements

For the convenience of the reader, we repeat here some definitions pertaining to levels in (planar) arrangements. Let Γ be a set of n x-monotone, unbounded Jordan curves, each pair of which intersects in at most s points. The *level* of a point $p \in \mathbb{R}^2$ in $\mathcal{A}(\Gamma)$ is the number of curves of Γ lying strictly below p, and the level of an edge $e \in \mathcal{A}(\Gamma)$ is the common level of all the points lying in the relative interior of e. For a nonnegative integer $k < n$, the *k-level* (respectively, *($\leq k$)-level*) of $\mathcal{A}(\Gamma)$ is (the closure of) the union of all edges in $\mathcal{A}(\Gamma)$ whose level is k (respectively, at most k). Note that the graph of the lower envelope E_Γ is the 0-level, so the complexity of the 0-level is at most $\lambda_s(n)$. As reviewed in Chapter 2, there are still big gaps between the known upper and lower bounds on the complexity of an arbitrary single level, even for arrangements of lines (which, as reviewed earlier, is the dual version of the *k-set* problem for point sets); see [51, 65, 102, 117, 149]. However, tight bounds are known for the complexity of ($\leq k$)-levels in arrangements of curves:

Theorem 3.6.11 ([46, 139]). *Let Γ be a set of n x-monotone curves, each pair intersecting in at most s points, and let $0 < k < n$ be an integer. The number of edges in $\mathcal{A}(\Gamma)$ of level at most k is $O(k^2\lambda_s(\lfloor n/k\rfloor))$, and this bound is tight in the worst case.*

The proof of the theorem is based on an elegant probabilistic analysis technique, due to Clarkson and Shor [46], which has been applied to a variety of other problems as well. An immediate corollary of the above theorem is the following claim.

Corollary 3.6.12 ([16]). *The number of edges in the ($\leq k$)-level of an arrangement of n lines in the plane is $\Theta(nk)$.*

Corollary 3.6.12 can be extended to arrangements of hyperplanes in higher dimensions as well, where the number of vertices in $(\leq k)$-level is $\Theta(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil})$ [46]. Efficient algorithms for computing $(\leq k)$-levels in arrangements are given in [5, 189, 313].

Theorem 3.6.11 can be extended to a more general setting. Let $\mathcal{K} = \{K_1, \ldots, K_n\}$ be a collection of n regions in $\mathbb{R}^2$ so that the boundaries of any two them intersect in at most s points. Let $f(r)$ denote the expected number of vertices on the boundary of the union of a random subset of r regions of $\mathcal{K}$. For example, if the boundary of each K_i is an x-monotone curve, then $f(r) = O(\lambda_s(r))$ (see Corollary 3.2.3). Using the same probabilistic technique of [46], the following theorem can be proved.

Theorem 3.6.13 ([139]). *Let $\mathcal{K} = \{K_1, \ldots, K_n\}$ be a collection of n regions in $\mathbb{R}^2$ so that the boundaries of any two them intersect in at most s points. For any integer $1 \leq k \leq n-2$, the number of intersection points of the boundaries of regions in $\mathcal{K}$ that lie in the interior of at most k regions of $\mathcal{K}$ is $O(k^2 f(\lfloor n/k \rfloor))$. If each ∂K_i is an x-monotone curve, then the number of such vertices is $O(k^2 \lambda_s(\lfloor n/k \rfloor))$.*

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Chapter 4

Incidences and Their Relatives

4.1 Introduction

The problem and its relatives. Let P be a set of m distinct points, and let L be a set of n distinct lines in the plane. Let $I(P, L)$ denote the number of *incidences* between the points of P and the lines of L, i.e.,

$$I(P, L) = |\{(p, \ell) \mid p \in P,\ \ell \in L,\ p \in \ell\}|.$$

How large can $I(P, L)$ be? More precisely, determine or estimate $\max_{|P|=m,|L|=n} I(P, L)$.

This simplest formulation of the incidence problem, due to Erdős and first settled by Szemerédi and Trotter, has been the starting point of extensive research that has picked up considerable momentum during the past two decades. It is the purpose of this survey to review the results obtained so far, describe the main techniques used in the analysis of this problem, and discuss many variations and extensions.

The problem can be generalized in many natural directions. One can ask the same question when the set L of lines is replaced by a set C of n curves of some other simple shape; the two cases involving respectively unit circles and arbitrary circles are of particular interest—see below.

A related problem involves the same kind of input—a set P of m points and a set C of n curves, but now we assume that no point of P lies on any curve of C. Let $\mathcal{A}(C)$ denote the *arrangement* of the curves of C (see Chapters 2 and 3). The combinatorial complexity of a single *face* is defined as the number of lower dimensional cells (i.e., vertices and edges) belonging to its boundary. The points of P then mark certain faces in the arrangement $\mathcal{A}(C)$ of the curves (assume for simplicity that there is at most one point of P in each face), and the goal is to establish an upper bound on $K(P, C)$, the combined combinatorial complexity of the marked faces. This problem is often referred to in the literature as the *Many-Faces Problem*; see Section 2.7.

One can extend the above questions to d-dimensional spaces, for $d > 2$. Here we can either continue to consider incidences between points and *curves*, or incidences between points and *surfaces* of any larger dimension $k, 1 < k < d$. In the special case when $k = d-1$, we may also wish to study the natural generalization of the 'many-faces problem' described in the previous paragraph: to estimate the total combinatorial complexity of m marked (d-dimensional) cells in the arrangement of n given surfaces.

All of the above problems have many algorithmic variants. Perhaps the simplest question of this type is *Hopcroft's problem*: Given m points and n lines in the plane, how fast can one determine whether there exists any point that lies on any line? One can consider more general problems, like counting the number of incidences or reporting all of them, doing the same for a collection of curves rather than lines, computing m marked faces in an arrangement of n curves, and so on. See also Section 2.13.4.

It turned out that two exciting *metric* problems (involving interpoint distances) proposed by Erdős in 1946 are strongly related to problems involving incidences.

1. *Repeated Distances Problem*: Given a set P of n points in the plane, what is the maximum number of pairs that are at distance exactly 1 from each other? To see the connection, let C be the set of unit circles centered at the points of P. Then two points $p, q \in P$ are at distance 1 apart if and only if the circle centered at p passes through q and vice versa. Hence, $I(P, C)$ is twice the number of unit distances determined by P.

2. *Distinct Distances Problem*: Given a set P of n points in the plane, at least how many distinct distances must there always exist between its point pairs? Later we will show the connection between this problem and the problem of incidences between P and an appropriate set of circles of different radii.

Some other applications of the incidence problem will be reviewed at the end of this chapter. They include the analysis of the maximum number of isosceles triangles, or triangles with a fixed area or perimeter, whose vertices belong to a planar point set; estimating the maximum number of mutually congruent simplices determined by a point set in higher dimensions; etc.

Historical perspective and overview. The first derivation of the tight upper bound

$$I(P, L) = \Theta(m^{2/3}n^{2/3} + m + n)$$

was given by Szemerédi and Trotter in their 1983 seminal paper [55]. They proved Erdős' conjecture, who found the matching lower bound (which was rediscovered many years later by Edelsbrunner and Welzl [26]). A slightly different lower bound construction was exhibited by Elekes [27] (see Section 4.2).

The original proof of Szemerédi and Trotter is rather involved, and yields a rather astronomical constant of proportionality hidden in the O-notation. A considerably simpler proof was found by Clarkson *et al.* [20] in 1990, using extremal graph theory combined with a geometric partitioning technique based on random sampling (see Section 4.3). Their paper contains many extensions and generalizations of the Szemerédi-Trotter theorem. Many further extensions can be found in subsequent papers by Edelsbrunner *et al.* [23, 24], by Agarwal and Aronov [1], by Aronov *et al.* [11], and by Pach and Sharir [44].

The next breakthrough occurred in 1997. In a surprising paper, Székely [54] gave an embarrassingly short proof (which we will review in Section 4.4) of the upper bound on $I(P, L)$ using a simple lower bound of Ajtai *et al.* [8] and of Leighton [36] on the *crossing number* of a graph G, i.e., the minimum number of edge crossings in the best drawing of G in the plane, where the edges are represented by Jordan arcs. In the literature this result is often referred to as the 'Crossing Lemma.' Székely's method can easily be extended to several other variants of the problem, but appears to be less general than the previous technique of Clarkson *et al.* [20].

Székely's paper has triggered an intensive re-examination of the problem. In particular, several attempts were made to improve the existing upper bound on the number of incidences between m points and n circles of arbitrary radii in the plane [45]. This was the simplest instance where Székely's proof technique failed. By combining Székely's method with a seemingly unrelated technique of Tamaki and Tokuyama [56] for cutting circles into 'pseudo-segments', Aronov and Sharir [14] managed to obtain an improved bound for this variant of the problem. Their work has then been followed by Agarwal *et al.* [2], who studied the complexity of many faces in arrangements of circles and pseudo-segments, and by Agarwal *et al.* [5], who extended this result to arrangements of pseudo-circles (see Section 4.5). A combinatorial result of Marcus and Tardos [38] (see below) has led to slight improvements and generalizations of these results. Aronov *et al.* [12] generalized the problem to higher dimensions, while Sharir and Welzl [50] studied incidences between points and *lines* in three dimensions (see Section 4.6).

The related problems involving distances in a point set have had mixed success in recent studies. As for the Repeated Distances Problem in the plane, the best known upper bound on the number of times the same distance can occur among n points is $O(n^{4/3})$, which was obtained more than 20 years ago by Spencer *et al.* [53] (with progressively simpler proofs in [20, 54]). This is far from the best known lower bound of Erdős, which is slightly super-linear (see [43]). The best known upper bound for the 3-dimensional case, due to Clarkson *et al.* [20], is roughly $O(n^{3/2})$, while the corresponding lower bound of Erdős is $\Omega(n^{4/3}\log\log n)$ (see [42]). Many variants of the problem have been studied; see, e.g., [29].

While the Repeated Distances problem has been "stuck" for quite some time, more progress has been made on the companion problem of Distinct Distances. In the planar case, L. Moser [41], Chung [18], and Chung *et al.* [19] proved that the number of distinct distances determined by n points in the plane is $\Omega(n^{2/3})$, $\Omega(n^{5/7})$, and $\Omega(n^{4/5}/\text{polylog}(n))$, respectively. Székely [54] managed to get rid of the polylogarithmic factor, while Solymosi and Tóth [51] improved this bound to $\Omega(n^{6/7})$. This was a real breakthrough. Their analysis was subsequently refined by Tardos [57] and then by Katz and Tardos [35], who obtained the current record of $\Omega(n^{(48-14e)/(55-16e)-\varepsilon})$, for any $\varepsilon > 0$, which is $\Omega(n^{0.8641})$. In spite of this steady improvement, there is still a considerable gap to the best known upper bound of $O(n/\sqrt{\log n})$, due to Erdős [28] (see Section 4.7). In three dimensions, a recent result of Aronov *et al.* [13] yields a lower bound of $\Omega(n^{77/141-\varepsilon})$, for any $\varepsilon > 0$, which is $\Omega(n^{0.546})$. This is still far from the best known upper bound of $O(n^{2/3})$. A better lower bound of $\Omega(n^{0.5794})$ in a special case (involving "homogeneous" point sets) has recently been given by Solymosi and Vu [52]. Their analysis also applies to higher-dimensional homogeneous point sets, and yields the bound $\Omega(n^{2/d-1/d^2})$. In a sill unpublished manuscript, the same authors have tackled the general case, and obtained a lower bound of $\Omega(n^{2/d-1/d(d+2)})$.

For other surveys on related subjects, consult [16], Chapter 4 of [39], [42], and [43].

4.2 Lower Bounds

We describe a simple construction due to Elekes [27] of a set P of m points and a set L of n lines, so that $I(P,L) = \Omega(m^{2/3}n^{2/3} + m + n)$. We fix two integer parameters ξ, η. We take P to be the set of all lattice points in $\{1, 2, \ldots, \xi\} \times \{1, 2, \ldots, 2\xi\eta\}$. The set L consists of all lines of the form $y = ax + b$, where a is an integer in the range $1, \ldots, \eta$, and b is an integer in the range $1, \ldots, \xi\eta$. Clearly, each line in L passes through exactly ξ points of P. See Figure 4.1.

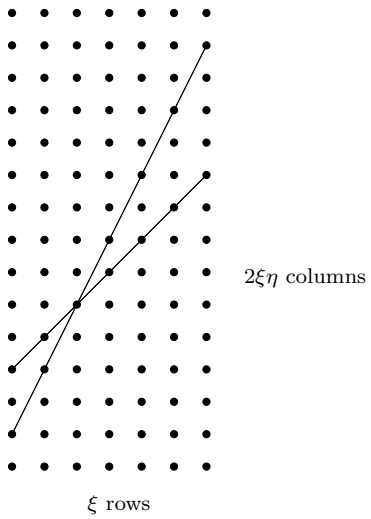

FIGURE 4.1. Elekes' construction.

We have $m = |P| = 2\xi^2\eta$, $n = |L| = \xi\eta^2$, and

$$I(P, L) = \xi|L| = \xi^2\eta^2 = \Omega(m^{2/3}n^{2/3}).$$

Given any sizes m, n so that $n^{1/2} \leq m \leq n^2$, we can find ξ, η that give rise to sets P, L whose sizes are within a constant factor of m and n, respectively. If m lies outside this range then $m^{2/3}n^{2/3}$ is dominated by $m + n$, and then it is trivial to construct sets P, L of respective sizes m, n so that $I(P, L) = \Omega(m + n)$. We have thus shown that

$$I(P, L) = \Omega(m^{2/3}n^{2/3} + m + n).$$

We note that this construction is easy to generalize to incidences involving other curves. For example, we can take P to be the grid $\{1, 2, \ldots, \xi\} \times \{1, 2, \ldots, 3\xi^2\eta\}$, and define C to be the set of all parabolas of the form $y = ax^2 + bx + c$, where $a \in \{1, \ldots, \eta\}$, $b \in \{1, \ldots, \xi\eta\}$, $c \in \{1, \ldots, \xi^2\eta\}$. Now we have $m = |P| = 3\xi^3\eta$, $n = |C| = \xi^3\eta^3$, and

$$I(P, C) = \xi|C| = \xi^4\eta^3 = \Omega(m^{1/2}n^{5/6}).$$

Note that in the construction we have $m = O(n)$. When m is larger, we use the preceding construction for points and lines, which can be easily transformed into a construction for points and parabolas, to obtain the overall lower bound for points and parabolas:

$$I(P, C) = \begin{cases} \Omega(m^{2/3}n^{2/3} + m), & \text{if } m \geq n \\ \Omega(m^{1/2}n^{5/6} + n), & \text{if } m \leq n. \end{cases}$$

These constructions can be generalized to incidences involving graphs of polynomials of higher degrees.

From incidences to many faces. Let P be a set of m points and L a set of n lines in the plane, and put $I = I(P, L)$. Fix a sufficiently small parameter $\varepsilon > 0$, and replace each line $\ell \in L$ by two

lines ℓ^+, ℓ^-, obtained by translating ℓ parallel to itself by distance ε in the two possible directions. We obtain a new collection L' of $2n$ lines. If ε is sufficiently small then each point $p \in P$ that is incident to $k \geq 2$ lines of L becomes a point that lies in a small face of $\mathcal{A}(L')$ that has $2k$ edges; note also that the circle of radius ε centered at p is tangent to all these edges. Moreover, these faces are distinct for different points p, when ε is sufficiently small.

We have thus shown that $K(P, L') \geq 2I(P, L) - 2m$ (where the last term accounts for points that lie on just one line of L). In particular, in view of the preceding construction, we have, for $|P| = m$, $|L| = n$,

$$K(P, L) = \Omega(m^{2/3}n^{2/3} + m + n).$$

An interesting consequence of this construction is as follows. Take $m = n$ and sets P, L that satisfy $I(P, L) = \Theta(n^{4/3})$. Let C be the collection of the $2n$ lines of L' and of the n circles of radius ε centered at the points of P. By applying an inversion,[1] we can turn all the curves in C into circles. We thus obtain a set C' of $3n$ circles with $\Theta(n^{4/3})$ tangent pairs. If we replace each of the circles centered at the points of P by circles with a slightly larger radius, we obtain a collection of $3n$ circles with $\Theta(n^{4/3})$ *empty lenses*, namely faces of degree 2 in their arrangement. Empty lenses play an important role in the analysis of incidences between points and circles; see Section 4.5.

Lower bounds for incidences with unit circles. As noted, this problem is equivalent to the problem of Repeated Distances. Erdős [28] has shown that, for the vertices of an $n^{1/2} \times n^{1/2}$ grid, there exists a distance that occurs $\Omega(n^{1+c/\log\log n})$ times, for an appropriate absolute constant $c > 0$. The details of this analysis, based on number-theoretic considerations, can be found in the monographs [39] and [43].

Lower bounds for incidences with arbitrary circles. As we will see later, we are still far from a sharp bound on the number of incidences between points and circles, especially when the number of points is small relative to the number of circles.

By taking sets P of m points and L of n lines with $I(P, L) = \Theta(m^{2/3}n^{2/3} + m + n)$, and by applying inversion to the plane, we obtain a set C of n circles and a set P' of m points with $I(P', C) = \Theta(m^{2/3}n^{2/3} + m + n)$. Hence the maximum number of incidences between m points and n circles is $\Omega(m^{2/3}n^{2/3} + m + n)$. However, we can slightly increase this lower bound, as follows.

Let P be the set of vertices of the $m^{1/2} \times m^{1/2}$ integer lattice. As shown by Erdős [28], there are $t = \Theta(m/\sqrt{\log m})$ distinct distances between pairs of points of P. Draw a set C of mt circles, centered at the points of P and having as radii the t possible inter-point distances. Clearly, the number of incidences $I(P, C)$ is exactly $m(m-1)$. If the bound on $I(P, C)$ were $O(m^{2/3}n^{2/3} + m + n)$, then we would have

$$m(m-1) = I(P, C) = O(m^{2/3}(mt)^{2/3} + mt) = O(m^2/((\log m)^{1/3}),$$

a contradiction. This shows that, under the most optimistic conjecture, the maximum value of $I(P, C)$ should be larger than the corresponding bound for lines by at least some polylogarithmic factor.

[1] An inversion about, say, the unit circle centered at the origin, maps each point (x, y) to the point $\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$. It maps lines to circles passing through the origin.

4.3 Upper Bounds for Incidences via the Partition Technique

The approach presented in this section is due to Clarkson *et al.* [20]. It predated Székely's method, but it seems to be more flexible, and suitable for generalizations. It can also be used for the refinement of some proofs based on Székely's method.

We exemplify this technique by establishing an upper bound for the number of point-line incidences. Let P be a set of m points and L a set of n lines in the plane. First, we give a weaker bound on $I(P, L)$, as follows. Consider the bipartite graph $H \subseteq P \times L$ whose edges represent all incident pairs (p, ℓ), for $p \in P$, $\ell \in L$. Clearly, H does not contain $K_{2,2}$ as a subgraph. By the Kővari-Sós-Turán Theorem in extremal graph theory (see [43]), we have

$$I(P, L) = O(mn^{1/2} + n). \tag{4.3.1}$$

To improve this bound, we partition the plane into subregions, apply this bound within each subregion separately, and sum up the bounds. We fix a parameter $r, 1 \leq r \leq n$, whose value will be determined shortly, and construct a so-called $(1/r)$-*cutting* of the arrangement $\mathcal{A}(L)$ of the lines of L. This is a decomposition of the plane into $O(r^2)$ vertical trapezoids with pairwise disjoint interiors, such that each trapezoid is crossed by at most n/r lines of L. The existence of such a cutting has been established by Chazelle and Friedman [17], following earlier and somewhat weaker results of Clarkson and Shor [21]. See Chapter 2, [39] and [49] for more details.

For each cell τ of the cutting, let P_τ denote the set of points of P that lie in the interior of τ, and let L_τ denote the set of lines that cross τ. Put $m_\tau = |P_\tau|$ and $n_\tau = |L_\tau| \leq n/r$. Using (4.3.1), we have

$$I(P_\tau, L_\tau) = O(m_\tau n_\tau^{1/2} + n_\tau) = O\left(m_\tau \left(\frac{n}{r}\right)^{1/2} + \frac{n}{r}\right).$$

Summing this over all $O(r^2)$ cells τ, we obtain a total of

$$\sum_\tau I(P_\tau, L_\tau) = O\left(m \left(\frac{n}{r}\right)^{1/2} + nr\right)$$

incidences. This does not quite complete the count, because we also need to consider points that lie on the boundary of the cells of the cutting. A point p that lies in the relative interior of an edge e of the cutting lies on the boundary of at most two cells, and any line that passes through p, with the possible exception of the single line that contains e, crosses both cells. Hence, we may simply assign p to one of these cells, and its incidences (except for at most one) will be counted within the subproblem associated with that cell. Consider then a point p which is a vertex of the cutting, and let ℓ be a line incident to p. Then ℓ either crosses or bounds some adjacent cell τ. Since a line can cross the boundary of a cell in at most two points, we can charge the incidence (p, ℓ) to the pair (ℓ, τ), use the fact that no cell is crossed by more than n/r lines, and conclude that the number of incidences involving vertices of the cutting is at most $O(nr)$.

We have thus shown that

$$I(P, L) = O\left(m \left(\frac{n}{r}\right)^{1/2} + nr\right).$$

Choose $r = m^{2/3}/n^{1/3}$. This choice makes sense provided that $1 \leq r \leq n$. If $r < 1$, then $m < n^{1/2}$ and (4.3.1) implies that $I(P, L) = O(n)$. Similarly, if $r > n$ then $m > n^2$ and (4.3.1) implies that

$I(P, L) = O(m)$. If r lies in the desired range, we get $I(P, L) = O(m^{2/3}n^{2/3})$. Putting all these bounds together, we obtain the bound

$$I(P, L) = O(m^{2/3}n^{2/3} + m + n),$$

as required.

Remark. An equivalent statement is that, for a set P of m points in the plane, and for any integer $k \leq m$, the number of lines that contain at least k points of P is at most

$$O\left(\frac{m^2}{k^3} + \frac{m}{k}\right).$$

Discussion. The cutting-based method is quite powerful, and can be extended in various ways. The crux of the technique is to derive somehow a weaker (but easier) bound on the number of incidences, construct a $(1/r)$-cutting of the set of curves, obtain the corresponding decomposition of the problem into $O(r^2)$ subproblems, apply the weaker bound within each subproblem, and sum up the bounds to obtain the overall bound. The work by Clarkson *et al.* [20] contains many such extensions.

Let us demonstrate this method to obtain an upper bound for the number of incidences between a set P of m points and a set C of n arbitrary circles in the plane. Here the forbidden subgraph property is that the incidence graph $H \subseteq P \times C$ does not contain $K_{3,2}$ as a subgraph, and thus (see [43])

$$I(P, C) = O(mn^{2/3} + n).$$

We construct a $(1/r)$-cutting for C, apply this weak bound within each cell τ of the cutting, and handle incidences that occur on the cell boundaries exactly as above, to obtain

$$I(P, C) = \sum_{\tau} I(P_\tau, C_\tau) = O\left(m\left(\frac{n}{r}\right)^{2/3} + nr\right).$$

With an appropriate choice of $r = m^{3/5}/n^{1/5}$, this becomes

$$I(P, C) = O(m^{3/5}n^{4/5} + m + n).$$

However, as we shall see later, in Section 4.5, this bound can be considerably improved.

The case of a set C of n unit circles is handled similarly, observing that in this case the intersection graph H does not contain $K_{2,3}$. This yields the same upper bound $I(P, C) = O(mn^{1/2} + n)$, as in (4.3.1). The analysis then continues exactly as in the case of lines, and yields the bound

$$I(P, C) = O(m^{2/3}n^{2/3} + m + n).$$

We can apply this bound to the Repeated Distances Problem, recalling that the number of pairs of points in an n-element set of points in the plane that lie at distance exactly 1 from each other, is half the number of incidences between the points and the unit circles centered at them. Substituting $m = n$ in the above bound, we thus obtain that the number of repeated distances is at most $O(n^{4/3})$. This bound is far from the best known lower bound, mentioned in Section 4.2, and no improvement has been obtained since its original derivation in [53] in 1984.

As a matter of fact, this approach can be extended to any collection C of curves that have "d degrees of freedom", in the sense that any d points in the plane determine at most $t = O(1)$ curves

from the family that pass through all of them, and any pair of curves intersect in only $O(1)$ points. The incidence graph does not contain $K_{d,t+1}$ as a subgraph, which implies that

$$I(P,C) = O(mn^{1-1/d} + n).$$

Combining this bound with a cutting-based decomposition yields the bound

$$I(P,C) = O(m^{d/(2d-1)}n^{(2d-2)/(2d-1)} + m + n).$$

Note that this bound extrapolates the previous bounds for the cases of lines ($d = 2$), unit circles ($d = 2$), and arbitrary circles ($d = 3$). See [45] for a slight generalization of this result, using Székely's method, outlined in the following section.

4.4 Incidences via Crossing Numbers—Székely's Method

A graph G is said to be *drawn* in the plane if its vertices are mapped to distinct points in the plane, and each of its edges is represented by a Jordan arc connecting the corresponding pair of points. It is assumed that no edge passes through any vertex other than its endpoints, and that when two edges meet at a common interior point, they properly *cross* each other there, i.e., each curve passes from one side of the other curve to the other side. Such a point is called a *crossing*. In the literature, a graph drawn in the plane with the above properties is often called a *topological graph*. If, in addition, the edges are represented by straight-line segments, then the drawing is said to be a *geometric graph*.

As we have indicated before, Székely discovered that the analysis outlined in the previous section can be substantially simplified, applying the following so-called Crossing Lemma for graphs drawn in the plane.

Lemma 4.4.1 (Leighton [36], Ajtai et al. [8]). *Let G be a simple graph drawn in the plane with V vertices and E edges. If $E > 4V$ then there are $\Omega(E^3/V^2)$ crossing pairs of edges.*

To establish the lemma, denote by $\mathrm{cr}(G)$ the minimum number of crossing pairs of edges in any 'legal' drawing of G. Since G contains too many edges, it is not planar, and therefore $\mathrm{cr}(G) \geq 1$. In fact, using Euler's formula, a simple counting argument shows that $\mathrm{cr}(G) \geq E - 3V + 6 > E - 3V$. We next apply this inequality to a random sample G' of G, which is an induced subgraph obtained by choosing each vertex of G independently with some probability p. By applying expectations, we obtain $\mathbf{E}[\mathrm{cr}(G')] \geq \mathbf{E}[E'] - 3\mathbf{E}[V']$, where E', V' are the numbers of edges and vertices in G', respectively. This can be rewritten as $\mathrm{cr}(G)p^4 \geq Ep^2 - 3Vp$, and choosing $p = 4V/E$ completes the proof of Lemma 4.4.1.

We remark that the constant of proportionality in the asserted bound, as yielded by the preceding proof, is $1/64$, but it has been improved by Pach and Tóth [47]. They proved that $\mathrm{cr}(G) \geq E^3/(33.75V^2)$ whenever $E \geq 7.5V$. In fact, the slightly weaker inequality $\mathrm{cr}(G) \geq E^3/(33.75V^2) - 0.9V$ holds without any extra assumption. We also note that it is crucial that the graph G be *simple* (i.e., any two vertices be connected by at most one edge), for otherwise no crossing can be guaranteed, regardless of how large E is.

Let P be a set of m points and L a set of n lines in the plane. We associate with P and L the following plane drawing of a graph G. The vertices of (this drawing of) G are the points of P. For each line $\ell \in L$, we connect each pair of points of $P \cap \ell$ that are consecutive along ℓ by an edge of

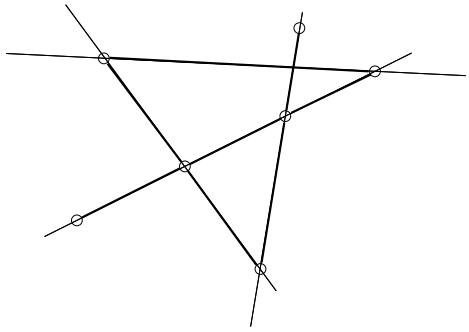

FIGURE 4.2. Székely's graph for points and lines in the plane.

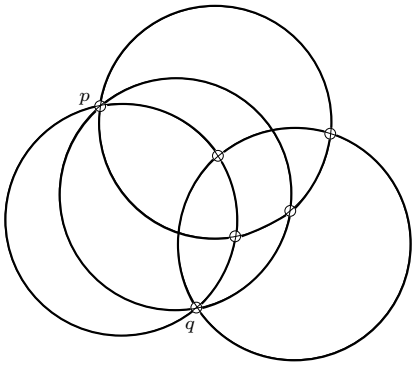

FIGURE 4.3. Székely's graph for points and unit circles in the plane: The maximum edge multiplicity is two—see the edges connecting p and q.

G, drawn as the straight segment between these points (which is contained in ℓ). See Figure 4.2 for an illustration. Clearly, G is a simple graph, and, assuming that each line of L contains at least one point of P, we have $V = m$ and $E = I(P, L) - n$ (the number of edges along a line is smaller by 1 than the number of incidences with that line). Hence, either $E < 4V$, and then $I(P, L) < 4m + n$, or $\mathrm{cr}(G) \geq E^3/(cV^2) = (I(P, L) - n)^3/(cm^2)$. However, we have, trivially, $\mathrm{cr}(G) \leq \binom{n}{2}$, implying that $I(P, L) \leq (c/2)^{1/3} m^{2/3} n^{2/3} + n \leq 2.57 m^{2/3} n^{2/3} + n$.

Extensions: Many faces and unit circles. The simple idea behind Székely's proof is quite powerful, and can be applied to many variants of the problem, as long as the corresponding graph G is simple, or, alternatively, has a bounded edge multiplicity. For example, consider the case of incidences between a set P of m points and a set C of n unit circles. Draw the graph G exactly as in the case of lines, but only along circles that contain more than two points of P, to avoid loops and multiple edges along the same circle. We have $V = m$ and $E \geq I(P, C) - 2n$. In this case, G need not be simple, but the maximum edge multiplicity is at most two; see Figure 4.3. Hence, by deleting at most half of the edges of G we make it into a simple graph. Moreover, $\mathrm{cr}(G) \leq n(n-1)$, so we get $I(P, C) = O(m^{2/3} n^{2/3} + m + n)$, again with a rather small constant of proportionality.

We can also apply this technique to obtain an upper bound on the complexity of many faces

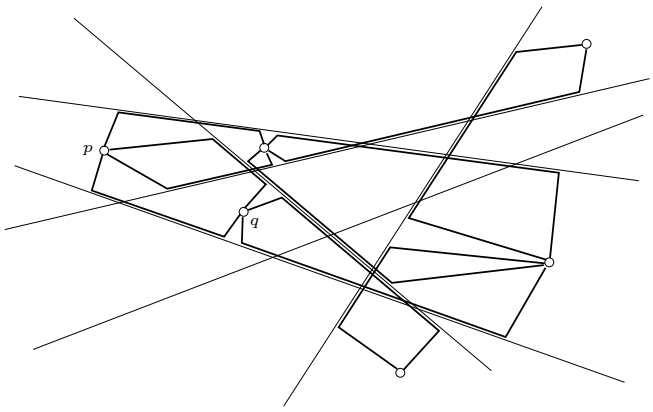

FIGURE 4.4. Székely's graph for face-marking points and lines in the plane. The maximum edge multiplicity is two—see, e.g., the edges connecting p and q.

in an arrangement of lines. Let P be a set of m points and L a set of n lines in the plane, so that no point lies on any line and each point lies in a distinct face of $\mathcal{A}(L)$. The graph G is now constructed in the following slightly different manner. Its vertices are the points of P. For each $\ell \in L$, we consider all faces of $\mathcal{A}(L)$ that are marked by points of P, are bounded by ℓ and lie on a fixed side of ℓ. For each pair f_1, f_2 of such faces that are consecutive along ℓ (the portion of ℓ between ∂f_1 and ∂f_2 does not meet any other marked face on the same side), we connect the corresponding marking points p_1, p_2 by an edge, and draw it as a polygonal path $p_1q_1q_2p_2$, where $q_1 \in \ell \cap \partial f_1$ and $q_2 \in \ell \cap \partial f_2$. We actually shift the edge slightly away from ℓ so as to avoid its overlapping with edges drawn for faces on the other side of ℓ. The points q_1, q_2 can be chosen in such a way that a pair of edges meet each other only at intersection points of pairs of lines of L. See Figure 4.4. Here we have $V = m$, $E \geq K(P, L) - 2n$, and $\mathrm{cr}(G) \leq 2n(n-1)$ (each pair of lines can give rise to at most four pairs of crossing edges, near the same intersection point). Again, G is not simple, but the maximum edge multiplicity is at most two, because, if two faces f_1, f_2 are connected along a line ℓ, then ℓ is a common external tangent to both faces. Since f_1 and f_2 are disjoint convex sets, they can have at most two external common tangents. Hence, arguing as above, we obtain $K(P, L) = O(m^{2/3}n^{2/3} + m + n)$. We remark that the same upper bound can also be obtained via the partition technique, as shown by Clarkson *et al.* [20]. Moreover, in view of the discussion in Section 4.2, this bound is tight.

However, Székely's technique does not always apply. The simplest example where it fails is when we want to establish an upper bound on the number of incidences between points and circles of arbitrary radii. If we follow the same approach as for equal circles, and construct a graph analogously, we may now create edges with arbitrarily large multiplicities, as is illustrated in Figure 4.5. We will tackle this problem in the next section.

Another case where the technique fails is when we wish to bound the total complexity of many faces in an arrangement of line *segments*. If we try to construct the graph in the same way as we did for full lines, the faces may not be convex any more, and we can create edges of high multiplicity; see Figure 4.6.

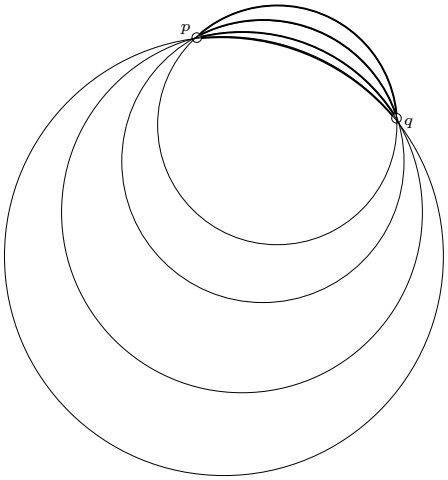

FIGURE 4.5. Székely's graph need not be simple for points and arbitrary circles in the plane.

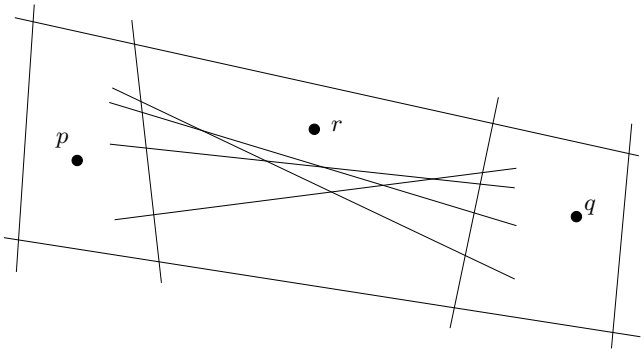

FIGURE 4.6. Székely's graph need not be simple for marked faces and segments in the plane: An arbitrarily large number of segments bounds all three faces marked by the points p, q, r, so the edges (p, r) and (r, q) in Székely's graph have arbitrarily large multiplicity.

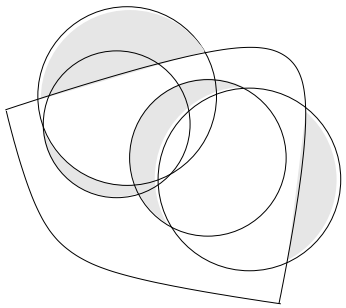

FIGURE 4.7. The boundaries of the shaded regions are nonoverlapping lenses in an arrangement of pseudo-circles. (Observe that the *regions* bounded by nonoverlapping lenses can overlap, as is illustrated here).

4.5 Improvements by Cutting into Pseudo-segments

Consider the case of incidences between points and circles of arbitrary radii. One way to overcome the technical problem in applying Székely's technique in this case is to cut the given circles into arcs so that any two of them intersect at most once. We refer to such a collection of arcs as a collection of *pseudo-segments*.

The first step in this direction has been taken by Tamaki and Tokuyama [56], who have shown that any collection C of n *pseudo-circles*, namely, closed Jordan curves, each pair of which intersect at most twice, can be cut into $O(n^{5/3})$ arcs that form a family of pseudo-segments. The union of two arcs that belong to distinct pseudo-circles and connect the same pair of points is called a *lens*. Let $\chi(C)$ denote the minimum number of points that can be removed from the curves of C, so that any two members of the resulting family of arcs have at most one point in common. Clearly, every lens must contain at least one of these cutting points, so Tamaki and Tokuyama's problem asks in fact for an upper bound on the number of points needed to "stab" all lenses. Equivalently, this problem can be reformulated, as follows.

Consider a hypergraph H whose vertex set consists of the edges of the arrangement $\mathcal{A}(C)$, i.e., the arcs between two consecutive crossings. Assign to each lens a *hyperedge* consisting of all arcs that belong to the lens. We are interested in finding the *transversal number* (or the size of the smallest "hitting set") of H, i.e., the smallest number of vertices of H that can be picked with the property that every hyperedge contains at least one of them. Based on Lovász' analysis [37] (see also [43]) of the greedy algorithm for bounding the transversal number from above (i.e., for constructing a hitting set), this quantity is not much bigger than the size of the largest *matching* in H, i.e., the maximum number of pairwise disjoint hyperedges. This is the same as the largest number of pairwise non-overlapping lenses, that is, the largest number of lenses, no two of which share a common edge of the arrangement $\mathcal{A}(C)$ (see Figure 4.7). Viewing such a family as a graph G, whose edges connect pairs of curves that form a lens in the family, Tamaki and Tokuyama proved that G does not contain $K_{3,3}$ as a subgraph, and this leads to the asserted bound on the number of cuts.

In order to establish an upper bound on the number of incidences between a set of m points P and a set of n circles (or pseudo-circles) C, let us construct a modified version G' of Székely's graph: its vertices are the points of P, and its edges connect adjacent pairs of points along the new pseudo-segment arcs. That is, we do not connect a pair of points that are adjacent along an original curve, if the arc that connects them has been cut by some point of the hitting set. Moreover, as in

the original analysis of Székely, we do not connect points along pseudo-circles that are incident to only one or two points of P, to avoid loops and trivial multiplicities.

Clearly, the graph G' is simple, and the number E' of its edges is at least $I(P,C) - \chi(C) - 2n$. The crossing number of G' is, as before, at most the number of crossings between the original curves in C, which is at most $n(n-1)$. Using the Crossing Lemma (Lemma 4.4.1), we thus obtain

$$I(P,C) = O(m^{2/3}n^{2/3} + \chi(C) + m + n).$$

Hence, applying the Tamaki-Tokuyama bound on $\chi(C)$, we can conclude that

$$I(P,C) = O(m^{2/3}n^{2/3} + n^{5/3} + m).$$

An interesting property of this bound is that it is tight when $m \geq n^{3/2}$. In this case, the bound becomes $I(P,C) = O(m^{2/3}n^{2/3} + m)$, matching the lower bound for incidences between points and lines, which also serves as a lower bound for the number of incidences between points and circles or parabolas. However, for smaller values of m, the term $O(n^{5/3})$ dominates, and the dependence on m disappears. This can be rectified by combining this bound with a cutting-based problem decomposition, similar to the one used in the preceding section, and we shall do so shortly.

Before proceeding, though, we note that Tamaki and Tokuyama's bound is not tight. The best known lower bound is $\Omega(n^{4/3})$, which follows from the lower bound construction for incidences between points and lines. (That is, we have already seen that this construction can be modified so as to yield a collection C of n circles with $\Theta(n^{4/3})$ empty lenses. Clearly, each such lens requires a separate cut, so $\chi(C) = \Omega(n^{4/3})$.) Recent work by Alon *et al.* [9], Aronov and Sharir [14], Agarwal *et al.* [5], and Marcus and Tardos [38] has led to improved bounds, where the currently best upper bound, due to Marcus and Tardos, is $\chi(C) = O(n^{3/2}\log n)$, for families C of *pseudo-parabolas* (graphs of continuous everywhere defined functions, each pair of which intersect at most twice), or of *pseudo-circles* (closed Jordan curves with the same property).

With the aid of this improved bound on $\chi(C)$, the modification of Székely's method reviewed above yields, for a set C of n circles and a set P of m points,

$$I(P,C) = O(m^{2/3}n^{2/3} + n^{3/2}\log n + m).$$

As already noted, this bound is tight when it is dominated by the first or last terms, which happens when $m = \Omega(n^{5/4}\log^{3/2} n)$. For smaller values of m, we decompose the problem into subproblems, using the following so-called "dual" partitioning technique. We map each circle $(x-a)^2+(y-b)^2 = \rho^2$ in C to the "dual" point $(a, b, \rho^2 - a^2 - b^2)$ in 3-space, and map each point (ξ, η) of P to the "dual" plane $z = -2\xi x - 2\eta y + (\xi^2 + \eta^2)$. As is easily verified, each incidence between a point of P and a circle of C is mapped to an incidence between the dual plane and point. We now fix a parameter r, and construct a $(1/r)$-cutting of the arrangement of the dual planes, which partitions $\mathbb{R}^3$ into $O(r^3)$ cells (which is a tight bound in the case of planes), each crossed by at most m/r dual planes and containing at most n/r^3 dual points (the latter property, which is not an intrinsic property of the cutting, can be enforced by further partitioning cells that contain more than n/r^3 points). We apply, for each cell τ of the cutting, the preceding bound for the set P_τ of points of P whose dual planes cross τ, and for the set C_τ of circles whose dual points lie in τ. (Some special handling of circles whose dual points lie on boundaries of cells of the cutting is needed, as in Section 4.3, but we omit the treatment of this special case.) This yields the bound

$$I(P,C) = O(r^3) \cdot O\left(\left(\frac{m}{r}\right)^{2/3}\left(\frac{n}{r^3}\right)^{2/3} + \left(\frac{n}{r^3}\right)^{3/2}\log\left(\frac{n}{r^3}\right) + \frac{m}{r}\right) =$$

$$O\left(m^{2/3}n^{2/3}r^{1/3} + \frac{n^{3/2}}{r^{3/2}}\log\left(\frac{n}{r^3}\right) + mr^2\right).$$

Assume that m lies between $n^{1/3}$ and $n^{5/4}$, and choose $r = n^{5/11}/m^{4/11}\log^{6/11} n$ in the last bound, to obtain

$$I(P,C) = O(m^{2/3}n^{2/3} + m^{6/11}n^{9/11}\log^{2/11}(m^3/n) + m + n).$$

It is not hard to see that this bound also holds for the complementary ranges of m.

4.6 Incidences in Higher Dimensions

It is natural to extend the study of incidences to instances involving points and curves or surfaces in higher dimensions. The case of incidences between points and (hyper)surfaces (mainly hyperplanes) has been studied earlier. Edelsbrunner *et al.* [24] considered incidences between points and planes in three dimensions. It is important to note that, without imposing some restrictions either on the set P of points or on the set H of planes, one can easily obtain $|P|\cdot|H|$ incidences, simply by placing all the points of P on a line, and making all the planes of H pass through that line. Some natural restrictions are to require that no three points be collinear, or that no three planes be collinear, or that the points be vertices of the arrangement $\mathcal{A}(H)$, and so on. Different assumptions lead to different bounds. For example, Agarwal and Aronov [1] proved an asymptotically tight bound $\Theta(m^{2/3}n^{d/3} + n^{d-1})$ for the number of incidences between n hyperplanes in d dimensions and $m > n^{d-2}$ vertices of their arrangement (see also [24]), as well as for the number of facets bounding m distinct cells in such an arrangement. Edelsbrunner and Sharir [25] considered the problem of incidences between points and hyperplanes in four dimensions, under the assumption that all points lie on the upper envelope of the hyperplanes. They obtained the bound $O(m^{2/3}n^{2/3} + m + n)$ for the number of such incidences, and applied the result to obtain the same upper bound on the number of bichromatic minimal distance pairs between a set of m blue points and a set of n red points in three dimensions. Another set of bounds and related results are obtained by Brass and Knauer [15], for incidences between m points and n planes in 3-space, and also for incidences in higher dimensions. See also the recent related work of Apfelbaum and Sharir [10].

The case of incidences between points and *curves* in higher dimensions has been studied only recently. There are only two papers that address this problem. One of them, by Sharir and Welzl [50], studies incidences between points and lines in 3-space. The other, by Aronov *et al.* [12], is concerned with incidences between points and circles in higher dimensions. Both works were motivated by problems asked by Elekes. We briefly review these results in the following two subsections.

4.6.1 Points and lines in three dimensions

Let P be a set of m points and L a set of n lines in 3-space. Without making some assumptions on P and L, the problem is trivial, for the following reason. Project P and L onto some generic plane. Incidences between points of P and lines of L are bijectively mapped to incidences between the projected points and lines, so we have $I(P,L) = O(m^{2/3}n^{2/3} + m + n)$. Moreover, this bound is tight, as is shown by the planar lower bound construction. (As a matter of fact, this reduction holds in any dimension $d \geq 3$.)

There are several ways in which the problem can be made interesting. First, suppose that the points of P are *joints* in the arrangement $\mathcal{A}(L)$, namely, each point is incident to at least three non-

coplanar lines of L. In this case, one has $I(P,L) = O(n^{5/3})$ [50]. Note that this bound is independent of m. In fact, it is known that the number of joints is at most $O(n^{112/69}\log^{6/23} n)$, which is $O(n^{1.6232})$ [33] (see also [48]); the best lower bound, based on lines forming a cube grid, is only $\Omega(n^{3/2})$. More details on joints can be found in Chapter 7.

For general point sets P, one can use a new measure of incidences, which aims to ignore incidences between a point and many incident coplanar lines. Specifically, we define the *plane cover* $\pi_L(p)$ of a point p to be the minimum number of planes that pass through p so that their union contains all lines of L incident to p, and define $I_c(P,L) = \sum_{p\in P}\pi_L(p)$. It is shown in [50] that

$$I_c(P,L) = O(m^{4/7}n^{5/7} + m + n),$$

which is smaller than the planar bound of Szemerédi and Trotter.

Another way in which we can make the problem "truly 3-dimensional" is to require that all lines in L be *equally inclined*, meaning that each of them forms a fixed angle (say, 45°) with the z-direction. In this case, every point of P that is incident to at least three lines of L is a joint, but this special case admits better upper bounds. Specifically, we have

$$I(P,L) = O(\min\left\{m^{3/4}n^{1/2}\log m,\ m^{4/7}n^{5/7}\right\} + m + n).$$

The best known lower bound is

$$I(P,L) = \Omega(m^{2/3}n^{1/2}).$$

Let us briefly sketch the proof of the upper bound $O(m^{3/4}n^{1/2}\log m)$. For each $p \in P$ let C_p denote the (double) cone whose apex is p, whose symmetry axis is the vertical line through p, and whose opening angle is 45°. Fix some generic horizontal plane π_0, and map each $p \in P$ to the circle $C_p \cap \pi_0$. Each line $\ell \in L$ is mapped to the point $\ell \cap \pi_0$, coupled with the projection ℓ^* of ℓ onto π_0. Note that an incidence between a point $p \in P$ and a line $\ell \in L$ is mapped to the configuration in which the circle dual to p is incident to the point dual to ℓ and the projection of ℓ passes through the center of the circle; see Figure 4.8. Hence, if a line ℓ is incident to several points $p_1, \ldots, p_k \in P$, then the dual circles $p_1^*, \ldots, p_k^*$ are all tangent to each other at the common point $\ell \cap \pi_0$. Viewing these tangencies as a collection of degenerate lenses, we can bound the overall number of these tangencies, which is equal to $I(P,L)$, by $O(n^{3/2}\log n)$. By a slightly more careful analysis, again based on cutting, one can obtain the bound $O(m^{3/4}n^{1/2}\log m)$.

4.6.2 Points and circles in three and higher dimensions

Let C be a set of n circles and P a set of m points in 3-space. Unlike in the case of lines, there is no obvious reduction of the problem to a planar one, because the projection of C onto some generic plane yields a collection of ellipses, rather than circles, which can cross each other at four points per pair. However, using a more refined analysis, Aronov *et al.* [12] have obtained the same asymptotic bound of $I(P,C) = O(m^{2/3}n^{2/3} + m^{6/11}n^{9/11}\log^{2/11}(m^3/n) + m + n)$ for $I(P,C)$. The same bound applies in any dimension $d \geq 3$.

4.7 Applications

The problem of bounding the number of incidences between various geometric objects is elegant and fascinating, and it has been mostly studied for its own sake. However, it is closely related to a

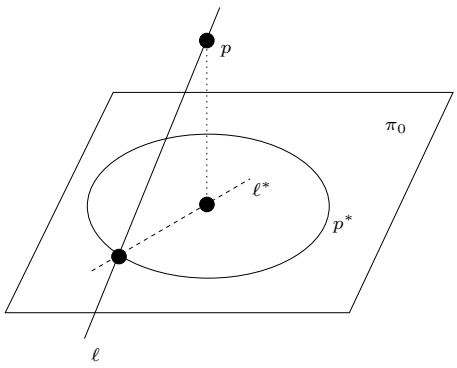

FIGURE 4.8. Transforming incidences between points and equally inclined lines to tangencies between circles in the plane.

variety of questions in combinatorial and computational geometry. In this section, we briefly review some of these connections and applications.

4.7.1 Algorithmic issues

There are two types of algorithmic problems related to incidences. The first group includes problems where we wish to actually determine the number of incidences between certain objects, e.g., between given sets of points and curves, or we wish to compute (describe) a collection of marked faces in an arrangement of curves or surfaces. The second group contains completely different questions whose solution requires tools and techniques developed for the analysis of incidence problems.

In the simplest problem of the first kind, known as Hopcroft's problem, we are given a set P of m points and a set L of n lines in the plane, and we ask whether there exists at least one incidence between P and L. The best running time known for this problem is $O(m^{2/3}n^{2/3} \cdot 2^{O(\log^*(m+n))})$ [40] (see [32] for a matching lower bound). Similar running time bounds hold for the problems of counting or reporting all the incidences in $I(P, L)$. The solutions are based on constructing cuttings of an appropriate size and thereby obtaining a decomposition of the problem into subproblems, each of which can be solved by a more brute-force approach. In other words, the solution can be viewed as an implementation of the cutting-based analysis of the combinatorial bound for $I(P, L)$, as presented in Section 4.3.

The case of incidences between a set P of m points and a set C of n circles in the plane is more interesting, because the analysis that leads to the current best upper bound on $I(P, C)$ is not easy to implement. In particular, suppose that we have already cut the circles of C into roughly $O(n^{3/2})$ pseudo-segments (an interesting and non-trivial algorithmic task in itself), and we now wish to compute the incidences between these pseudo-segments and the points of P. Székely's technique is non-algorithmic, so instead we would like to apply the cutting-based approach to these pseudo-segments and points. However, this approach, for the case of lines, after decomposing the problem into subproblems, proceeds by duality. Specifically, it maps the points in a subproblem to dual lines, constructs the arrangement of these dual lines, and locates in the arrangement the points dual to the lines in the subproblem. When dealing with the case of pseudo-segments, there is no obvious incidence-preserving duality that maps them to points and maps the points to pseudo-lines.

Nevertheless, such a duality has been recently defined by Agarwal and Sharir [7] (refining an older and less efficient duality given by Goodman [34]), which can be implemented efficiently and thus yields an efficient algorithm for computing $I(P, C)$, whose running time is comparable with the bound on $I(P, C)$ given above. A similar approach can be used to compute many faces in arrangements of pseudo-circles; see [2] and [7]. Algorithmic aspects of incidence problems have also been studied in higher dimensions; see, e.g., Brass and Knauer [15].

The cutting-based approach has by now become a standard tool in the design of efficient geometric algorithms in a variety of applications in range searching, geometric optimization, ray shooting, and many others. It is beyond the scope of this survey to discuss these applications, and the reader is referred, e.g., to the survey of Agarwal and Erickson [3] and to the references therein.

4.7.2 Distinct distances

The above techniques can be applied to obtain some nontrivial results concerning the Distinct Distances problem of Erdős [28] formulated in the Introduction: what is the minimum number of distinct distances determined by n points in the plane? As we have indicated after presenting the proof of the Crossing Lemma (Lemma 4.4.1), Székely's idea can also be applied in several situations where the underlying graph is not *simple*, i.e., two vertices can be connected by more than one edge. However, for the method to work it is important to have an upper bound for the multiplicity of the edges. Székely [54] formulated the following natural generalization of Lemma 4.4.1.

Lemma. *Let G be a multigraph drawn in the plane with V vertices, E edges, and with maximal edge-multiplicity M. Then there are $\Omega\left(\frac{E^3}{MV^2}\right) - O(M^2V)$ crossing pairs of edges.*

Székely applied this statement to the Distinct Distances problem, and improved by a polylogarithmic factor the best previously known lower bound of Chung et al. [19] on the minimum number of distinct distances determined by n points in the plane. His new bound was $\Omega(n^{4/5})$. However, Solymosi and Tóth [51] have realized that, combining Székely's analysis of distinct distances with the Szemerédi-Trotter theorem for the number of incidences between m points and n lines in the plane, this lower bound can be substantially improved. They managed to raise the bound to $\Omega(n^{6/7})$. Later, Tardos and Katz have further improved this result, using the same general approach, but improving upon a key algebraic step of the analysis. In their latest paper [35], they combined their methods to prove that the minimum number of distinct distances determined by n points in the plane is $\Omega(n^{(48-14e)/(55-16e)-\varepsilon})$, for any $\varepsilon > 0$, which is $\Omega(n^{0.8641})$. This is the best known result so far. A close inspection of the general Solymosi-Tóth approach shows that, without any additional geometric idea, it can never lead to a lower bound better than $\Omega(n^{8/9})$.

4.7.3 Equal-area, equal-perimeter, and isoceles triangles

Let P be a set of n points in the plane. We wish to bound the number of triangles spanned by the points of P that have a *given area*, say 1. To do so, we note that if we fix two points $a, b \in P$, any third point $p \in P$ for which $\mathrm{Area}(\Delta abp) = 1$ lies on the union of two fixed lines parallel to ab. Pairs (a, b) for which such a line ℓ_{ab} contains fewer than $n^{1/3}$ points of P generate at most $O(n^{7/3})$ unit area triangles. For the other pairs, we observe that the number of lines containing more than $n^{1/3}$ points of P is at most $O(n^2/(n^{1/3})^3) = O(n)$, which, as already mentioned, is an immediate consequence of the Szemerédi-Trotter theorem. The number of incidences between these lines and the points of P is at most $O(n^{4/3})$. We next observe that any line ℓ can be equal to one of the two

lines ℓ_{ab} for at most n pairs a, b, because, given ℓ and a, there can be at most two points b for which $\ell = \ell_{ab}$. It follows that the lines containing more than $n^{1/3}$ points of P can be associated with at most $O(n \cdot n^{4/3}) = O(n^{7/3})$ unit area triangles. Hence, overall, P determines at most $O(n^{7/3})$ unit area triangles. The best known lower bound is $\Omega(n^2 \log\log n)$ (see [16]).

Next, consider the problem of estimating the number of *unit perimeter* triangles determined by P. Here we note that if we fix $a, b \in P$, with $|ab| < 1$, any third point $p \in P$ for which Perimeter$(\Delta abp) = 1$ lies on an ellipse whose foci are a and b and whose major axis is $1 - |ab|$. Clearly, any two distinct pairs of points of P give rise to distinct ellipses, and the number of unit perimeter triangles determined by P is equal to one third of the number of incidences between these $O(n^2)$ ellipses and the points of P. The set of these ellipses has four degrees of freedom, in the sense of Pach and Sharir [45] (see also Section 4.3), and hence the number of incidences between them and the points of P, and consequently the number of unit perimeter triangles determined by P, is at most

$$O(n^{4/7}(n^2)^{6/7}) = O(n^{16/7}).$$

Here the best known lower bound is very weak—only $\Omega(ne^{c\frac{\log n}{\log\log n}})$ [16].

Finally, consider the problem of estimating the number of *isosceles* triangles determined by P. Recently, Pach and Tardos [46] proved that the number of isosceles triangles induced by triples of an n-element point set in the plane is $O(n^{(11-3\alpha)/(5-\alpha)})$ (where the constant of proportionality depends on α), provided that $0 < \alpha < \frac{10-3e}{24-7e}$. In particular, the number of isoceles triangles is $O(n^{2.136})$. The best known lower bound is $\Omega(n^2 \log n)$ [16]. The proof proceeds through two steps, interesting in their own right.

(i) Let P be a set of n distinct points and let C be a set of ℓ distinct circles in the plane, with $m \leq \ell$ distinct centers. Then, for any $0 < \alpha < 1/e$, the number I of incidences between the points in P and the circles of C is

$$O\left(n + \ell + n^{\frac{2}{3}}\ell^{\frac{2}{3}} + n^{\frac{4}{7}}m^{\frac{1+\alpha}{7}}\ell^{\frac{5-\alpha}{7}} + n^{\frac{12+4\alpha}{21+3\alpha}}m^{\frac{3+5\alpha}{21+3\alpha}}\ell^{\frac{15-3\alpha}{21+3\alpha}} + n^{\frac{8+2\alpha}{14+\alpha}}m^{\frac{2+2\alpha}{14+\alpha}}\ell^{\frac{10-2\alpha}{14+\alpha}}\right),$$

where the constant of proportionality depends on α.

(ii) As a corollary, we obtain the following statement. Let P be a set of n distinct points and let C be a set of ℓ distinct circles in the plane such that they have at most n distinct centers. Then, for any $0 < \alpha < 1/e$, the number of incidences between the points in P and the circles in C is

$$O\left(n^{\frac{5+3\alpha}{7+\alpha}}\ell^{\frac{5-\alpha}{7+\alpha}} + n\right).$$

In view of a recent result of Katz and Tardos [35], both statements extend to all $0 < \alpha < \frac{10-3e}{24-7e}$, which easily implies the above bound on the number of isosceles triangles.

4.7.4 Congruent simplices

Bounding the number of incidences between points and circles in higher dimensions can be applied to the following interesting question asked by Erdős and Purdy [30, 31] and discussed by Agarwal and Sharir [6]. Determine the largest number of simplices congruent to a fixed simplex σ, which can be spanned by an n-element point set $P \subset \mathbb{R}^k$?

Here we consider only the case when $P \subset \mathbb{R}^4$ and $\sigma = abcd$ is a 3-simplex. Fix three points $p, q, r \in P$ such that the triangle pqr is congruent to the face abc of σ. Then any fourth point $v \in P$

for which $pqrv$ is congruent to σ must lie on a circle whose plane is orthogonal to the triangle pqr, whose radius is equal to the height of σ from d, and whose center is at the foot of that height. Hence, bounding the number of congruent simplices can be reduced to the problem of bounding the number of incidences between circles and points in 4-space. (The actual reduction is slightly more involved, because the same circle can arise for more than one triangle pqr; see [6] for details.) Using the bound of [12], mentioned in Section 4.6, one can deduce that the number of congruent 3-simplices determined by n points in 4-space is $O(n^{20/9+\varepsilon})$, for any $\varepsilon > 0$.

This is just one instance of a collection of bounds obtained in [6] for the number of congruent copies of a k-simplex in an n-element point set in $\mathbb{R}^d$, whose review is beyond the scope of this survey.

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Chapter 5

Crossing Numbers of Graphs: Graph Drawing and its Applications

Drawing is one of the most ancient human activities. Our ancestors drew their pictures (pictographs or, simply, "*graphs*") on walls of caves, nowadays we use mostly computer screens for this purpose. From the mathematical point of view, there is not much difference: both surfaces are "flat," they are topologically equivalent.

5.1 Crossings – the Brick Factory Problem

Let G be a finite graph with vertex set $V(G)$ and edge set $E(G)$. By a *drawing* of G, we mean a representation of G in the plane such that each vertex is represented by a distinct point and each edge by a simple (non-selfintersecting) continuous arc connecting the corresponding two points. If it is clear whether we talk about an "abstract" graph G or its planar representation, these points and arcs will also be called vertices and edges, respectively. For simplicity, we assume that in a drawing (a) no edge passes through any vertex other than its endpoints, (b) no two edges touch each other (i.e., if two edges have a common interior point, then at this point they properly cross each other), and (c) no three edges cross at the same point.

Every graph has many different drawings. If G can be drawn in such a way that no two edges cross each other, then G is *planar*. According to an observation of István Fáry [17], if G is planar then it has a drawing, in which every edge is represented by a straight-line segment.

It is well known that K_5, the *complete graph* with 5 vertices, and $K_{3,3}$, the *complete bipartite graph* with 3 vertices in its classes are not planar. According to Kuratowski's theorem, a graph is planar if and only if it has no subgraph that can be obtained from K_5 or from $K_{3,3}$ by subdividing some (or all) of its edges with distinct new vertices. In the next section, we give a completely different representation of planar graphs (see Theorem 5.2.4).

If G is not planar then it cannot be drawn in the plane without crossing. Paul Turán [54] raised the following problem: find a drawing of G, for which the number of crossings is minimum. This number is called the *crossing number* of G and is denoted by $\mathrm{CR}(G)$. More precisely, Turán's

(still unsolved) original problem was to determine $\mathrm{CR}(K_{n,m})$, for every $n, m \geq 3$. According to an assertion of Zarankiewicz, which was down-graded from theorem to conjecture [21], we have

$$\mathrm{CR}(K_{n,m}) = \lfloor \frac{m}{2} \rfloor \cdot \lfloor \frac{m-1}{2} \rfloor \cdot \lfloor \frac{n}{2} \rfloor \cdot \lfloor \frac{n-1}{2} \rfloor,$$

but we do not even know the limits

$$\lim_{n\to\infty} \frac{\mathrm{CR}(K_{n,n})}{n^4}, \quad \lim_{n\to\infty} \frac{\mathrm{CR}(K_n)}{n^4}$$

(cf. [48], [28]).

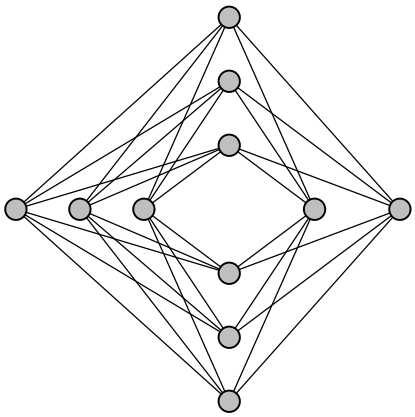

FIGURE 5.1. $K_{5,6}$ drawn with minimum number of crossings.

Turán used to refer to this question as the "brick factory problem," because it occurred to him at a factory yard, where, as forced labor during World War II, he moved wagons filled with bricks from kilns to storage places. According to his recollections, it was not a very tough job, except that they had to push much harder at the crossings. Had this been the only "practical application" of crossing numbers, much fewer people would have tried to estimate $\mathrm{CR}(G)$ during the past quarter of a century. In the early eighties, it turned out that the chip area required for the realization (VLSI layout) of an electrical circuit is closely related to the crossing number of the underlying graph [30]. This discovery gave an impetus to research in the subject.

5.2 Thrackles – Conway's Conjecture

A drawing of a graph is called a *thrackle*, if any two edges that do not share an endpoint cross precisely once, and if two edges share an endpoint then they have no other point in common.

It is easy to verify that e.g. C_4, a cycle of length 4, cannot be drawn as a thrackle, but any other cycle can [58]. If a graph cannot be drawn as a thrackle, then the same is true for all graphs that contain it as a subgraph. Thus, a thrackle does not contain a cycle of length 4, and, according to an old theorem of Erdős in extremal graph theory, the number of its edges cannot exceed $n^{3/2}$, where n denotes the number of its vertices.

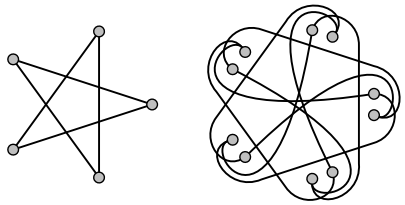

FIGURE 5.2. Cycles C_5 and C_{10} drawn as thrackles.

The following old conjecture states much more.

Conjecture 5.2.1 (J. Conway). *Every thrackle has at most as many edges as vertices.*

The first upper bound on the number of edges of a thrackle, which is linear in n, was found in [31]: Every thrackle has at most twice as many edges as vertices. The constant two has been improved to one and a half.

Theorem 5.2.2 ([11]). *Every thrackle has at most one and a half times as many edges as vertices.*

Thrackle and planar graph are, in a certain sense, opposite notions: in the former any two edges intersect, in the latter there is no crossing pair of edges. Yet the next theorem shows how similar these concepts are.

A drawing of a graph is said to be a *generalized thrackle* if every pair of its edges intersect an odd number of times. Here the common endpoint of two edges also counts as a point of intersection. Clearly, every thrackle is a generalized thrackle, but not the other way around. For example, a cycle of length 4 can be drawn as a generalized thrackle, but not as a thrackle.

Theorem 5.2.3 ([31]). *A bipartite graph can be drawn as a thrackle if and only if it is planar.*

According to an old observation of Erdős, every graph has a bipartite subgraph which contains at least half of its edges. Clearly, every planar graph of $n \geq 3$ vertices has at most $2n - 4$ edges. Hence, Theorem 5.2.3 immediately implies that every thrackle with $n \geq 3$ vertices has at most $2(2n - 4) = 4n - 8$ edges. This statement is weaker than Theorem 5.2.2, but the bound is linear in n.

In a drawing of a graph, a triple of internally disjoint paths $(P_1(u, v), P_2(u, v), P_3(u, v))$ between the same pair of vertices (u, v) is called a *trifurcation*. (The three paths cannot have any vertices in common, other than u and v, but they can cross at points different from their vertices.) A trifurcation $(P_1(u, v), P_2(u, v), P_3(u, v))$ is said to be a *converter* if the cyclic order of the initial pieces of P_1, P_2, and P_3 around u is opposite to the cyclic order of their final pieces around v.

Theorem 5.2.4 ([31]). *A graph is planar if and only if it has a drawing, in which every trifurcation is a converter.*

The second half of the theorem is trivial: if a graph is planar, then it can be drawn without crossing, and, clearly, every trifurcation in this drawing is a converter. The first half of the statement can be proved using Kuratowski's theorem.

5.3 Different Crossing Numbers?

As is illustrated by Theorem 5.2.4, the investigation of crossings in graphs often requires parity arguments. This phenomenon can be partially explained by the "banal" fact that if we start out from the interior of a simple (non-selfintersecting) closed curve in the plane, then we find ourselves inside or outside of the curve depending on whether we crossed it an even or an odd number of times.

Next we define three variants of the notion of crossing number.

(1) The *rectilinear crossing number*, $\text{LIN-CR}(G)$, of a graph G is the minimum number of crossings in a drawing of G, in which every edge is represented by a straight-line segment.

(2) The *pairwise crossing number* of G, $\text{PAIR-CR}(G)$, is the minimum number of crossing pairs of edges over all drawings of G. (Here the edges can be represented by arbitrary continuous curves, so that two edges may cross more than once, but every pair of edges can contribute to $\text{PAIR-CR}(G)$ at most one.)

(3) The *odd-crossing number* of G, $\text{ODD-CR}(G)$, is the minimum number of those pairs of edges which cross an odd number of times, over all drawings of G.

It readily follows from the definitions that

$$\text{ODD-CR}(G) \leq \text{PAIR-CR}(G) \leq \text{CR}(G) \leq \text{LIN-CR}(G).$$

Bienstock and Dean [10] exhibited a series of graphs with crossing number 4, whose rectilinear crossing numbers are arbitrary large. Very recently, Pelsmajer, Schaefer, and Štefankovič [47] have shown that for any $\varepsilon > 0$ there exists a graph G with

$$\text{ODD-CR}(G) \leq \left(\frac{\sqrt{3}}{2} + \varepsilon\right) \text{CR}(G).$$

However, we cannot rule out the possibility that

Conjecture 5.3.1 ([47]). *There is a constant $c > 0$ such that $\text{ODD-CR}(G) \geq c\text{CR}(G)$, for every graph G.*

Conjecture 5.3.2. *For every graph G, we have $\text{PAIR-CR}(G) = \text{CR}(G)$.*

The determination of the odd-crossing number can be rephrased as a purely combinatorial problem, thus the above two conjectures may offer a spark of hope that there exists an efficient approximation algorithm for estimating their values.

According to a remarkable theorem of Hanani (alias Chojnacki) [13] and William Tutte [55], if a graph G can be drawn in the plane so that any pair of its edges cross an even number of times, then it can also be drawn without any crossing. In other words, $\text{ODD-CR}(G) = 0$ implies that $\text{CR}(G) = 0$. Note that in this case, by the observation of Fáry mentioned in Section 5.1, we also have that $\text{LIN-CR}(G) = 0$.

The main difficulty in this problem is that a graph has so many essentially different drawings that the computation of any of the above crossing numbers, for a graph of only 15 vertices, appears to be a hopelessly difficult task even for very fast computers [16].

Theorem 5.3.3 ([19], [43], [49]). *The computation of the crossing number, the pairwise crossing number, and the odd-crossing number are NP-complete problems.*

We can show that the growth rates of the three parameters in Theorem 5.3.3, $\text{CR}(G)$, $\text{PAIR-CR}(G)$, and $\text{ODD-CR}(G)$, are not completely unrelated.

Theorem 5.3.4 ([43]). *For any graph G, we have*

$$\text{CR}(G) \leq 2(\text{ODD-CR}(G))^2.$$

The proof of the last statement is based on the following sharpening of the Hanani–Tutte Theorem.

Theorem 5.3.5 ([43]). *Any drawing of any graph in the plane can be redrawn in such a way that no edge, which originally crossed every other edge an even number of times, would participate in any crossing.*

In [40], we apply the original form of the Hanani–Tutte Theorem to answer a question about the "complexity" of the boundary of the union geometric figures [26].

In the original definition of the crossing number we assume that no three edges pass through the same point. Of course, this can be always achieved by slightly perturbing the drawing. Equivalently, we can say that *k-fold crossings* are permitted, but they are counted $\binom{k}{2}$ times.

G. Rote, M. Sharir, and others asked what happens if multiple crossings are counted only *once*? To what extent does this modification effect the notion of crossing number? It is important to assume here that *no tangencies are allowed* between the edges. Indeed, otherwise, given a complete graph with n vertices, one can easily draw it with only *one* crossing point p so that every pair of vertices is connected by an edge passing through p.

Let $\text{CR}'(G)$ denote the *degenerate crossing number* of G, that is, the minimum number of crossing *points* over all drawings of G satisfying this condition, where k-fold crossings are also allowed. Of course, we have

$$\text{CR}'(G) \leq \text{CR}(G),$$

and the two crossing numbers are not necessarily equal. For example, Kleitman [28] proved that the crossing number of the complete bipartite graph $K_{5,5}$ with five vertices in its classes is 16. On the other hand, the degenerate crossing number of $K_{5,5}$ in the plane is at most 15. Another example is depicted in Figure 1.

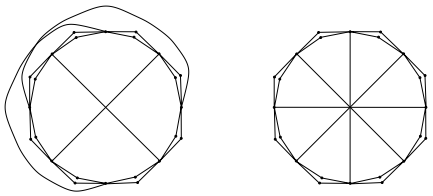

FIGURE 5.3. $\mathrm{cr}(G) = 2$, $\underline{\mathrm{cr}}(G) = 1$..

Let $n = n(G)$ and $e = e(G)$ denote the number of vertices and the number of edges of a graph G. Ajtai, Chvátal, Newborn, Szemerédi [4] and, independently, Leighton [30] proved

Theorem 5.3.6. *For every graph G with $e(G) \geq 4n(G)$, we have*

$$\mathrm{cr}(G) \geq \frac{1}{64}\frac{e^3(G)}{n^2(G)}.$$

This statement has many interesting applications in combinatorial geometry. Does it remain asymptotically true for the *degenerate* crossing number? It is not hard to show [44] that the answer is "no" if we permit drawings in which two edges may cross an arbitrary number of times. More precisely, any graph G with $n(G)$ vertices and $e(G)$ edges has a proper drawing with fewer than $e(G)$ crossings, where each crossing point that belongs to the interior of several edges is counted only once. That is, we have $\mathrm{cr}'(G) < e(G)$. The order of magnitude of this bound cannot be improved if $e \geq 4n$.

Therefore, we restrict our attention to so-called *simple* drawings, i.e., to proper drawings in which two edges are allowed to cross at most once. Let $\mathrm{cr}^*(G)$ denote the minimum number of crossings over all simple drawings. We prove that in this sense the degenerate crossing number of very "dense" graphs is at least $\Omega(e^3/n^2)$. More precisely, we have

Theorem 5.3.7 ([44]). *There exists a constant $c^* > 0$ such that*

$$\mathrm{cr}^*(G) \geq c^*\frac{e^4(G)}{n^4(G)}$$

holds for any graph G with $e(G) \geq 4n(G)$.

It is a challenging question to decide whether here the term $\frac{e^4(G)}{n^4(G)}$ can be replaced by $\frac{e^3(G)}{n^2(G)}$, just like in Theorem 5.3.6.

5.4 Straight-Line Drawings

For "straight-line thrackles," Conway's conjecture discussed in Section 5.2 had been settled by H. Hopf–E. Pannwitz [22] and (independently) by Paul Erdős much before the problem was raised!

If every edge of a graph is drawn by a straight-line segment, then we call the drawing a *geometric graph* [33], [34], [35]. We assume for simplicity that no three vertices of a geometric graph are collinear and that no segment representing an edge passes through any vertex other than its endpoints.

Theorem 5.4.1 (Hopf–Pannwitz–Erdős theorem). *If any two edges of a geometric graph intersect (in an endpoint or an internal point), then it can have at most as many edges as vertices.*

Proof. (Perles) We say that an edge uv of a geometric graph is a *leftmost* edge at its endpoint u if turning the ray uv around u through 180 degrees in the counterclockwise direction, it never contains any other edge uw. For each vertex u, delete the leftmost edge at u, if such an edge exists. We claim that at the end of the procedure, no edge is left. Indeed, if at least one edge uv remains, it follows that we did not delete it when we visited u and we did not delete it when we visited v. Thus, there exist two edges uw and vz such that the ray uw can be obtained from uv by a counterclockwise rotation about u through less than 180 degrees, and the ray vz can be obtained from vu by a counterclockwise rotation about v through less than 180 degrees. This implies that uw and vz cannot intersect, which contradicts our assumption. □

The systematic study of extremal problems for geometric graphs was initiated by S. Avital–H. Hanani [7], Erdős, Micha Perles, and Yaakov Kupitz [29]. In particular, they asked the following question: what is the maximum number of edges of a geometric graph of n vertices, which does not have k pairwise disjoint edges? (Here, by "disjoint" we mean that they cannot cross and cannot even share an endpoint.) Denote this maximum by $e_k(n)$.

Using this notation, the above theorem says that $e_2(n) = n$, for every $n > 2$. Noga Alon and Erdős [5] proved that $e_3(n) \leq 6n$. T'his bound was first reduced by a factor of two [20], and not long ago Černý [12] showed that $e_3(n) = 2.5n + O(1)$. It had been an open problem for a long time to decide whether $e_k(n)$ is linear in n for every fixed $k > 3$. Pach and Törőcsik [45] were the first to show that this is indeed the case. More precisely, they used Dilworth's theorem for partial orders to prove that $e_k(n) \leq (k-1)^4 n$. This bound was improved successively by G. Tóth–P. Valtr [53], and by Tóth [52].

Theorem 5.4.2 ([52]). *For every k and every n, we have $e_k(n) \leq 2^{10}(k-1)^2 n$.*

It is very likely that the dependence of $e_k(n)$ on k is also (roughly) linear.

Analogously, one can try to determine the maximum number of edges of a geometric graph with n vertices, which does not have k pairwise crossing edges. Denote this maximum by $f_k(n)$. It follows from Euler's Polyhedral Formula that, for $n > 2$, every planar graph with n vertices has at most $3n - 6$ edges. Equivalently, we have $f_2(n) = 3n - 6$.

Theorem 5.4.3 ([3]). $f_3(n) = O(n)$.

Better proofs and generalizations were found in [38], [37], [2]

Recently, Ackerman [1] proved that $f_4(n) = O(n)$. Plugging this bound into the result of [39], we obtain

Theorem 5.4.4. *For a fixed $k > 4$, we have $f_k(n) = O(n \log^{2k-8} n)$.*

Valtr [56] has shown that $f_k(n) = O(n \log n)$, for any $k > 4$, but it can be conjectured that $f_k(n) = O(n)$. Moreover, it cannot be ruled out that there exists a constant c such that $f_k(n) \leq ckn$, for every k and n. However, we cannot even decide whether every complete geometric graph with n

vertices contains at least (a positive) constant times n pairwise crossing edges. We are ashamed to admit that the strongest result in this direction is the following

Theorem 5.4.5 ([6]). *Every complete geometric graph with n vertices contains at least $\lfloor\sqrt{n/12}\rfloor$ pairwise crossing edges.*

In a series of papers [23], [24], [25], we established some *Ramsey-type* results for geometric graphs, closely related to the subject of this section. In [14], we generalized the above results for *geometric hypergraphs* (systems of simplices).

5.5 Angular Resolution and Slopes

It is one of the major goals of graphic visualization to improve the readability of diagrams. If the angle between two adjacent edges is too small, it causes "blob effects" and it is hard to tell those edges apart. The minimum angle between two edges in a straight-line drawing of a graph G is called the *angular resolution* of the drawing. Of course, if the maximum degree of a vertex of G is d, then the angular resolution of any drawing of G is at most $\frac{2\pi}{d}$. It was shown by Formann et al. [18] that every graph G of maximum degree d can be drawn by straight-line edges with angular resolution at least constant times $\frac{1}{d^2}$ and that this bound is best possible up to a logarithmic factor. For straight-line drawings, one can achieve the asymptotically optimal resolution $\Omega\left(\frac{1}{d}\right)$, but then the optimal drawing is not necessarily crossing-free. In the case we insist on crossing-free (planar) straight-line drawings, Malitz and Papakostas [32] proved that there exists a constant $\alpha >)$ such that any planar graph of maximum degree d permits a good drawing of angular resolution at least α^d.

Wade and Chu [57] defined the *slope number* (G) of G as the smallest number of distinct edge slopes used in a straight-line drawing of G. Dujmović et al. [15] asked whether the slope number of a graph of maximum degree d can be arbitrarily large. The following short argument of Pach and Pálvölgyi shows that the answer is yes for $d \geq 5$.

Proof. Define a "frame" graph F on the vertex set $\{1, \ldots, n\}$ by connecting vertex 1 to 2 by an edge and connecting every $i > 2$ to $i-1$ and $i-2$. Adding a perfect matching M between these n points, we obtain a graph $G_M := F \cup M$. The number of different matchings is at least $(n/3)^{n/2}$. Let G denote the huge graph obtained by taking the union of disjoint copies of all G_M. Clearly, the maximum degree of the vertices of G is *five*. Suppose that G can be drawn using at most S slopes, and fix such a drawing.

For every edge $ij \in M$, label the points in G_M corresponding to i and j by the slope of ij in the drawing. Furthermore, label each frame edge ij $(|i-j| \leq 2)$ by its slope. Notice that no two components of G receive the same labelling. Indeed, up to translation and scaling, the labelling of the edges uniquely determines the positions of the points representing the vertices of G_M. Then the labelling of the vertices uniquely determines the edges belonging to M. Therefore, the number of different possible labelling, which is $S^{|F|+n} < S^{3n}$, is an upper bound for the number of components of G. On the other hand, we have seen that the number of components (matchings) is at least $(n/3)^{n/2}$. Thus, for any S we obtain a contradiction, provided that n is sufficiently large. □

A more complicated proof has been found independently by Barát, Matoušek, and Wood [8].

With some extra care one can refine the first argument to obtain

Theorem 5.5.1 ([36]). *For any $d \geq 5$, there exist graphs G with n vertices of maximum degree d, whose slope numbers satisfy $(G) \geq n^{\frac{1}{2}-\frac{1}{d-2}-o(1)}$.*

On the other hand, for *cubic* graphs we have

Theorem 5.5.2 ([27]). *Any graph G of maximum degree three can be drawn with edges of at most five different slopes, and if G has a vertex of degree less than three, then four slopes suffice.*

5.6 An Application in Computer Graphics

It is a pleasure for the mathematician to see his research generate some interest outside his narrow field of studies. During the past twenty years, combinatorial geometers have been fortunate enough to experience this feeling quite often. Automated production lines revolutionized *robotics*, and started an avalanche of questions whose solution required new combinatorial geometric tools [50]. *Computer graphics*, whose group of users encompasses virtually everybody from engineers to film-makers, has had a similar effect on our subject [9].

Most graphics packages available on the market contain some (so-called *warping* or *morphing*) program suitable for deforming figures or pictures. Originally, these programs were written for making commercials and animated movies, but today they are widely used.

An important step in programs of this type is to fix a few basic points of the original picture (say, the vertices of the straight-line drawing of a planar graph), and then to choose new locations for these points. We would like to redraw the graph without creating any crossing. In general, now we cannot insist that the edges be represented by segments, because such a drawing may not exist. Our goal is to produce a drawing with polygonal edges, in which the total number of segments is small. The complexity and the running time of the program is proportional to this number.

Theorem 5.6.1 ([46]). *Every planar graph with n vertices can be re-drawn in such a way that the new positions of the vertices are arbitrarily prescribed, and each edge is represented by a polygonal path consisting of at most $24n$ segments. There is an $O(n^5)$–time algorithm for constructing such a drawing.*

The next result shows that Theorem 5.6.1 cannot be substantially improved.

Theorem 5.6.2 ([46]). *For every n, there exist a planar graph G_n with n vertices and an assignment of new locations for the vertices such that in any polygonal drawing of G_n there are at least $n/100$ edges composed of at least $n/100$ segments.*

The proof of this theorem is based on a generalization of a result of Leighton [30], found independently by Pach et al. [39] and by Sýkora et al. [51]. It turned out to play a crucial role in the solution of several extremal and algorithmic problems related to graph embeddings.

The *bisection width* of a graph is the minimum number of edges whose removal splits the graph into two pieces such that there are no edges running between them and the larger piece has at most twice as many vertices as the smaller.

Theorem 5.6.3 ([51], [39]). *Let G be a graph of n vertices whose degrees are $d_1, d_2, \ldots, d_n$. Then the bisection width of G is at most*

$$1.58\sqrt{16\text{CR}(G) + \sum_{i=1}^{n} d_i^2}.$$

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Chapter 6

Extremal Combinatorics: Repeated Patterns and Pattern Recognition

6.1 Models and Problems

Today massive amounts of pictures and other geometric data is collected by digital cameras, laser scanners, electron microscopes, telescopes, etc. A digital survey underway maps in detail one-quarter of the entire sky, determining the positions of more than 100 million celestial objects. To find regularities in this huge digital archive, one needs new mathematical and computational methods. This challenge has not found the mathematics community completely unprepared. Several extremal problems raised by Erdős, Hadwiger, and others, that half-a-century ago may have appeared to be curiosities in recreational mathematics, have turned out to be instrumental in this area of research. They have motivated the discovery of important combinatorial tools, including the probabilistic method, Szemerédi's Regularity Lemma, Ramsey Theory.

We discuss some extremal problems on repeated geometric patterns in finite point sets in Euclidean space. Throughout this chapter, a *geometric pattern* is an equivalence class of point sets in d-dimensional space under some fixed geometrically defined equivalence relation. Given such an equivalence relation and the corresponding concept of patterns, one can ask several natural questions:

(1) *What is the maximum number of occurrences of a given pattern among all subsets of an n-point set?*

(2) *How does the answer to the previous question depend on the particular pattern?*

(3) *What is the minimum number of distinct k-element patterns determined by a set of n points?*

These questions make sense for many specific choices of the underlying set and the equivalence relation. Hence it is not surprising that several basic problems of combinatorial geometry can be studied in this framework [33].

In the simplest and historically first examples, due to Erdős [17], the underlying set consists of point pairs in the plane and the defining equivalence relation is the isometry (congruence). That is, two point pairs, $\{p_1, p_2\}$ and $\{q_1, q_2\}$, determine the same pattern if and only if $|p_1 - p_2| = |q_1 - q_2|$. In this case,(1) becomes the well known *Unit Distance Problem*: What is the maximum number of unit distance pairs determined by n points in the plane? It follows by scaling that the answer does not depend on the particular distance (pattern). For most other equivalence relations, this is not the case: different patterns may have different maximal multiplicities. For $k = 2$, question (3) becomes the *Problem of Distinct Distances*: What is the minimum number of distinct distances that must occur among n points in the plane? In spite of many efforts, we have no satisfactory answers to these questions. The best known results are the following.

Theorem 6.1.1 ([40]). *Let $f(n)$ denote the maximum number of times the same distance can be repeated between n points in the plane. We have*

$$ne^{\Omega\left(\frac{\log n}{\log\log n}\right)} \leq f(n) \leq O\left(n^{\frac{4}{3}}\right).$$

Theorem 6.1.2 ([28]). *Let $g(n)$ denote the minimum number of distinct distances determined by n points in the plane. We have*

$$\Omega\left(n^{0.8641}\right) \leq g(n) \leq O\left(\frac{n}{\sqrt{\log n}}\right).$$

In Theorems 6.1.1 and 6.1.2, the lower and upper bounds, respectively, are conjectured to be asymptotically sharp. See more about these questions in Section 6.3. See also Chapter 4 for more details.

Erdős and Purdy [22], [23] initiated the investigation of the analogous problems with the difference that, instead of pairs, we consider *triples* of points, and call two of them *equivalent* if the corresponding triangles have the same angle, or area, or perimeter. This leads to questions about the maximum number of equal angles, or unit-area resp. unit-perimeter triangles, that can occur among n points in the plane, and to questions about the minimum number of distinct angles, triangle areas, and triangle perimeters, respectively. Erdős' Unit Distance Problem and his Problem of Distinct Distances has motivated a lot of research in extremal graph theory. The above mentioned questions of Erdős and Purdy, and, in general, problems (1), (2), and (3) for larger than 2-element patterns, require the extension of graph theoretic methods to hypergraphs. This appears to be one of the most important trends in modern combinatorics.

It is most natural to define two sets to be *equivalent* if they are congruent or similar to, or translates, homothets or affine images of each other. This justifies the choice of the word 'pattern' for the arising equivalence classes. Indeed, the algorithmic aspects of these problems have also been studied in the context of geometric pattern matching [4, 3, 10]. A typical algorithmic question is the following.

(4) *Design an efficient algorithm for finding all occurrences of a given pattern in a set of n points?*

It is interesting to compare the equivalence classes that correspond to the same relation applied to sets of different sizes. If A and A' are equivalent under congruence (or under some other group of

transformations mentioned above), and a is a point in A, then there exists a point $a' \in A'$ such that $A \setminus \{a\}$ is equivalent to $A' \setminus \{a'\}$. On the other hand, if A is equivalent (congruent) to A' and A is large enough, then usually its possible extensions are also determined: for each a, there exist only a small number of distinct elements a' such that $A \cup \{a\}$ is equivalent to $A' \cup \{a'\}$. Therefore, in order to bound the number of occurrences of a large pattern, it is usually sufficient to study small pattern fragments.

We have indicated above that one can rephrase many extremal problems in combinatorial geometry as questions of type (1) (so-called *Turán-type* questions). Similarly, many *Ramsey-type* geometric coloring problems can also be formulated in this general setting.

(5) *Is it possible to color space with k colors such that there is no monochromatic occurrence of a given pattern?*

For point pairs in the plane under congruence, we obtain the famous Hadwiger-Nelson problem [Ha61]: What is the smallest number of colors $\chi(\mathbb{R}^2)$ needed to color all points of the plane so that no two points at unit distance from each other get the same color?

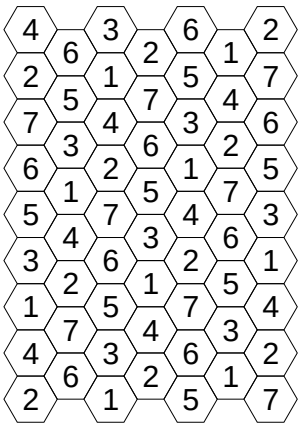


FIGURE 6.1. Seven coloring of the plane showing that $\chi(\mathbb{R}^2) \leq 7$..

Theorem 6.1.3. $4 \leq \chi(\mathbb{R}^2) \leq 7$.

Another instance of question (5) is the following open problem from [19]: Is it possible to color all points of the three-dimensional Euclidean space with three colors so that no color class contains two vertices at distance one and the midpoint of the segment determined by them? It is known that four colors suffice, but there is no such coloring with two colors. In fact, Erdős et al. [19] proved that, for every d, the Euclidean d-space can be colored with four colors without creating a monochromatic triple of this kind.

6.2 A Simple Sample Problem: Equivalence under Translation

We illustrate our framework by analyzing the situation when two point sets are considered equivalent if and only if they are translates of each other. In this case, we know the (almost) complete solution to problems (1)–(5) in Section 6.1.

Theorem 6.2.1. *Any set B of n points in d-dimensional space has at most $n+1-k$ subsets that are translates of a fixed set A of k points. This bound is attained if and only if $A = \{p, p+v, \ldots, p+(k-1)v\}$ and $B = \{q, q+v, \ldots, q+(n-1)v\}$ for some $p, q, v \in \mathbb{R}^d$.*

The proof is simple. Notice first that any linear mapping φ that keeps all points of B distinct also preserves the number of translates: if $A + t \subset B$, then $\varphi(A) + \varphi(t) \subset \varphi(B)$. Thus, we can use any projection into $\mathbb{R}$, and the question reduces to a one-dimensional problem: Given real numbers $a_1 < \cdots < a_k$, $b_1 < \cdots, b_n$, what is the maximum number of values t such that $t + \{a_1, \ldots, a_k\} \subset \{b_1, \ldots b_n\}$. Clearly, $a_1 + t$ must be one of $b_1, \ldots, b_{n-k+1}$, so there are at most $n+1-k$ translates. If there are $n+1-k$ translates $t + \{a_1, \ldots, a_k\}$ that occur in $\{b_1, \ldots b_n\}$, for translation vectors $t_1 < \cdots < t_{n-k+1}$, then $t_i = b_i - a_1 = b_{i+1} - a_2 = b_{i+j} - a_{1+j}$, for $i = 1, \ldots, n-k+1$ and $j = 0, \ldots, k-1$. But then $a_2 - a_1 = b_{i+1} - b_i = a_{j+1} - a_j = b_{i+j} - b_{i+j-1}$, so all differences between consecutive a_j and b_i are the same. For higher dimensional sets, this holds for every one-dimensional projection, which guarantees the claimed structure. In other words, the maximum is attained only for sets of a very special type, which answers question (1). An asymptotically tight answer to (2), describing the dependence on the particular pattern, was obtained in [10].

Theorem 6.2.2. *Let A be a set of points in d-dimensional space, such that the rational affine space spanned by A has dimension k. Then the maximum number of translates of A that can occur among n points in d-dimensional space is $n - \Theta(n^{\frac{k-1}{k}})$.*

Any set of the form $\{p, p+v, \ldots, p+(k-1)v\}$ spans a 1-dimensional rational affine space. An example of a set spanning a two-dimensional rational affine space is $\{0, 1, \sqrt{2}\}$, so for this set there are at most $n - \Theta(n^{1/2})$ possible translates. This bound is attained, e.g., for the set $\{i + j\sqrt{2} \mid 1 \leq i, j \leq \sqrt{n}\}$.

In this case, it is also easy to answer question (3), i.e., to determine the minimum number of distinct patterns (translation-inequivalent subsets) determined by an n-element set.

Theorem 6.2.3. *Any set of n points in d-dimensional space has at least $\binom{n-1}{k-1}$ distinct k-element subsets, no two of which are translates of each other. This bound is attained only for sets of the form $\{p, p+v, \ldots, p+(n-1)v\}$ for some $p, v \in \mathbb{R}^d$.*

By projection, it is again sufficient to prove the result on the line. Let $f(n,k)$ denote the minimum number of translation inequivalent k-element subsets of a set of n real numbers. Considering the set $\{1, \ldots, n\}$, we obtain that $f(n,k) \leq \binom{n-1}{k-1}$, since every equivalence class has a unique member that contains 1. To establish the lower bound, observe that, for any set of n real numbers, there are $\binom{n-2}{k-2}$ distinct subsets that contain both the smallest and the largest numbers, and none of them is translation equivalent to any other. On the other hand, there are at least $f(n-1,k)$ translation inequivalent subsets that do not contain the last element. So we have $f(n,k) \geq f(n-1,k) + \binom{n-2}{k-2}$, which, together with $f(n,1) = 1$, proves the claimed formula. To verify the structure of the extremal

set, observe that, in the one-dimensional case, an extremal set minus its first element, as well as the same set minus its last element, must again be extremal sets, and for $n = k+1$ it follows from Theorem 6.1.1 that all extremal sets must form arithmetic progressions. Thus, the whole set must be an arithmetic progression, which holds, in higher dimensional cases, for each one-dimensional projection. The corresponding algorithmic problem (4) has a natural solution: Given two sets, $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_n\}$, we can fix any element of A, say, a_1, and try all possible image points b_i. Each of them specifies a unique translation $t = b_i - a_1$, so we simply have to test for each set $A + (b_i - a_1)$ whether it is a subset of B. This takes $\Theta(kn \log n)$ time. The running time of this algorithm is not known to be optimal.

Problem 6.2.4. *Does there exist a $o(kn)$ time algorithm for finding all real numbers t such that $t + A \subset B$, for every pair of input sets, A and B, consisting of k and n reals, respectively?*

The Ramsey-type problem (5) is trivial for translates: Given any set A of at least two points $a_1, a_2 \in A$, we can two-color $\mathbb{R}^d$ without generating any monochromatic translate of A. Indeed, the space can be partitioned into arithmetic progressions with difference $a_2 - a_1$, and each of them can be colored separately with alternating colors.

6.3 Equivalence under Congruence in the Plane

Problems (1)–(5) are much more interesting and difficult under congruence. In the plane, considering two-element subsets, the congruence class of a pair of points is determined by their distance. Questions (1) and (3) become the Erdős' famous problems, mentioned in Section 6.1.

Problem 6.3.1. *What is the maximum number of times the same distance can occur among n points in the plane?*

Problem 6.3.2. *What is the minimum number of distinct distances determined by n points in the plane?*

The best known results concerning these questions are summarized in Theorems 6.1.1 and 6.1.2, respectively. There are several different proofs known for the currently best upper bound in Theorem 6.1.1 (see [40], [15], [33], and [42]), which, obviously, does not depend on the particular distance (congruence). This answers question (2). As for the lower bound of Katz and Tardos [28] in Theorem 6.1.2, it represents the latest improvement over a series of previous results [38], [42], [14], [13], [6], [32]. Again, see Chapter 4 for more details.

The algorithmic problem (4) can now be stated as follows.

Problem 6.3.3. *How fast can we find all unit distance pairs among n points in the plane?*

Some of the methods developed to establish the $O(n^{\frac{4}{3}})$ bound for the number of unit distances can also be used to design an algorithm for finding all unit distance pairs in time $O(n^{\frac{4}{3}} \log n)$ (similarly to the algorithms for detecting point-line incidences [31]).

The corresponding Ramsey-type problem (5) for patterns of size two is the famous Hadwiger-Nelson problem, see Theorem 6.1.3 above.

Problem 6.3.4. *What is the minimum number of colors necessary to color all points of the plane so that no pair of points at unit distance receive the same color?*

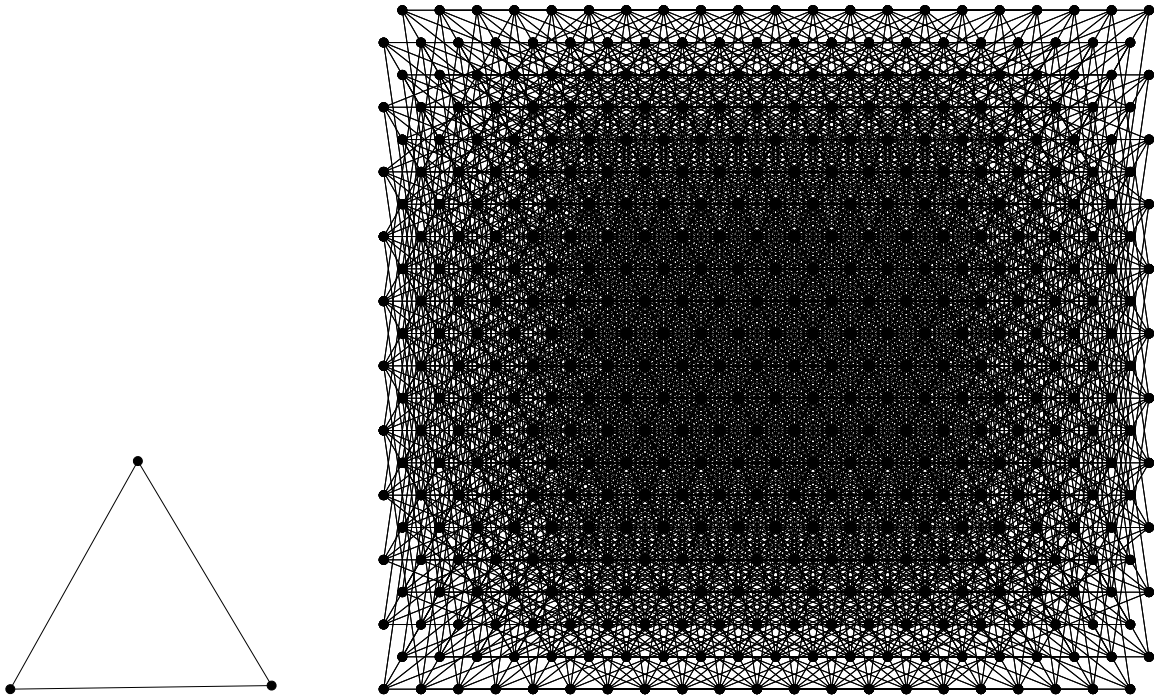


FIGURE 6.2. A unit equilateral triangle and a lattice section containing many congruent copies of the triangle directions.

If we ask the same questions for patterns of size k rather than point pairs, but still in the plane, the answer to (1) does not change: given a pattern $A = \{a_1, \ldots, a_k\}$, any congruent image of A is already determined, up to reflection, by the images of a_1 and a_2. Thus, the maximum number of congruent copies of a set is at most twice the maximum number of (ordered) unit distance pairs. Depending on the given set, this maximum number may be smaller, but no results of this kind are known. As n tends to infinity, the square and triangular lattice constructions that realize $ne^{\frac{c \log n}{\log \log n}}$ unit distances among n points also contain asymptotically the same number of congruent copies of *any* fixed set that is a subset of a square or triangular lattice. However, it is likely that this asymptotics cannot be attained for most other patterns.

Problem 6.3.5. *Does there exist, for every finite set A, a positive constant $c(A)$ with the following property? For every n, there is a set of n points in the plane containing at least $ne^{\frac{c(A) \log n}{\log \log n}}$ congruent copies of A.*

Problem (3) on the minimum number of distinct congruence classes of k-element subsets of a point set, is strongly related to the Problem of Distinct Distances, just like the maximum number

of pairwise congruent subsets was related to the Unit Distance Problem. For if we consider ordered k-tuples instead of k-subsets (counting each subset $k!$ times), then two such k-tuples are certainly incongruent if their first two points determine distinct distances. For each distance s, fix a point pair that determines s. Clearly, any two different ways one can extend it by filling the remaining $k-2$ positions results in incongruent k-tuples. This leads to a lower bound of $\Omega(n^{k-2+0.8641})$ for the minimum number of distinct congruence classes of k-element subsets. Since a regular n-gon has $O(n^{k-1})$ pairwise incongruent k-element sets, this problem becomes less interesting for large k.

The algorithmic question (4) can also be reduced to the corresponding problem on unit distances. Given the sets A and B, we first fix $a_1, a_2 \in A$ and use our algorithm developed for detecting unit distance pairs to find all possible image pairs $b_1, b_2 \in B$ whose distance is the same as that of a_1 and a_2. Then we check for each of these pairs whether the rigid motion that takes a_i to b_i $(i = 1, 2)$ maps the whole set A into a subset of B. This takes $O^*(n^{4/3}k)$ time, and we cannot expect any substantial improvement in the dependence on n, unless we apply a faster algorithm for finding unit distance pairs. (In what follows, we write O^* to indicate suppression of lower order factors, i.e., $O^*(n^\alpha) = o(n^{\alpha+\varepsilon})$ for every $\varepsilon > 0$).

Many problems of Euclidean Ramsey theory can be interpreted as special cases of question (5) in our model. We particularly like the following problem raised in [20].

Problem 6.3.6. *Is it true that, for any triple $A = \{a_1, a_2, a_3\} \subset \mathbb{R}^2$ that does not span an equilateral triangle, and for any coloring of the plane with two colors, one can always find a monochromatic congruent copy of A?*

It was conjectured in [20] that the answer to this question is in the affirmative. It is easy to see that the statement is not true for equilateral triangles A. Indeed, decompose the plane into half-open parallel strips whose width is equal to the height of A, and color them red and blue, alternately. On the other hand, the seven-coloring of the plane, with no two points at unit distance whose colors are the same, shows that any given pattern can be avoided with seven colors. Nothing is known about coloring with three colors.

Problem 6.3.7. *Does there exist a triple $A = \{a_1, a_2, a_3\} \subset \mathbb{R}^2$ such that any three-coloring of the plane contains a monochromatic congruent copy of A?*

6.4 Equivalence under Congruence in Higher Dimensions

All questions discussed in the previous section can also be asked in higher dimensions. There are two notable differences. In the plane, the image of a fixed pair of points was sufficient to specify a congruence. Therefore, the number of congruent copies of any larger set was bounded from above by the number of congruent pairs. In d-space, however, one has to specify d image points to determine a congruence, up to reflection. Hence, estimating the maximum number of congruent copies of a k-point set is a different problem for each $k = 2, \ldots, d$.

The second difference from the planar case is that starting from 4 dimensions, there exists another type of construction, discovered by Lenz, that provides asymptotically best answers to some of the above questions. For $k = \lfloor \frac{d}{2} \rfloor$, choose k concentric circles of radius $\frac{1}{\sqrt{2}}$ in pairwise orthogonal planes in $\mathbb{R}^d$ and distribute n points on them as equally as possible. Then any two points from distinct circles are at distance one, so the number of unit distance pairs is $(\frac{1}{2} - \frac{1}{2k} + o(1))n^2$, which is

a positive fraction of all point pairs. It is known [Er60] that the here the constant of proportionality cannot be improved. Similarly, in this construction, any choice of three points from distinct circles span a unit equilateral triangle, so if $d \geq 6$, a positive fraction of all triples can be congruent. In general, for each $k \leq \lfloor \frac{1}{2}d \rfloor$, Lenz' construction shows that a positive fraction of all k-element subsets can be congruent. Obviously, this gives the correct order of magnitude for question (1). With some extra work, even the exact maxima can be determined, as shown for $k = 2, d = 4$ in [8], [45].

Even for $k > \frac{d}{2}$, we do not know any construction better than Lenz', but for these parameters the problem is not trivial: now one is forced to select several points from the same circle, and only one of them can be selected freely. So, for $d = 3$, in the interesting versions of (1), we have $k = 2$ or 3 (now there is no Lenz construction). For $d \geq 4$, the cases $\lfloor \frac{d}{2} \rfloor < k \leq d$ are nontrivial.

Problem 6.4.1. *What is the maximum number of unit distances among n points in three-dimensional space?*

Here, the currently best bounds are $\Omega(n^{\frac{4}{3}} \log\log n)$ [18] and $O^*(n^{\frac{3}{2}})$ [15].

Problem 6.4.2. *What is the maximum number of pairwise congruent triangles spanned by a set of n points in three-dimensional space?*

Here the lower bound is $\Omega(n^{\frac{4}{3}})$ [21], [2] and the upper bound is $O^*(n^{\frac{5}{3}})$ [3], improving previous bounds in [4], [9]. For higher dimensions, Lenz' construction or, in the odd-dimensional cases, a combination of Lenz' construction with the best known three-dimensional point set [21], [2], is most likely to be optimal. The only results in this direction, given in [3], are for $d \leq 7$ and do not quite attain this bound.

Problem 6.4.3. *Is it true that, for any $\lfloor \frac{d}{2} \rfloor \leq k \leq d$, the maximum number of congruent k-dimensional simplices among n points in d-dimensional space is $O(n^{\frac{d}{2}})$ if d is even, and $O(n^{\frac{d}{2}-\frac{1}{6}})$ if d is odd?*

Very little is known about problem (2) in this setting. For point pairs, scaling again shows that all 2-element patterns can occur the same number of times. For 3-element patterns (triangles), the above mentioned $\Omega(n^{\frac{4}{3}})$ lower bound in [21] was originally established only for right-angle isosceles triangles. It was later extended in [2] to any fixed triangle. However, the problem is already open for full-dimensional simplices in 3-space. An especially interesting special case is the following.

Problem 6.4.4. *What is the maximum number of orthonormal bases that can be selected from n distinct unit vectors?*

The upper bound $O(n^{\frac{4}{3}})$ is simple, but the construction of [21] that gives $O(n^{\frac{4}{3}})$ orthogonal pairs does not extend to orthogonal triples.

Question (3) on the minimum number of distinct patterns is largely open. For 2-element patterns, we obtain higher dimensional versions of the Problem of Distinct Distances. Here the upper bound $O(n^{\frac{2}{d}})$ is realized, e.g., by a cubic section of the d-dimensional integer lattice. The general lower bound of $\Omega(n^{\frac{1}{d}})$ was observed already in [17]. For $d = 3$, this was subsequently improved to $\Omega^*(n^{\frac{77}{141}})$ [5] and to $\Omega(n^{0.564})$ [39]. For large values of d, Solymosi and Vu [39] got very close to

finding the best exponent by establishing the lower bound $\Omega\left(n^{\frac{2}{d}-\frac{2}{d(d+2)}}\right)$. This extends, in the same way as in the planar case, to a bound of $\Omega\left(n^{k-2+\frac{2}{d}-\frac{2}{d(d+2)}}\right)$ for the minimum number of distinct k-point patterns of an n-element set, but even for triangles, nothing better is known. Lenz-type constructions are not useful in this context, because they span $\Omega(n^{k-1})$ distinct k-point patterns, as do regular n-gons.

As for the algorithmic problem (4), it is easy to find all congruent copies of a given k-point pattern A in an n-point set. For any $k \geq d$, this can be achieved in $O(n^d k \log n)$ time: fix a d-tuple C in A, and test all d-tuples of the n-point set B, whether they could be an image of C. If yes, test whether the congruence specified by them maps all the remaining $k-d$ points to elements of B. It is very likely that there are much faster algorithms, but, for general d, the only published improvement is by a factor of $\log n$ [37].

The Ramsey-type question (5) includes a number of problems of Euclidean Ramsey theory, as special cases.

Problem 6.4.5. *Is it true that, for every two-coloring of the three-dimensional space, there are four vertices of the same color that span a unit square?*

It is easy to see that if we divide the plane into half-open strips of width one and color them alternately by two colors, then no four vertices that span a unit square will receive the same color. On the other hand, it is known that any two-coloring of four-dimensional space will contain a monochromatic unit square [20]. Actually, the (vertex set of a) square is one of the simplest examples of a *Ramsey-set*, i.e., a set B with the property that, for every positive integer c, there is a constant $d = d(c)$ such that under any c-coloring of the points of $\mathbb{R}^d$ there exists a monochromatic congruent copy of B. All boxes and all triangles [24] are known to be Ramsey. It is a longstanding open problem to decide whether all finite subsets of finite dimensional spheres are Ramsey. If the answer is in the affirmative, this would provide a perfect characterization of Ramsey sets, for all Ramsey sets are known to be subsets of a sphere [19].

The simplest non-spherical example, consisting of an equidistant sequence of three points along the same line, was mentioned at the end of the Section 6.1.

6.5 Equivalence under Similarity

If we consider problems (1)–(5) with similarity (congruence and scaling) as the equivalence relation, again we find that many of the arising questions have been extensively studied. Since any two point pairs are similar to each other, we can restrict our attention to patterns of size at least three. The first interesting instance of problem (1) is to determine or to estimate the maximum number of pairwise similar triangles spanned by n points in the plane. This problem was almost completely solved in [ElE94]. For any given triangle, the maximum number of similar triples in a set n point in the plane is $\Theta(n^2)$. If the triangle is equilateral, we even have fairly good bounds on the multiplicative constants hidden in the Θ-notation [AbF00]. In this case, most likely, suitable sections of the triangular lattice are close to being extremal. In general, the following construction from [ElE94] always gives a quadratic number of similar copies of a given triangle $\{a, b, c\}$. Interpreting a, b, c as complex numbers $0, 1, z$, consider the points $\frac{i_1}{n}z$, $\frac{i_2}{n} + (1 - \frac{i_2}{n})z$, and $\frac{i_3}{n}z + (1 - \frac{i_2}{n})z^2$. Then

any triangle $(\beta - \alpha)z$, $\alpha + (1 - \alpha)z$, $\beta z + (1 - \beta)z^2$ is similar to $0, 1, z$, which can be checked by computing the quotient of the sides.

The answer to question (1) for k-point patterns, $k > 3$, is more or less the same as for $k = 3$. Certain patterns, including all k-element subsets of a regular triangular lattice, permit $\Theta(n^2)$ similar copies, and in this case a suitable section of the triangular lattice is probably close to being extremal. For some other patterns, the order $\Theta(n^2)$ cannot be attained. All patterns of the former type were completely characterized in [LaR97]: for any pattern A of $k \geq 4$ points, one can find n points containing $\Theta(n^2)$ similar copies of A if and only if the cross ratio of every quadruple of points in A, interpreted as complex numbers, is algebraic. Otherwise, the maximum is slightly subquadratic. This result also answers question (2).

In higher dimensions, the situation is entirely different: we do not have good bounds for question (1) in any nontrivial case. The first open question is to determine the maximum number of triples in a set of n points in 3-space, which induce pairwise similar triangles. The trivial upper bound, $O(n^3)$, was reduced to $O(n^{2.2})$ in [4]. On the other hand, we do not have any better lower bound than $\Omega(n^2)$, which is already valid in the plane. These estimates extend to similar copies of k-point patterns, $k > 3$, provided that they are planar.

Problem 6.5.1. *What is the maximum number of pairwise similar triangles induced by n points in three-dimensional space?*

For full-dimensional patterns, no useful constructions are known. The only lower bound we are aware of follows from the lattice L which, in 3 dimensions, spans $\Omega(n^{\frac{4}{3}})$ similar copies of the full-dimensional simplex formed by its basis vectors or, in fact, of any k-element subset of lattice points. However, to attain this bound, we do not need to allow rotations: L spans $\Omega(n^{\frac{4}{3}})$ *homothetic* copies.

Problem 6.5.2. *In three-dimensional space, what is the maximum number of quadruples in an n-point set that span pairwise similar tetrahedra?*

For higher dimensions and for larger pattern sizes, the best known lower bound follows from Lenz' construction for congruent copies, which again does not use the additional freedom of scaling. Since, for $d \geq 3$, we do not know the answer to question (1) on the maximum number occurrences, there is no hope that we would be able to answer question (2) on the dependence of this maximum number on the pattern.

Problem (3) on the minimum number of pairwise inequivalent patterns under similarity is an interesting problem even in the plane.

Problem 6.5.3. *What is the minimum number of similarity classes of triangles spanned by a set of n points in the plane?*

There is a trivial lower bound of $\Omega(n)$: if we choose two arbitrary points, and consider all of their $n - 2$ possible extensions to a triangle, then among these triangles each (oriented) similarity class will be represented only at most three times. Alternatively, we obtain asymptotically the same lower bound $\Omega(n)$ by just using the pigeonhole principle and the fact that the maximum size of a similarity class of triangles is $O(n^2)$. On the other hand, as shown by the example of a regular n-gon,

the number of similarity classes of triangles can be $O(n^2)$. This leaves a huge gap between the lower and upper bounds.

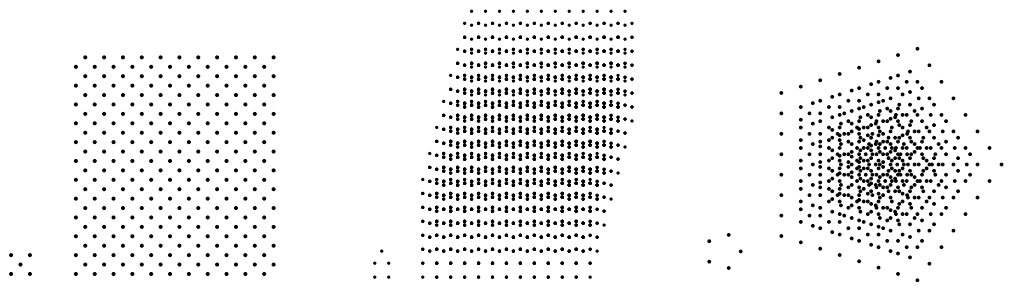
For higher dimensions and for larger sets, our knowledge is even more limited. In 3-dimensional space, for instance, we do not even have an $\Omega(n)$ lower bound for the number of similarity classes of triangles, while the best known upper bound, $O(n^2)$, remains the same. For four-element patterns, we have a linear lower bound (fix any triangle, and consider its extensions), but we have no upper bound better than $O(n^3)$ (consider again a regular n-gon). Here we have to be careful in the statement of the problem: are we counting similarity classes of *full-dimensional* simplices only, or all similarity classes of possibly degenerate four-tuples? A regular $n-1$-gon with an additional point on its axis has only $\Theta(n^2)$ similarity classes of full-dimensional simplices, but $\Theta(n^3)$ similarity classes of four-tuples. In dimensions larger than three, nothing nontrivial is known.

In the plane, the algorithmic question (4) of finding all similar copies of a fixed k-point pattern is not hard: it can be achieved in time $O(n^2 k \log n)$, which is tight up to the $\log n$-factor, because the output complexity is this large in the worst case. For dimensions three and higher, we have no nontrivial algorithmic results. Obviously, the problem can always be solved in $O(n^d k \log n)$ time, by testing all possible d-tuples of the underlying set, but this is probably far from optimal.

The Ramsey-type question (5) has a negative answer, for any finite number of colors, even for homothetic copies. Indeed, for any finite set A and for any coloring of space with a finite number of colors, one can always find a monochromatic set similar (even homothetic) to A. This follows from the Hales-Jewett theorem [26], which implies that every coloring of the integer lattice $\mathbf{Z}^d$ with a finite number of colors contains a monochromatic homothetic copy of the lattice cube $\{1, \ldots, m\}^d$ (Gallai-Witt theorem [36], [46]).

6.6 Equivalence under Homothety or Affine Transformations

For homothety-equivalence, questions (1) and (2) have been completely answered in all dimensions [29], [16], [10]. The maximum number of homothetic copies of a set that can occur among n points is $\Theta(n^2)$; the upper bound $O(n^2)$ is always trivial, since the image of a set under a homothety is specified by the images of two points; and a lower bound of $\Omega(n^2)$ is attained by the homothetic copies of $\{1, \ldots, k\}$ in $\{1, \ldots, n\}$. The maximum order is attained only for this 1-dimensional example. If the dimension of the rational affine space induced by a given pattern A is k, then the maximum number of homothetic copies of A that can occur among n points is $\Theta(n^{1+\frac{1}{k}})$, which answers question (2).



Three five-point patterns of different rational dimensions and three sets containing many of their translates

Question (3) on the minimum number of distinct homothety classes of k-point subsets among n points, seems to be still open. As in the case of translations, by projection, we can restrict our attention to the one-dimensional case, where a sequence of equidistant points $\{0, \ldots, n-1\}$ should be extremal. This gives $\Theta(n^{k-1})$ distinct homothety classes. To see this, notice that as the size of the sequence increases from $n-1$ to n, the number of additional homothety classes, which were not already present in $\{0, \ldots n-2\}$, is $\Theta(n^{k-2})$. (The increment certainly includes the classes of all k-tuples that containing 0 and $n-1$, and a third number coprime to $n-1$.) Unfortunately, the pigeonhole principle gives only an $\Omega(n^{k-2})$ lower bound for the number of pairwise dissimilar k-point patterns spanned by a set of n numbers.

Problem 6.6.1. *What is the minimum number of distinct homothety classes among all k-element subsets of a set of n numbers?*

The algorithmic problem (4) was settled in [29], [10]. All homothetic copies of a given full-dimensional k-point pattern in an n-element set in d-space can be found in $O(n^{1+\frac{1}{d}} k \log n)$ time. This is asymptotically tight up to the $\log n$-factor. As mentioned in the previous section, the answer to the corresponding Ramsey-type question (5), is in the negative: one cannot avoid monochromatic homothetic copies of any finite pattern with any finite number of colors.

The situation is very similar for affine images. The maximum number of affine copies of a set among n points in d-dimensional space is $\Theta(n^{d+1})$. The upper bound is trivial, since an affine image is specified by the images of $d+1$ points. On the other hand, the d-dimensional "lattice cube," $\{1, \ldots, n^{\frac{1}{d}}\}^d$, contains $\Omega(n^{d+1})$ affine images of $\{0,1\}^d$ or of any other small lattice-cube of fixed size.

However, the answer to question (2) is not so clear.

Problem 6.6.2. *Do there exist, for every full-dimensional pattern A in d-space, n-element sets containing $\Omega(n^{d+1})$ affine copies of A?*

Problem 6.6.3. *What is the minimum number of affine equivalence classes among all k-element subsets of a set of n points in d-dimensional space?*

For the algorithmic problem (4), the brute force method of trying all possible $(d+1)$-tuples of image points is already optimal. The Ramsey-type question (5) has again a negative answer, since every homothetic copy is also an affine copy.

6.7 Other Equivalence Relations for Triangles in the Plane

For triples in the plane, several other equivalence relations have been studied. An especially interesting example is the following. Two ordered triples are considered equivalent if they determine the same angle. It was proved in [34] that the maximum number of triples in a set of n points in the plane that determine the same angle α is $\Theta(n^2 \log n)$. This order of magnitude is attained for a dense set of angles α. For every other angle α, distribute as evenly as possible $n-1$ points on two rays that emanate from the origin and enclose angle α, and place the last point at the origin. Clearly, the number of triples determining angle α is $\Omega(n^2)$, which "almost" answers question (2). As for the minimum number of distinct angles determined by n points in the plane, Erdős conjectured that the answer to the following question is in the affirmative.

Problem 6.7.1. *Is it true that every set of n points in the plane, not all on a line, determine at least $n-2$ distinct angles?*

This number is attained by a regular n-gon and by several other configurations.

The corresponding algorithmic question (4) is easy: list, for each point p of the set, all lines ℓ through p, together with the points on ℓ. Then we can find all occurrences of a given angle in time $O(n^2 \log n + a)$, where a is the number of occurrences of that angle. Thus, by the above bound from [34], the problem can be solved in $O(n^2 \log n)$ time, which is optimal. The negative answer to the Ramsey-type question (5) again follows from the analogous result for homothetic copies: no coloring with a finite number of colors can avoid a given angle. Another natural equivalence relation classifies triangles according to their areas.

Problem 6.7.2. *What is the maximum number of unit-area triangles that can be determined by n points in the plane?*

An upper bound of $O(n^{\frac{7}{3}})$ was established in [34], while it was pointed out in [22] that a section of the integer lattice gives the lower bound $\Omega(n^2 \log\log n)$. By scaling, we see that all areas allow the same multiplicities, which answers (2). However, problem (3) is open in this case.

Problem 6.7.3. *Is it true that every set of n points in the plane, not all on a line, spans at least $\left\lfloor \frac{n-1}{2} \right\rfloor$ triangles of pairwise different areas?*

This bound is attained by placing on two parallel lines two equidistant point sets whose sizes differ by at most one. This construction is conjectured to be extremal [23], [41]. The best known lower bound, $0.4142n - O(1)$, follows from [11], using [44].

The corresponding algorithmic problem (4) is to find all unit-area triangles. Again, this can be done in $O((n^2 \log n + a)$ time, where a denotes the number of unit area triangles. First, dualize the points to lines, and construct their arrangement, together with a point location structure. Next, for each pair (p, q) of original points, consider the two parallel lines that contain all points r such that pqr is a triangle of unit area. These lines correspond to points in the dual arrangement, for which we can perform a point location query to determine all dual lines containing them. They correspond to points in the original set that together with p and q span a triangle of area one. Each such query takes $\log n$ time.

Concerning the Ramsey-type problem (4), it is easy to see that, for any 2-coloring of the plane, there is a monochromatic triple that spans a triangle of unit area. The same statement may hold for any coloring with a finite number of colors.

Problem 6.7.4. *Is it true that, for any coloring of the plane with a finite number of colors, there is a monochromatic triple that spans a triangle of unit area?*

The *perimeter* of triangles was also discussed in the same paper [34], and later in [35], where an upper bound of $O(n^{\frac{16}{7}})$ was established, but there is no nontrivial lower bound. The lattice section has $\Omega(ne^{\frac{c \log n}{\log\log n}})$ pairwise *congruent* triangles, which, of course, also have the same perimeter, but this bound is probably far from being sharp.

Problem 6.7.5. *What is the maximum number of unit perimeter triangles spanned by n points in the plane?*

By scaling, all perimeters are equivalent, answering (2). By the pigeonhole principle, we obtain an $\Omega(n^{\frac{5}{7}})$ lower bound for the number of distinct perimeters, but again this is probably far from the truth.

Problem 6.7.6. *What is the minimum number of distinct perimeters assumed by all* $\binom{n}{3}$ *triangles spanned by a set of n points in the plane?*

Here neither the algorithmic problem 4 nor the Ramsey-type problem (5) has an obvious solution. For the latter question, it is clear that with a sufficiently large number of colors, one can avoid unit perimeter triangles: color the plane "cellwise," where each cell is too small to contain a unit perimeter triangle, and two cells of the same color are far apart. The problem of determining the minimum number of colors required seems to be similar to the question addressed by Theorem 6.1.3.

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Chapter 7

Lines in Space: From Ray Shooting to Geometric Transversals

7.1 Introduction

In this chapter we address certain combinatorial and algorithmic problems about lines in three-dimensional Euclidean space. Algorithmic questions about lines in 3-space arise in numerous applications, including the hidden surface removal and ray tracing problems in computer graphics, motion planning, placement and assembly problems in robotics, object recognition using three-dimensional range data in computer vision, interaction of solids and of surfaces in solid modeling and CAD, and terrain analysis and reconstruction in geography. In addition, lines in 3-space present many challenging problems in their own right, whose solution requires the use of many sophisticated tools from the theory of higher-dimensional arrangements that has been reviewed in the preceding chapters.

Not surprisingly, straight lines, one of the simplest types of geometric objects, already present many conceptual difficulties when studied in a spatial context. In fact, as we will see below, lines in space are modeled best by non-linear objects. For a classical treatment of the geometry of lines in 3-space see the book by Sommerville [104], or various kinematics texts [22, 64].

We begin this chapter by studying several combinatorial problems involving *arrangements of lines in space*. By the arrangement of a given set of lines we mean the partitioning of the space of all lines according to their interaction with the given lines; this is a special case of a more general situation—see below. We first provide a combinatorial and algorithmic analysis of what we call an *orientation class* of a collection of lines in space, i.e., the subspace of lines having a specified orientation with each of the given lines. We show how to express the "above/below" relationship of lines in space by means of the orientation relationship, and use this reduction to analyze various problems concerning the vertical relationship of lines in space. Even though (as discussed below) the "natural complexity" of an arrangement of n lines in space is $\Theta(n^4)$, there are many natural problems that can be solved much faster, typically in nearly-quadratic time, and sometimes even faster.

In order to introduce and summarize in more detail results involving lines in space, we review some basic geometric properties of these lines. A line in space requires four real parameters to specify

it, so it is natural to study arrangements of lines in space within an appropriate 4-dimensional parametric space. Unfortunately, any reasonable such representation introduces non-linear surfaces. For example, in many standard parametrizations, the space of all lines that meet a given line is a quadric in 4-space. To obtain a combinatorial representation of an arrangement of n lines in space, one therefore needs to construct an arrangement of n quadrics in 4-space. This arrangement has complexity $O(n^4)$, which is usually unacceptable for practical applications. The same reasoning applies when we want to partition the set of lines in space according to their interaction with more complex objects, say, balls. In this case, each ball induces a surface in $\mathbb{R}^4$, which is the locus of all points representing lines tangent to that ball. The arrangement of these surfaces decompose $\mathbb{R}^4$ into cells, so that for all points within a cell, all the corresponding lines in 3-space meet the same set of balls (and if the balls are disjoint, in the same order too). These observations indicate why "naive" solutions to visibility and related problems involving arbitrary collections of lines (or other simply shaped objects) in space produce bounds like $O(n^4)$ or worse. Moreover, for some of these problems, no better solutions are known to date—see below.

Fortunately, there are two lucky breaks that we are able to exploit, which lead to improved solutions in many applications. The first is that there is an alternative way to represent lines, using *Plücker coordinates* (see e.g. [106, 22, 64]; the original reference is [94]). These coordinates transform (oriented) lines into either points or hyperplanes in homogeneous 6-space (more precisely, in oriented projective 5-space) in such a way that the property of one line ℓ_1 intersecting another line ℓ_2 is transformed into the property of the Plücker point of ℓ_1 lying on the Plücker hyperplane of ℓ_2 (or vice versa). Thus, at the cost of passing to five dimensions, we can linearize the incidence relationship between lines. This Plücker machinery is developed in Section 7.2.

In studying arrangements of lines in space, it is more important to analyze the *relative orientation* of two lines rather than only the incidence between them, as the latter is a degenerate case of the former. Intuitively, it distinguishes between right-handed pairs and left-handed pairs of oriented lines. We develop this concept of relative orientation in Section 7.3 and show how to efficiently determine if a query line is of a particular orientation class with respect to n given lines. We give a method that takes preprocessing and storage of $O(n^{2+\varepsilon})$ and allows query time of $O(\log n)$. In the process we show that the total combinatorial description complexity of any particular orientation class is in the worst case $\Theta(n^2)$. We get this bound by mapping our lines to hyperplanes in oriented projective 5-space using Plücker coordinates. Our orientation class then corresponds to a convex polyhedron defined by the intersection of n halfspaces based on these hyperplanes. Our second lucky break now comes from the Upper Bound Theorem (see e.g., [81]), stating that the complexity of such a polyhedron is only $O(n^{\lfloor 5/2 \rfloor}) = O(n^2)$.

For many applications, however, we need to analyze the property of one line lying above or below another. In Section 7.3.1 we show how, by adding certain auxiliary lines, we can express "above/below" relationships by means of orientation relationships. This leads, using standard machinery, to an efficient algorithm for testing if a query line lies above the n given lines, using nearly-quadratic preprocessing time and storage, and $O(\log n)$ query time. The algorithm can also output the line of $\mathcal{L}$ that lies "immediately below" ℓ, i.e., the first line of $\mathcal{L}$ that is hit as ℓ is translated downwards. We also observe in Section 7.3.2 that the worst case combinatorial complexity of the "upper envelope" of n lines is $\Theta(n^3)$—the main observation being that such an envelope can be expressed as the union of n orientation classes of the kind discussed in the previous paragraph. With some modifications, we can also apply this technique to the batched version of the problem: Given m blue lines and n red lines, determine whether all blue lines lie above all red lines (we call this the "towering property"), and, if so, find for each blue line the red line lying immediately below it (in

the above sense). The resulting algorithm runs in $O((m+n)^{4/3+\varepsilon})$ time, for any $\varepsilon > 0$.

Having introduced this basic machinery involving Plücker coordinates, we then go on to review other topics involving lines in $\mathbb{R}^3$. We first address, in Section 7.4, the problem of detecting and eliminating all cycles in the "depth order" of n lines in space, namely, the relation consisting of those pairs (ℓ_1, ℓ_2) where ℓ_1 passes below ℓ_2. The goal is to eliminate all cycles by cutting the lines into a small number of pieces, for which the depth order is indeed a (partial) order. The goal is to achieve this using a subquadratic number of cuts (a quadratic number is trivial). This seems to be a hard problem, and so far there are only very few results that provide some partial solutions. A related (and simpler) problem is to obtain bounds on the number of *joints* in a line arrangement in 3-space, namely, points incident to at least three non-coplanar lines. In spite of some recent progress, there is still a gap between the best known upper and lower bounds.

We then study, in Section 7.5, the *ray shooting* problem, a classic in computer graphics and modelling. The basic problem is to preprocess a set T of n triangles in space, so that, give a query ray ρ, the first triangle of T hit by ρ can be found efficiently. Using a clever trick of Agarwal and Matoušek [8], the problem is reduced to testing whether a query segment intersects any triangle in T. This latter problem can be solved by standard range searching techniques. We review the basic results, and discuss some related visibility problems.

The next topic that we study, in Section 7.6, is the theory of *line transversals* in 3-space, which is a special topic in the general theory of geometric transversals, as reviewed in the surveys [56, 90, 110]. A line transversal to a set $\mathcal{F}$ of compact convex sets in $\mathbb{R}^3$ is a line that intersects every member of $\mathcal{F}$. Analyzing the structure of the space $T^3(\mathcal{F})$ of all line transversals to $\mathcal{F}$ has been the subject of extensive study, which has generated many interesting (and hard) problems. We review some of those, such as bounding the number of *geometric permutations* of $\mathcal{F}$ (the number of orders in which a transversal line can meet the members of $\mathcal{F}$), bounding the number of connected components (isotopy classes) of $T^3(\mathcal{F})$, and several other problems. Here we have not attempted a fully comprehensive survey, and many topics, such as Helly-type and Hadwiger-type theorems for line transversals, have not been treated; see the surveys [56, 90, 110] for more details.

We conclude in Section 7.7 with a list of the most obvious open problems involving lines in space.

7.2 Geometric Preliminaries

The main geometric object studied in this chapter is a line in 3-space. Such a line ℓ can be specified by four real parameters in many ways. For example, we can take two fixed parallel planes (e.g. $z = 0$ and $z = 1$) and specify ℓ by its two intersections with these planes. We can therefore represent all lines in 3-space, except those parallel to the two given planes, as points in four dimensions. An important observation is in order: The space of lines in 3-space is projective (a special case of the so-called *Grassmanian manifold* [61], so any attempt to represent *all* lines in 3-space as points in *real* 4-space is doomed to failure—there will always be a sub-manifold (of lower dimension) of lines that will have to be excluded (like the horizontal lines in the above parametrization). We will tend to ignore this issue, since it will not affect the combinatorial or algorithmic analysis of most of the problems that we will study.

However, as already noted, even simple relationships between lines, such as incidence between a pair of lines, become non-linear in 4-space. More specifically, a collection $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$ of

n lines, induces a corresponding collection of hypersurfaces $\mathcal{S} = \{s^1, \ldots, s^n\}$ in 4-space, where s^i represents the locus of all lines that intersect, or are parallel to, ℓ_i (it is easily checked that each s^i is a quadratic hypersurface). The *arrangement* $\mathcal{A} = \mathcal{A}(\mathcal{S})$ induced by these hypersurfaces represents the arrangement of the lines in $\mathcal{L}$, in the sense that each (4-dimensional) cell of $\mathcal{A}$ represents an isotopy class of lines in 3-space (i.e. any such line in the class can be moved continuously to any other line in the same class without crossing, or becoming parallel to, any line in $\mathcal{L}$).

This arrangement can be understood in three dimensions as follows. Given three lines in general position[1] in 3-space, they define a quadratic ruled surface, called a *regulus*, which is the locus of all lines incident with the given three lines. A fourth line will in general cut this surface in two, or zero, points. Thus four lines in general position will have either two or zero lines incident with all four of them. These quadruplets of lines with a common stabber correspond to vertices of the arrangement $\mathcal{A}$. (In other words, a line moving within an isotopy class comes to rest when it is in contact with four of the given arrangement lines—each of them removing one of the four degrees of freedom that the moving line has. Note that some of these contacts can be at infinity, corresponding to the moving line becoming parallel to one of the given lines.) Similarly, the edges of $\mathcal{A}$ correspond to motion of a line while incident with three given lines, in the regulus fashion described earlier. In the general case each vertex of $\mathcal{A}$ has eight incident edges. Higher dimensional faces of $\mathcal{A}$ can be obtained similarly, by letting the common stabber move away from two, three, or all four of the lines defining a vertex. This shows that the number of these higher dimensional faces of $\mathcal{A}$ is, in each case, related by at most a constant factor to the number of vertices of $\mathcal{A}$. This last statement remains valid even if the given lines are not in general position, as follows from a standard perturbation argument.

By the discussion in the preceding paragraph, we can conclude that the combinatorial complexity of the arrangement of n quadratic surfaces in 4-space that represent n lines in 3-space is $O(n^4)$, and this bound is attainable. In particular, $\mathcal{A}$ has $O(n^4)$ vertices, where each such vertex represents a line that meets four of the lines in $\mathcal{L}$.

One approach to many problems involving lines in space is to construct $\mathcal{A}$ explicitly, in time close to $O(n^4)$, examine each of its cells, and look for a (point representing a) line in that cell that satifies some desired property. Here is a simple example: Let T be a set of n triangles in 3-space. We want to find a line that *stabs* the maximum number of triangles in T. To solve this problem, let $\mathcal{L}$ denote the set of the $3n$ lines supporting the edges of the triangles in T, and let $\mathcal{A} = \mathcal{A}(\mathcal{L})$ denote the corresponding arrangement in 4-space. For each 4-dimensional cell c of $\mathcal{A}$, all points in c represent lines that stab the same subset of triangles: A line ℓ starts/stops crossing a triangle when it touches (the line containing) one of its edges, in which case the point that represents ℓ will have to cross one of the surfaces of $\mathcal{A}$, and then leave c. We thus obtain $O(n^4)$ cells c, and can easily compute the number of triangles crossed by lines in each c, but we can observe that when we cross from one cell c to a neighboring cell c', the number of stabbed triangles changes by at most ± 1, so by traversing the cells of $\mathcal{A}$ in an appropriate fashion, we can find the optimal line in a total of $O(n^4)$ time. No faster algorithm for this problem is known. The same approach can be used to solve Open Problem (f) mentioned in Section 7.7, in time close to $O(n^4)$.

However, in many problems faster solutions do exist. To obtain them, we exploit another representation of lines, using *Plücker coordinates and coefficients* (see [22, 64, 106] for a review of these concepts). Let ℓ be an oriented line, and a, b two points on ℓ such that the line is oriented from a to b. Let $[a_0, a_1, a_2, a_3]$ and $[b_0, b_1, b_2, b_3]$ be the homogeneous coordinates of a and b, with $a_0, b_0 > 0$ being the homogenizing weights. (By this we mean that the Cartesian coordinates of a

[1] We take this to mean that the lines are pairwise non-intersecting and non-parallel. For more lines we add the condition that no five of our lines can be simultaneously incident with another line (not necessarily of our collection).

are $(a_1/a_0, a_2/a_0, a_3/a_0)$ when $a_0 \neq 0$; the case $a_0 = 0$ represents lines at infinity.) By definition, the Plücker coordinates of ℓ are the six real numbers

$$\pi(\ell) = [\pi_{01}, \pi_{02}, \pi_{12}, \pi_{03}, \pi_{13}, \pi_{23}],$$

where $\pi_{ij} = a_i b_j - a_j b_i$ for $0 \leq i < j \leq 3$. Similarly, the Plücker coefficients of ℓ are

$$\varpi(\ell) = \langle \pi_{23}, -\pi_{13}, \pi_{03}, \pi_{12}, -\pi_{02}, \pi_{01} \rangle,$$

i.e. the Plücker coordinates listed in reverse order with two signs flipped. The most important property of Plücker coordinates and coefficients is that incidence between lines is a bilinear predicate. Specifically, ℓ^1 is incident to ℓ^2 if and only if their Plücker coordinates π^1, π^2 satisfy the relationship

$$\pi^1_{01}\pi^2_{23} - \pi^1_{02}\pi^2_{13} + \pi^1_{12}\pi^2_{03} + \pi^1_{03}\pi^2_{12} - \pi^1_{13}\pi^2_{02} + \pi^1_{23}\pi^2_{01} = 0. \tag{7.2.1}$$

This formula follows from expanding the four-by-four determinant whose rows are the coordinates of four distinct points a, b, c, d, with a, b on ℓ^1 and c, d on ℓ^2. This determinant is equal to 0 if and only if the two lines are incident (or parallel). In general, the absolute value of the quantity in equation (7.2.1) is six times the volume of the tetrahedron *abcd*,[2] and its sign gives the orientation of the tetrahedron *abcd*. As long as ℓ^1 is oriented from a to b and ℓ^2 from c to d, this sign is independent of the choice of the four points, and defines the *relative orientation* of the pair ℓ^1, ℓ^2, which we denote by $\ell^1 \diamond \ell^2$ [106].

It is easily checked that any positive scalar multiple of $\pi(\ell)$ is also a valid set of Plücker coordinates for the same oriented line ℓ, corresponding to a different choice of the defining points a and b, or to a positive scaling of their homogeneous coordinates. Also, any negative multiple of $\pi(\ell)$ is a representation of ℓ with the opposite orientation. Therefore, we can regard the Plücker coordinates $\pi(\ell)$ as the homogeneous coordinates of a point in projective oriented 5-space $\mathcal{P}^5$, which is a double covering of ordinary projective 5-space.[3] Dually, we can regard the Plücker coefficients $\varpi(\ell)$ as the homogeneous coefficients of an oriented hyperplane of $\mathcal{P}^5$. Equation (7.2.1) merely states that line ℓ^1 is incident to line ℓ^2 if and only if the Plücker point $\pi(\ell^1)$ lies on the Plücker hyperplane $\varpi(\ell^2)$ (or the other way around). In fact, the relative orientation $\ell^1 \diamond \ell^2$ of the two lines is $+1$ if $\pi(\ell^1)$ lies on the positive side of the hyperplane $\varpi(\ell^2)$, and -1 if it lies on the negative side.

We observe that not every point of $\mathcal{P}^5$ is the Plücker image of some line. Specifically, the real six-tuple (π_{ij}) is such an image if and only if it satisfies the quadratic equation

$$\pi_{01}\pi_{23} - \pi_{02}\pi_{13} + \pi_{12}\pi_{03} = 0. \tag{7.2.2}$$

which states (using (7.2.1)) that every line is incident to itself. Thus among the six Plücker coordinates *two* are redundant. Equation (7.2.2) defines a four-dimensional subset of $\mathcal{P}^5$, called the *Plücker hypersurface* (or the *Grassmannian* or the *Klein hypersurface* Π. Notice that the relative orientation of a line relative to a sixtuplet of numbers that does not correspond to a point on the Plücker hypersurface still makes perfect sense—simply plug the appropriate numbers into the left-hand side of Equation (7.2.1). It turns out that such "imaginary" lines do have a natural geometric interpretation in 3-space. They are known as *linear complexes* and their properties are studied in the literature [50, 66].

We close this review by emphasizing that the projective representation of lines in space via Plücker coordinates is indeed a representation of all lines, including those at infinity.

[2] It also equals the product $|ab| \cdot |cd| \cdot D \sin\alpha$, where D is the distance between ℓ^1 and ℓ_2, and α is the angle between the two lines.

[3] The points of $\mathcal{P}^5$ can be viewed as the oriented lines through the origin of $\mathbb{R}^6$, with the geometric structure induced by the linear subspaces of $\mathbb{R}^6$; or, equivalently, as the points of the 5-dimensional sphere $\mathbb{S}^5$, with the geometric structure induced by its great circles. See reference [106] for more details on the theory of oriented projective spaces.

7.3 The Orientation of a Line Relative to n Given Lines, with Applications

We wish to analyze the set $\overline{\mathcal{C}}(\mathcal{L}, \sigma)$, consisting of all lines ℓ in 3-space that have specified orientations $\sigma = (\sigma^1, \sigma^2, \ldots, \sigma^n)$ relative to n given lines $\mathcal{L} = (\ell^1, \ell^2, \ldots, \ell^n)$. (We call this set the *orientation class* σ relative to $\mathcal{L}$.) Translated to Plücker space, the definition says that point $\pi(\ell)$ has to lie on side σ^i of every hyperplane $\varpi(\ell^i)$, and therefore inside the convex polyhedron $C(\mathcal{L}, \sigma)$ in $\mathcal{P}^5$ that is the intersection of those n halfspaces. The orientation class $\overline{\mathcal{C}}(\mathcal{L}, \sigma)$ is thus the intersection of the polyhedron $C(\mathcal{L}, \sigma)$ and the Plücker hypersurface Π. Note that since Π is of degree 2, it can interact in at most "a constant fashion" with each feature of the polyhedron $C(\mathcal{L}, \sigma)$.

For the purposes of this chapter, we consider the polyhedron $C(\mathcal{L}, \sigma)$ to be an adequate description of the orientation class $\overline{\mathcal{C}}(\mathcal{L}, \sigma)$. Computing the class then means computing all the features of this polyhedron, i.e. all its faces (of any dimension). The number of such features is the *combinatorial complexity* of the class—intersecting with Π can only increase this number by a constant factor. By the Upper Bound Theorem (see e.g. [46, 81]), this complexity is only $O(n^{\lfloor 5/2 \rfloor}) = O(n^2)$. It is not difficult to find configurations of lines $\mathcal{L}$ that attain this bound. Consider the regulus (actually hyperbolic paraboloid) $z = xy$ and two families of $n/2$ lines each on the regulus. One family consists of lines from one of the two rulings of the regulus, and the other of lines from the other ruling . By perturbing the lines of one family to be slightly off the regulus, we can make this a non-degenerate arrangement. It is simple to check that in every elementary square defined by two successive lines from one ruling and two successive lines from the other ruling there corresponds a line incident to all four of the lines defining the square and passing above all the rest. A more detailed construction of this kind will be given in Section 7.3.2.

A possible data structure for representing the polyhedron $C(\mathcal{L}, \sigma)$ is its face-incidence lattice, as described in [46]. Seidel's output-sensitive convex hull algorithm [96] constructs this representation in $O(\log n)$ amortized time per face. Alternatively, one may use the optimal but complex convex hull algorithm of Chazelle [33] or the simpler randomized incremental algorithm of Clarkson and Shor [41], to obtain the polyhedron C in $O(n^2)$ (deterministic or expected) time.

Theorem 7.3.1. *The set of all lines in 3-space that have specified orientations to n given lines has combinatorial complexity $\Theta(n^2)$ in the worst case, and can be calculated in time $O(n^2)$.*

It was shown by Neil White (see [80]) that the intersection of the convex polyhedron $C(\mathcal{L}, \sigma)$ and the Plücker hypersurface Π may consist of many connected components. In other words, an orientation class relative to the fixed lines $\mathcal{L}$ may contain multiple distinct isotopy classes; see also [56]. We note that the vertices of those isotopy classes are intersections of the Plücker hypersurface Π with the edges of the polyhedron $C(\mathcal{L}, \sigma)$. Since Π is a quadratic hypersurface, there are at most two such intersections per edge, and therefore the total number of vertices in all those isotopy classes is only $O(n^2)$. In other words, there are at most $O(n^2)$ lines that touch four of the lines of $\mathcal{L}$ and have specified orientations with all the others. A slightly more complicated argument shows that there are at most $O(n^2)$ isotopy classes in one orientation class. We do not know if this bound can be attained.

The above discussion easily leads to an efficient algorithm for deciding whether a given query line ℓ in 3-space lies in a particular orientation class σ relative to a set $\mathcal{L}$ of n fixed lines. This simply amounts to checking whether the Plücker point of ℓ lies in the polyhedron $C(\mathcal{L}, \sigma)$. Using, e.g., the algorithm of Matoušek and Schwarzkopf [78], this can be done using $O(n^2/\log^{2-\delta} n)$ storage and

preprocessing time, so that a query can be answered in $O(\log n)$ time.

Theorem 7.3.2. *Given n lines in space and an orientation class σ, we can preprocess these lines by a procedure whose running time and storage is $O(n^2/\log^{2-\delta} n)$, so that, given any query line ℓ, we can determine, in $O(\log n)$ time, whether ℓ lies in the orientation class σ with respect to the given lines.*

We note that a simple modification of this data structure allows us to actually *compute* in $O(\log n)$ time the orientation class of a line ℓ relative to n fixed ones, rather then merely test whether ℓ is in a predetermined class. The modification consists of locating the Plücker point of ℓ in the zone of the Plücker hypersurface Π in the entire arrangement of the Plücker hyperplanes of $\mathcal{L}$ (see Section 2.5). This leads to a data structure of size $O(n^{4+\varepsilon})$, which can be constructed in $O(n^{4+\varepsilon})$ time, that can be used to compute the orientation class of a given query line in logarithmic time.

7.3.1 Testing whether a line lies above n given lines

We will now consider a particular case of the general problem discussed in the previous subsection, which turns out to have significant applications on its own. We will be concerned with the property of one line lying above or below another. Formally, ℓ^1 lies above ℓ^2 if there exists a vertical line that meets both lines, and its intersection with ℓ^1 is higher than its intersection with ℓ^2. We are assuming that neither ℓ^1 nor ℓ^2 is vertical, and the two lines are not parallel. Our previous non-degeneracy assumptions already exclude concurrent or parallel lines; whenever we are discussing the "above/below" relation, we also exclude vertical lines from consideration.

We can express this notion in terms of the relative orientation of these lines, as follows. Assume the lines ℓ^1 and ℓ^2 have been oriented in an arbitrary way, and consider their (oriented) perpendicular projections $\ell^{1\prime}$, $\ell^{2\prime}$ onto the xy-plane, seen from above. Observe that ℓ^1 is above ℓ^2 if and only if

the direction of $\ell^{1\prime}$ is clockwise to that of $\ell^{2\prime}$ and $\ell^1 \diamond \ell^2 = +1$, or

the direction of $\ell^{1\prime}$ is counterclockwise to that of $\ell^{2\prime}$ and $\ell^1 \diamond \ell^2 = -1$.

Now let us introduce the line a infinity λ^2 that is parallel to ℓ^2 and passes through zenith point $z_\infty = (0, 0, 0, 1)$, the point at positive infinity on the z-axis. We orient the line λ^2 so that its projection on the xy-plane has the same direction as the projection of ℓ^2. It is easy to check that the direction of $\ell^{1\prime}$ is clockwise of $\ell^{2\prime}$ if and only if $\ell^1 \diamond \lambda^2 = -1$. Therefore, we conclude that ℓ^1 is above ℓ^2 if and only if

$$\ell^1 \diamond \ell^2 = -\ell^1 \diamond \lambda^2. \tag{7.3.3}$$

Intuitively, ℓ^1 passes above ℓ^2 if and only if ℓ^1 passes "between" the lines ℓ^2 and λ^2. Thus, to express the fact that one line lies above another we need to check consistency between two linear inequalities. This fact complicates the analysis of the above/below relationship, in particular when many lines are involved.

Now let $\mathcal{L}$ be a collection of n lines in 3-space, and consider the set $\mathcal{U}(\mathcal{L})$, *the upper envelope of* $\mathcal{L}$, consisting of all lines ℓ that pass above every line of $\mathcal{L}$. We introduce the auxiliary lines at infinity $\Lambda = \{\lambda^1, \lambda^2, \ldots, \lambda^n\}$, with each λ^i parallel to the corresponding ℓ^i and passing through the point z_∞. Then, according to Equation 7.3.3, a line ℓ is above all lines in $\mathcal{L}$ if and only if $\ell \diamond \ell^i = -\ell \diamond \lambda^i$;

that is, if the orientation class of ℓ relative to the set $\mathcal{L}$ is exactly opposite to its orientation with respect to the set Λ.

Therefore, the set $\mathcal{U}(\mathcal{L})$ is the union of all orientation classes $\overline{\mathcal{C}}(\mathcal{L} \cup \Lambda, \sigma \cdot \bar{\sigma})$ where $\sigma \cdot \bar{\sigma}$ is a sign sequence of the form $(\sigma^1, \sigma^2, \ldots, \sigma^n, -\sigma^1, -\sigma^2, \ldots, -\sigma^n)$. Luckily for us, only n of these classes are non-empty. To see why, let us assume that the x and y coordinate axes have been rotated and the lines oriented so that the projection of ℓ^1 coincides with the negative y-axis, and all other lines (including the query line ℓ) point towards increasing x. Let's assume also that the lines $\ell^2, \ldots, \ell^n$ are sorted in order of increasing xy slope. It is easy to see that if the xy slope of ℓ lies between those of ℓ^k and ℓ^{k+1}, then its orientation class relative to the set Λ is $(-^k+^{n-k})$. Therefore, we conclude that there are only n orientation classes relative to Λ.

This observation leads to a fast algorithm for deciding whether a query line ℓ passes above n fixed lines $\mathcal{L}$. For each of the n valid orientation classes $\sigma_k = (-^k+^{n-k})$, we build a data structure $\Sigma_k(\mathcal{L}) = \Sigma(\mathcal{L} \cup \Lambda, \sigma_k \cdot \bar{\sigma_k})$, as described in Section 7.3. Then, to test a given query line ℓ we first use binary search to locate its xy slope among the slopes of the n given lines. This information determines the orientation class σ_k of ℓ relative to the lines in Λ. Once this has been found, we use the data structure $\Sigma_k(\mathcal{L})$ to test whether ℓ has the opposite orientation class $\bar{\sigma_k}$ relative to the lines in $\mathcal{L}$.

This straightforward algorithm uses space approximately cubic in n. To reduce the amount of space, we will merge all the n data structures $\Sigma_k(\mathcal{L})$ into a single data structure $\Sigma^*(\mathcal{L})$, as follows. Assume all lines in $\mathcal{L}$ have been sorted by xy slope and oriented as described above. Let m be a parameter, to be chosen later. Partition $\mathcal{L}$ into m subsets $\mathcal{L}_1, \ldots, \mathcal{L}_m$, each subset consisting of approximately n/m consecutive lines in slope order. Prepare the data structures $\Sigma_j^p(\mathcal{L}) = \Sigma(\mathcal{L}_j^p, (+ + \cdots +))$ and $\Sigma_j^s(\mathcal{L}) = \Sigma(\mathcal{L}_j^s, (- - \cdots -))$ for each prefix set $\mathcal{L}_j^p = \bigcup_{1 \le k < j} \mathcal{L}_k$ $(1 \le j \le m)$ and each suffix set $\mathcal{L}_j^s = \bigcup_{j < k \le m} \mathcal{L}_k$ $(2 \le j \le m)$. The storage and preprocessing time for these steps amount to $O(mn^{2+\varepsilon})$, for any $\varepsilon > 0$. Then, recursively build the data structure $\Sigma^*(\mathcal{L}_j)$ for each subset $\mathcal{L}_j$ (using the same choice of the parameter m). Therefore $\Sigma^*(\mathcal{L})$ is a data structure tree whose degree is m and whose depth will be $O(\log n / \log m)$. Testing a query line ℓ now proceeds as follows. As before, we use binary search to locate the xy slope of ℓ between the slopes of two lines ℓ^k and ℓ^{k+1} of $\mathcal{L}$. (This step has to be performed only once). Let $\mathcal{L}_j$ be the subset containing the line ℓ^k. By construction, the xy slope of ℓ is greater than the slopes of all lines in $\mathcal{L}_j^p$ and less than the slopes of all lines in $\mathcal{L}_j^s$. Then we can test, in $O(\log n)$ time, whether ℓ lies above all lines in these two subsets, using the data structures Σ_j^p and Σ_j^s. If ℓ does not lie above all these lines we stop immediately; otherwise we recursively test ℓ against $\mathcal{L}_j$ using the data structure $\Sigma^*(\mathcal{L}_j)$. If we set $m = \lceil n^\nu \rceil$, for some fixed and very small $\nu > 0$, the entire procedure takes time $O((\log n)^2 / \log m)) = O(\log n)$. The storage and preprocessing time amount to $O(m(n^2 / \log^{2-\delta} n)(\log n)/\log m)$, which is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$.

We can also provide a modified version of this procedure, having the same complexity bounds, that can determine, for each query line ℓ lying above all lines of $\mathcal{L}$, which is the first line of $\mathcal{L}$ that ℓ will hit when translated vertically downwards. The key observation is that translation of ℓ downwards corresponds to motion of $\pi(\ell)$ along a straight line, say $\rho(\ell)$, on the Plücker hypersurface Π: the coordinates $\pi(\ell)$ change linearly with the altitude of ℓ, as follows

$$[\pi_{01}, \pi_{02}, \pi_{12}, \pi_{03}, \pi_{13} - t\pi_{01}, \pi_{23} - t\pi_{02}],$$

where t is a parameter denoting the altitude. As ℓ moves vertically, it will become incident with another line ℓ' exactly when $\rho(\ell)$ crosses the plane $\varpi(\ell')$, and the crossing point can be computed in constant time. Moreover, the crossing point determines the line $\rho(\ell)$ uniquely, since it corresponds to

a unique line in 3-space and the inverse of the downwards translation is a unique upwards translation. Note that as t tends to infinity (which corresponds to lifting the line up by an infinite amount), its Plücker image tends to a point of the form $[0, 0, 0, 0, -\pi_{01}, -\pi_{02}]$. These limit points constitute a line τ in $\mathcal{P}^5$, and correspond to lines at infinity of 3-space passing through the zenith point. In other words, finding the first line that ℓ hits amounts to performing a ray shooting query within the corresponding polyhedron in Plücker space.

To perform such queries efficiently, we slightly modify the construction of the data structure $\Sigma(\mathcal{L})$. We construct it in a hierarchical recursive manner. In each step we take a random sample $\mathcal{R}$ of r of the hyperplanes $\varpi(\ell^i)$, and construct the convex polyhedron $C(\mathcal{R})$. Instead of decomposing $C(\mathcal{R})$ into simplices, as is usually done, we divide its interior by a set of hypersurfaces with the property that no line $\rho(\ell)$ crosses one of these hypersurfaces, and the resulting cells still have constant complexity. Specifically, take a decomposition of the boundary of $C(\mathcal{R})$ into simplices, and back-project from each such simplex s along the lines ρ that terminate at points on s. The collection of these back-projections yield a decomposition of $C(\mathcal{R})$ into $O(r^2)$ cells. We argue that the combinatorial complexity of each cell is a constant independent of r. Indeed, the base of each cell is a 4-dimensional simplex, the walls of the cell are a lifting of the boundary of this simplex along the lines $\rho(\ell)$, and the roof of the cell is some interval on the line τ. Because of the way these cells are constructed, to each cell c there corresponds a unique line $\ell(c)$ of $\mathcal{R}$ that is first hit as we translate downwards any line whose Plücker point lies in c. Again, the ε-net theory tells us that we can find a subset of $O(\frac{n}{r}\log r)$ lines of $\mathcal{L}$ such that the downwards translation of any line ℓ, with $\pi(\ell)$ in c, will not meet any other line of $\mathcal{L}$ until it reaches λ.

Therefore, if we use this modified cell decomposition of $C(\mathcal{R})$ when constructing the data structure $\Sigma^*(\mathcal{L})$, then when we test a line ℓ for being above the n given lines, we can at the same time locate the nearest line below ℓ.

Theorem 7.3.3. *Given n lines in space, we can preprocess them by a procedure whose running time and storage is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, so that, given any query line ℓ, we can determine, in $O(\log n)$ time, whether ℓ lies above all the given lines, and, if so, which is the first line of $\mathcal{L}$ that ℓ will hit when translated downwards.*

7.3.2 The complexity of the upper envelope of n lines

In the previous subsection we saw that the upper envelope $\mathcal{U}(\mathcal{L})$ of a set of n lines in 3-space is the union of n orientation classes relative to the set $\mathcal{L} \cup \Lambda$. Each of these classes can be described as a polyhedron of $\mathcal{P}^5$ with at most $O(n^2)$ features. Therefore, the combinatorial complexity of $\mathcal{U}(\mathcal{L})$ is at most $O(n^3)$.

Notice that each of these n selected orientation classes relative to $\mathcal{L} \cup \Lambda$ defines a single isotopy class. This is so because any two lines ρ^1, ρ^2 in this class point in the same sector defined by the lines of $\mathcal{L}$ down in the xy-plane. Thus we can always continuously move ρ^1 to ρ^2 by first lifting it up high enough, then rotating it to align it with ρ^2, and then dropping it down onto ρ^2. In particular, this implies that in each of these n orientation classes, there are at most $O(n^2)$ lines that touch four lines of $\mathcal{L}$, and lie above all the remaining ones. (Each such line is the intersection of the Plücker hypersurface Π with an edge of polyhedron $C(\mathcal{L}, \sigma)$; since Π is a quadric, there are at most two such intersections per edge.)

It is also possible to exhibit a set of n lines that attains this cubic bound. Details of such a construction are given in [30]. That is, we have:

Theorem 7.3.4. *The maximum combinatorial complexity of the entire upper envelope of n lines in space is $\Theta(n^3)$.*

7.3.3 Testing the towering property

In this subsection we exhibit a reasonably efficient deterministic algorithm for testing whether n blue lines $b_1, \ldots, b_n$ in 3-space lie above m other red lines $r_1, \ldots, r_m$; this is what we call the "towering property". The algorithm runs in time $O\left((m+n)^{4/3+\varepsilon}\right)$, for any $\varepsilon > 0$, a substantial improvement over the obvious $O(mn)$ method.

We first consider the case where the xy-slope of every red line is at least as large as that of any blue line. In that case, if we map the blue lines (oriented so as to have xy-projections going from left to right) to points $\lambda_1, \ldots, \lambda_n$ in $\mathcal{P}^5$ via Plücker coordinates, and the red lines (similarly oriented) to hyperplanes $\rho_1, \ldots, \rho_m$ in $\mathcal{P}^5$ via Plücker coefficients, then the towering property is equivalent to asserting that all n blue points lie in the convex polyhedron $\overline{\mathcal{C}}$ obtained by intersecting the appropriate halfspaces bounded by the m red hyperplanes, as given by equation (7.2.1).

This latter property is easy to test, using standard space decomposition techniques: We choose a sufficiently large constant r, sample a subset $\mathcal{R}$ of r random elements of the Plücker hyperplanes, construct the intersection of the corresponding halfspaces, get a convex polyhedron $\overline{\mathcal{C}}_{\mathcal{R}}$ of $O(r^2)$ complexity, and decompose it into $O(r^2)$ simplices. If any Plücker point lies outside $\overline{\mathcal{C}}_{\mathcal{R}}$, the property does not hold and we stop. Otherwise, we distribute the points in the cells of the decomposition—by refining some cells as needed, we may also assume that each cell contains at most n/r^2 points.

We then need to solve $O(r^2)$ subproblems, each involving the points in some cell and the $O\left(\frac{m}{r}\log r\right)$ hyperplanes crossing the cell. We do this by simply exchanging the roles of points and hyperplanes, mapping the red lines into their Plücker points and the blue lines into their Plücker hyperplanes. We apply, to each of these $O(r^2)$ dual problems, the same sampling-and-decomposition approach that was used above, obtaining a grand total of $O(r^4)$ subproblems, each involving $O\left(\frac{n}{r^3}\log r\right)$ blue lines, and $O\left(\frac{m}{r^3}\log r\right)$ red lines. Assuming r to be a constant, the overhead in preparing these subproblems is only $O(m+n)$.

Unwinding the recursion in this manner, we obtain a total running time of $O\left((m+n)^{4/3+\varepsilon}\right)$, for any $\varepsilon > 0$ (where the choice of r depends on the desired ε).

Let us now return to the general towering problem and relax all assumptions on the slopes of the projections. Compute the median slope among the projections onto the xy-plane of all the red and blue lines together. This partitions the red lines into two sets, R_1 and R_2, and the blue lines into two sets, B_1 and B_2 such that each line in $R_2 \cup B_2$ projects onto the xy-plane into a line of slope at least as large as that of any projected line of $R_1 \cup B_1$; furthermore, the sizes of $R_1 \cup B_1$ and $R_2 \cup B_2$ are roughly equal. Now, we solve the towering problem recursively with respect to R_1 vs. B_1 and then for R_2 vs. B_2. If no negative answer has been produced yet, then we may apply the previous algorithm to the pairs (R_1, B_2) and (R_2, B_1). The correctness of the procedure follows from the fact that all pairs of red and blue lines are (implicitly) checked. Moreover, the asymptotic bound on the running time is not affected by this extra layer of processing, as is easy to check. (See [30] for more details, and for a slightly different approach.)

An additional computation similar to that detailed at the end of the previous section allows us to determine, within the same bounds, the red line immediately below each blue line, and thus also, if the towering property holds, the smallest vertical distance between the two groups of lines. So we

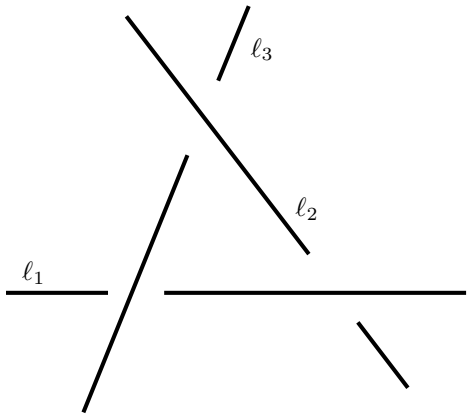

FIGURE 7.1. A triangular cycle.

conclude with our theorem:

Theorem 7.3.5. *Given n blue lines and m red lines in space, one can test that all blue the lines pass above all the red lines (the towering property) in time and space $O((m+n)^{4/3+\varepsilon})$, for any $\varepsilon > 0$. If so, within the same time bound, we can actually find the first red line below each blue line.*

Separating lines by translation. We close this part of the chapter, by mentioning the following curious fact, whose proof is given in [30].

Theorem 7.3.6. *There exists a set of nine mutually disjoint lines in 3-space that cannot be taken apart by continuously translating a proper subset off to infinity.*

7.4 Cycles and Depth Order

Let $\mathcal{L}$ be a collection of n lines in $\mathbb{R}^3$ in *general position*. In particular, assume that no two lines in $\mathcal{L}$ intersect, and that the xy-projections of the lines are all distinct, and every pair of these projections intersect. For any pair ℓ, ℓ' of lines in $\mathcal{L}$, we say, as in Section 7.3.1, that ℓ passes *above* ℓ' (equivalently, ℓ' passes *below* ℓ) if the unique vertical line λ that meets both ℓ and ℓ' intersects ℓ at a point that lies higher than its intersection with ℓ'. We denote this relation as $\ell' \prec \ell$. The relation $\prec$ is total, but in general it needs not be transitive and it can contain *cycles* of the form $\ell_1 \prec \ell_2 \prec \cdots \prec \ell_k \prec \ell_1$. We refer to k as the *length* of the cycle. Cycles of length 3 are called *triangular*. See Figure 7.1.

If we cut the lines of $\mathcal{L}$ at a finite number of points, we obtain a collection of lines, segments, and rays. We can extend the definition of the relation $\prec$ to the new collection in an obvious manner, except that it is now only a partial relation. Our goal is to cut the lines in such a way that $\prec$ becomes a *partial order*, and to do so with a small number of cuts. In this case we call $\prec$ a *depth order*. We note that it is trivial to construct a depth order with $\Theta(n^2)$ cuts: Simply cut each line near every point whose xy-projection is a crossing point with another projected line. A long standing conjecture is that one can always construct a depth order with a *subquadratic* number of cuts.

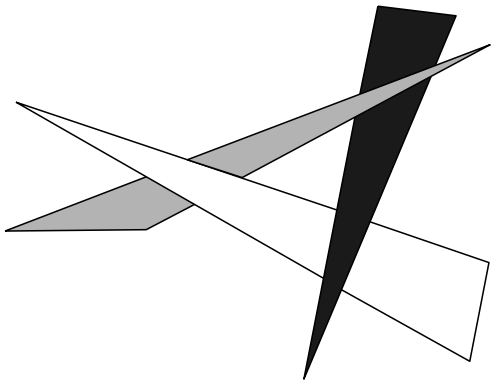

FIGURE 7.2. A depth cycle defined by three triangles.

Background. The main motivation for studying this problem comes from *hidden surface removal* (HSR) in computer graphics. We are given a collection of objects in $\mathbb{R}^3$, say pairwise disjoint triangles, and a viewing point, placed for convenience at $z = -\infty$, and we wish to compute and render all visible portions of the input objects; that is, for each object o we wish to compute the subset of all points p on o for which the downward-directed ray emanating from p meets no other object.

Until the 1970s, HSR was considered one of computer graphics' most important problems, and has received a substantial amount of attention; see [107] for a survey of the ten leading HSR algorithms circa 1974. Since then it has been solved in hardware, using the *z-buffer* technique [25], which produces a 'discrete' solution to the problem, by computing the nearest object at each pixel of the image. Nevertheless, there is still considerable interest in obtaining an *object-space* representation of the visible scene, which is a combinatorial description of the visible portions, independent of the pixel locations and resolution.

These considerations motivated an extensive study of hidden-surface removal in computational geometry, culminating in the early 1990s with a number of algorithms that provide both conceptual simplicity and satisfactory running-time bounds. See de Berg [18] and Dorward [45] for overviews of these developments, and Overmars and Sharir [85] for a simple HSR algorithm with good theoretical running-time bounds.

A common feature of most HSR algorithms is that they rely on the existence of a consistent *depth order* for the input objects, which is defined as in the case of lines: A pair of objects A, B satisfy $A \prec B$ if there exists a point on B so that the downward-directed ray emanating from it meets A. These algorithms begin by sorting the objects either front-to-back (e.g., the Overmars-Sharir algorithm [85]) or back-to-front (e.g., the classical Painter's Algorithm [107]).

A large number of algorithms have been developed for *testing* whether the depth relationship $\prec$ in a collection of triangles contains a cycle with respect to a specific viewpoint; see de Berg [18] and the references therein. However, while these algorithms help detect cycles, they do not provide strategies for dealing with cycles.

One such common strategy is to eliminate all depth cycles, by cutting the objects into portions that do not form cycles, and running an HSR algorithm on the resulting collection of pieces. In 1980, Fuchs *et al.* [52] introduced *binary space partition (BSP) trees*, which can be used to perform the desired cutting. However, a BSP tree may force up to a quadratic number of cuts [88], which

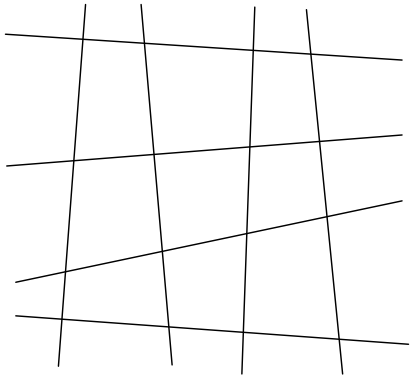

FIGURE 7.3. A collection of line segments that forms a grid.

brings us back to the original challenge: Devise an algorithm that, given a specific viewpoint and a collection of n triangles in $\mathbb{R}^3$, removes all depth cycles defined by this collection with respect to the viewpoint, using a subquadratic number of cuts. This has been an open problem since 1980.

We only consider the simpler problem mentioned above, by restricting the input to lines in space, rather than triangles. Note that since any cycle defined by a collection of *line segments* is also a cycle in the collection of lines spanned by these segments, the case of line segments is simpler than the case of lines, and we thus concentrate on the latter case. (The case of triangles, though, is more involved, since a depth cycle among triangles does not necessarily imply a depth cycle among their edges.)

The work of Solan [104] and of Har-Peled and Sharir [59] supply algorithms that achieve the above goal, provided a subquadratic number of cuts is always sufficient. In particular, these works present algorithms that, given a collection $\mathcal{L}$ of n lines (or segments) in 3-space, perform close to $O(n\sqrt{C})$ cuts (the precise bound is $O(n^{1+\varepsilon}\sqrt{C})$ for [104] and $O(n\sqrt{C}\alpha(n)\log n)$ for [59]) that eliminate all cycles defined by $\mathcal{L}$ as seen from $z = -\infty$, where C is the minimal required number of such cuts for $\mathcal{L}$. That is, if we can provide a subquadratic bound on the minimum number of cuts that suffice to eliminate all cycles defined by a collection of lines, then the aforementioned algorithms are guaranteed to find a collection of such cuts of (potentially larger but still) subquadratic size.

Such an upper bound however remains elusive. One major progress in this direction is due to Chazelle *et al.* [34], who in 1992 have analyzed the following special case of the problem. A collection of line segments in the plane is said to form a *grid* if it can be partitioned into two subcollections of 'red' and 'blue' segments, such that all red (resp., blue) segments are pairwise disjoint, and all red (resp., blue) segments intersect all blue (resp., red) segments in the same order; see Figure 7.3. Chazelle *et al.* [34] have shown that if the xy-projections of a collection of n segments in 3-space form a grid, then all cycles defined by this collection (again, as seen from $z = -\infty$) can be eliminated with $O(n^{9/5})$ cuts.

A more recent contribution, due to Aronov *et al.* [12] makes the first step towards obtaining subquadratic general upper bounds on the number of cuts that are sufficient to eliminate all cycles defined by an arbitrary collection of lines in space. Specifically, it shows that all *triangular cycles*, which are cycles formed by triples of lines, can be eliminated with $O(n^{2-1/69}\log^{16/69} n)$ cuts. This allows adapting the technique of Har-Peled and Sharir [59] or of Solan [104], to yield an algorithm

that eliminates all such cycles using close to $O(n^{2-1/138})$ cuts.

The best known lower bound for the number of required cuts is $\Omega(n^{3/2})$, due to Chazelle *et al.* [34]; it is obtained from an appropriately rotated and slightly perturbed grid of lines.

Both upper bound proofs in [12, 34] use, as a main ingredient, the unrealizability of certain 'weaving patterns' of lines in space. Specifically, a *weaving* is a finite collection of lines drawn in the plane, such that at each intersection of a pair of lines, it is specified which of the two is 'above' the other. A weaving Ψ is said to be *realizable* if there is a collection $\mathcal{L}$ of lines in 3-space (called the *realization* of Ψ) whose xy-projection forms a collection of lines that is combinatorially equivalent to the one that defines Ψ, and the lines in $\mathcal{L}$ adhere to the above-below constraints specified by Ψ. Otherwise, Ψ is said to be unrealizable. A growing, albeit still relatively small, body of work deals with the analysis and classification of realizable and unrealizable weavings [12, 53, 86, 95]. While it can be shown that, for a sufficiently large number of lines, most weavings are unrealizable, showing that specific (small-size) weavings are unrealizable is a rather nontrivial problem. Figures 7.4 and 7.5 show the unrealizable weavings that are used in the proofs of [12, 34], respectively.

7.4.1 Joints

Let $\mathcal{L}$ be a set of n lines in space. A *joint* of $\mathcal{L}$ is a point in $\mathbb{R}^3$ where at least three non-coplanar lines ℓ, ℓ', ℓ'' of $\mathcal{L}$ meet. We denote the joint by the triple (ℓ, ℓ', ℓ'').

Let $\mathcal{J}_{\mathcal{L}}$ denote the set of joints of $\mathcal{L}$, and put $J(n) = \max |\mathcal{J}_{\mathcal{L}}|$, taken over all sets $\mathcal{L}$ of n lines in space. A trivial upper bound on $J(n)$ is $O(n^2)$, as a joint is a point of intersection of (more than) two lines, but it was shown by Sharir [99], following a weaker subquadratic bound in [34], that $J(n)$ is only $O(n^{23/14}\text{polylog}(n)) = O(n^{1.643})$. An easy construction, based on lines forming an $n^{1/2} \times n^{1/2} \times n^{1/2}$ portion of the integer grid, shows that $|\mathcal{J}_{\mathcal{L}}|$ can be $\Omega(n^{3/2})$ (see Figure 7.6 and [34]).

Sharir's bound has been recently improved by Feldman and Sharir [51] to $O(n^{112/69}\log^{6/23} n) = O(n^{1.6232})$. The proof proceeds by mapping the lines of $\mathcal{L}$ into points and/or hyperplanes in projective 5-space, using Plücker coordinates. This is followed by a two-stage decomposition process, which partitions the problem into subproblems, using *cuttings* of arrangements of appropriate subsets of the Plücker hyperplanes. One then bounds the number of joints within each subproblem, and sums up the resulting bounds to obtain the above bound. The proof adapts and applies some of the tools used by Sharir and Welzl [102] and recently enhanced by Aronov *et al.* [12], related mainly to the connection between joints and *reguli* spanned by the lines of $\mathcal{L}$.

One of the main motivations for studying joints of a set $\mathcal{L}$ of lines in space is their connection to cycles of $\mathcal{L}$. Joints can be regarded as a degenerate case of cycles. In fact, a slight random perturbation of the lines in $\mathcal{L}$ turns any joint incident to $O(1)$ lines into a cycle with some constant probability, implying that the number of joints is strongly related to the number of *elementary cycles*, whose xy-projections form single faces in the arrangement of the projected $\mathcal{L}$.

The problem of joints is considerably simpler than the corresponding problems involving cycles, as witnessed by the much sharper upper bounds cited above (or just by the ability to prove *any* subquadratic bound). Still, it is a rather challenging open problem to tighten the gap between the upper and lower bounds. One hopes that better insights into the joints problem would lead to tools that could also be used to obtain subquadratic bounds for elementary cycles, and for many other problems that involve lines in space. Recently, Sharir and Welzl [102] have shown that the number

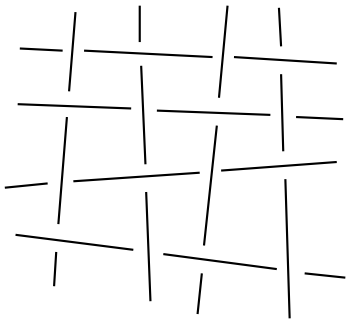

FIGURE 7.4. The unrealizable complete 4×4 weaving.

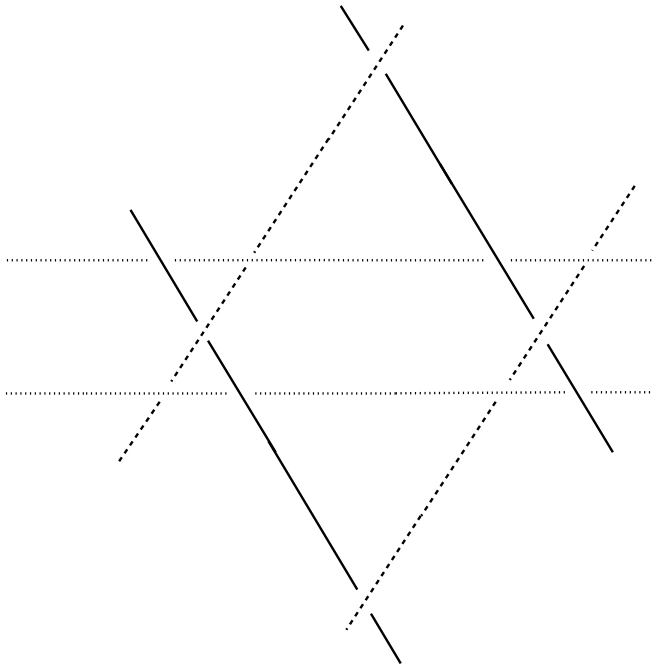

FIGURE 7.5. The unrealizable Magen-David weaving.

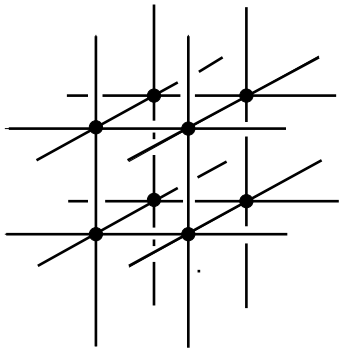

FIGURE 7.6. The lower bound construction for joints, illustrated with $n = 12$ lines.

of incidences between the points in $\mathcal{J}_{\mathcal{L}}$ and the lines in $\mathcal{L}$ is $O(n^{5/3})$.

7.5 Ray Shooting and Other Visibility Problems

The ray shooting problem is to preprocess a set of objects such that the first object hit by a query ray can be determined efficiently. The ray shooting problem has received considerable attention in the past because of its applications in computer graphics and other geometric problems. The planar case has been studied thoroughly. Optimal solutions, which answer a ray shooting query in $O(\log n)$ time using $O(n)$ space, have been proposed for some special cases [26, 36, 60]. For an arbitrary collection of segments in the plane, the best known algorithm answers a ray shooting query in time $O(\frac{n}{\sqrt{s}} \log^{O(1)} n)$, using $O(s^{1+\varepsilon})$ space and preprocessing [2, 9, 17], for any $\varepsilon > 0$, where s is a parameter that can vary between n and n^2.

The three-dimensional ray shooting problem seems much harder, and it seems to be still far from being fully solved. It constitutes a nice and useful application of the theory of lines in space. Most studies of this problem consider the case where the given set is a collection of triangles. If these triangles are the faces of a convex polyhedron, then an optimal algorithm (with $O(n)$ storage and $O(\log n)$ query time) can be obtained using the hierarchical decomposition scheme of Dobkin and Kirkpatrick [44]. If the triangles form a polyhedral terrain (an xy-monotone piecewise-linear surface), then the technique of Chazelle *et al.* [28, 30] yields an algorithm that requires $O(n^{2+\varepsilon})$ space and answers ray shooting queries in $O(\log n)$ time; we will give more details concerning this technique in the next subsection on visibility problems. The best known algorithm for the general ray shooting problem (involving triangles) is due to Agarwal and Matoušek [7]; it answers a ray shooting query in time $O(\frac{n^{1+\varepsilon}}{s^{1/4}})$, with $O(s^{1+\varepsilon})$ space and preprocessing, where the parameter s can range between n and n^4. A variant of this technique was presented in [10] for the case of ray shooting amid a collection of convex polyhedra. Thus, to achieve fast (logarithmic) query time, we need about n^4 storage, "in accordance" with the general observations made earlier in this chapter. Alternatively, to achieve near-linear storage, the best query time that we can achieve is close to $O(n^{3/4})$.

Here is a brief review of this technique. Let T be the set of n input triangles. The first tool is to reduce the problem, using *parametric search* [8], to the problem of testing whether a query segment (an arbitrary "prefix" of the query ray) intersects any triangle in T. Next, we take each triangle

$t \in T$ and orient the lines $\lambda_1, \lambda_2, \lambda_3$ supporting its edges in a cyclic manner, so that a line ℓ inersects t if and only if it has the same (positive or negative) orientation with respect to each of the λ_i's. Hence a segment ab intersects t if and only if (i) a and b lie on different sides of the plane supporting t, and (ii) the line containing ab has the same orientation with respect to each of $\lambda_1, \lambda_2, \lambda_3$. Testing whether T contains a triangle that satisfies (i) and (ii) can be done using a multi-level halfspace range searching data structure, as described, e.g., in [7].

Selecting triangles that satisfy (i) can be done using two levels of a 3-dimensional halfspace range searching structure (such as the efficient partition trees of [75]). Selecting triangles that satisfy (ii) can be done by first tranforming the problem into Plücker's 5-space. Since we only deal now with triangles that satisfy (i), we need to find all triangles that are crossed by the *line* ℓ supporting the query segment (or original ray), or, more precisely, determine whether any such triangle exists. In Plücker space, this amounts to finding all triangles t for which the Plücker point of ℓ lies, say, below the three Plücker hyperplanes of the lines supporting the edges of t. We perform this task using three more levels of a 5-dimensional halfspace range searching data structure. There is however a twist: The points that we query with lie on the 4-dimensional Plücker hypersurface Π, so, when we construct the partition tree of [75], using an arrangement of some subset of r hyperplanes, we do not need to consider all the cells of that arrangement, but only those that lie in the zone of Π. Since the overall complexity of that zone is $O(r^4 \log r)$ [13] (see Section 2.5), the partition tree has performance that is similar to that of partition trees in four dimensions. Putting everything together, and omitting several additional technical details (for which see [7, 9]), we obtain a solution that has the preformance asserted above.

On the other hand, there are certain special cases of the 3-dimensional ray shooting problem which can be solved more efficiently. de Berg [18] presents some of these instances. For example, if the objects are planes or halfplanes, ray shooting amid them can be performed in time $O(\frac{n^{1+\varepsilon}}{s^{1/3}})$, with $O(s^{1+\varepsilon})$ space and preprocessing, for $n \leq s \leq n^3$; see [8] for details. If the objects are horizontal fat triangles or axis-parallel polyhedra, ray shooting can be performed in time $O(\log n)$ using $O(n^{2+\varepsilon})$ space; see [20] for details. If the objects are spheres, ray shooting can be performed in time $O(\log^4 n)$ with $O(n^{3+\varepsilon})$ space; see [83].

Sharir and Shaul [101] have recently considered several special cases of the ray shooting problem, including the case of arbitrary fat triangles and the case of triangles stabbed by a common line. They present an improved solution for the case where only near-linear storage is allowed, improving the query time to $O(n^{2/3+\varepsilon})$, using $O(n^{1+\varepsilon})$ space and preprocessing. Curiously, at the other end of the trade-off, no better solutions than the general solution that requires $O(n^{4+\varepsilon})$ storage to guarantee logarithmic query time, are known.

7.5.1 Other visibility problems

Besides the ray shooting problem, computer graphics and modelling, as well as several other areas, are teeming with visibility problems in 3-space of all shapes and colors. We briefly mention some of them:

Visibility from a fixed point. In this kind of problems, we are typically given some polyhedral scene consisting of n non-crossing triangles, and a fixed viewing point v, and our goal is either to compute the portions of these triangles that are seen from v (the so-called *hidden surface removal* problem), or to preprocess the scene for efficient ray shooting queries, for rays emanating from v. In other

variants, v is not fixed, but its location is constrained, say, to lie on a fixed line.

By projecting the given triangles centrally from v onto some sphere (or a cube, or a pair of planes), we obtain a 2-dimensional arrangement of the projected triangles, and each face of the arrangement determines uniquely the triangle seen from v when looking in directions that lead to that face. Hence, the *visibility map*, which is the decomposition of the sphere of projection into regions, each being a maximal connected visible portion of a single triangle, has combinatorial complexity $O(n^2)$, and this bound is easily attained. However, in practice the complexity of the map is much smaller, so output-sensitive algorithms for constructing visibility maps are of interest. See O'Rourke [84] for a survey of some algorithms of this kind. When the input triangles form a *polyhedral terrain* (the graph of a continuous piecewise linear bivariate function), the visibility map can still have quadratic complexity, but, as shown by Cole and Sharir [42], it can be preprocessed into a data structure that uses only $O(n\alpha(n)\log n)$ storage, in $O(n\alpha(n)\log n)$ time, so that a ray shooting query from v can be answered in $O(\log n)$ time.

Aspect graphs. When we vary continuously the point of view v, the visibility map changes continuously, but its combinatorial structure generally remains unchanged, except at certain critical viewing points, for which a vertex of the visibility map coincides with an edge, or three edges become concurrent. In these cases, the visibility map undergoes a discrete combinatorial change. The *aspect graph* of the original 3-dimensional scene is the collection of all combinatorially different visibility maps (or *views*) of the given scene, as we vary the point of view v. This definition depends on how are we allowed to move v. Two main special cases are *orthographic aspect graphs*, where v lies on the plane at infinity, and *perspective aspect graphs*, where v can lie anywhere in 3-space. In the former (resp., latter) case, we also decompose the sphere $\mathbb{S}^2$ of viewing directions (resp., the entire 3-space of viewing points) into maximal connected regions, so that, for each region K, the visibility maps for all directions in K (resp., points in K) are combinatorially the same. The number of such regions is the complexity of the respective aspect graph. For background and a survey of earlier research on aspect graphs, see [23]; see also [84] for a more recent survey.

For unrestricted scenes, the size of their orthographic (resp., perspective) aspect graph is $O(n^6)$ (resp., $O(n^9)$), and these bounds can be attained [84, 92]. In certain special cases better bounds can be obtained. For example, for polyhedral terrains, the bounds are $O(n^{5+\varepsilon})$ in the orthograohic case, and $O(n^{8+\varepsilon})$ in the perspective case, and both bounds are nearly worst-case tight.

Here is a sketch of the argument. Let Σ be the given terrain. A pair of parallel rays (ρ_1, ρ_2) is called *critical* if for each $i = 1, 2$, the source point of ρ_i lies on an edge a_i of Σ, ρ_i passes through three edges of Σ (including a_i), and ρ_i does not intersect the (open) region lying below Σ. It can be shown that the number of topologically different orthographic views of Σ is $O(n^5)$ plus the number of critical pairs of parallel rays. Fix a pair a_1, a_2 of edges of Σ. Agarwal and Sharir [11] define, for each pair (a_1, a_2) of edges of Σ, a collection $\mathcal{F}_{a_1,a_2}$ of n trivariate functions, so that every pair (ρ_1, ρ_2) of critical rays, where ρ_i emanates from a point on a_i (for $i = 1, 2$), corresponds to a vertex of $M_{\mathcal{F}_{a_1,a_2}}$. They also show that the graphs of the functions in $\mathcal{F}_{a_1,a_2}$ satisfy assumptions (A1)–(A2). Using Theorem 2.3.1 and summing over all pairs of edges of Σ, we can conclude that the number of critical pairs of rays, and thus the number of topologically different orthographic views of Σ, is $O(n^{5+\varepsilon})$. Using a more careful analysis, Halperin and Sharir [58] proved that the number of different orthographic views is $n^5 2^{O(\sqrt{\log n})}$. de Berg *et al.* [19] have constructed a terrain for which there are $\Omega(n^5\alpha(n))$ topologically different orthographic views.

For perspective views, it is argued in [11] number of perspective views of a terrain. They argue

that the number of perspective views of Σ is proportional to the number of triples of rays emanating from a common point, each of which passes through three edges of Σ before intersecting the open region lying below Σ. Following a similar approach to the one sketched above, the problem is reduced to the analysis of lower envelopes of $O(n^3)$ families of 5-variate functions, each family consisting of $O(n)$ functions that satisfy assumptions (A1)–(A2). This leads to an overall bound of $O(n^{8+\varepsilon})$ for the number of topologically different perspective views of Σ. This bound is also known to be almost tight in the worst case, as follows from another lower-bound construction given by de Berg *et al.* [19].

There are some other special cases where better bounds can be obtained. de Berg *et al.* [19] showed that if Σ is a set of k pairwise-disjoint convex polytopes with a total of n vertices, then the number of orthographic views is $O(n^4k^2)$; the best known lower bound is $\Omega(n^2k^4)$. For translates of a cube, Aronov *et al.* [14] established the bounds are $O(n^{4+\varepsilon})$ for orthographic views and $O(n^{6+\varepsilon})$ for perspective views; both bounds are nearly tight in the worst case.

While all these bounds are huge, they do not reflect the real nature of the problem, because they are non-local: When the viewing point v crosses one critical surface σ, some local feature of the visibility map changes in a discrete manner, and when v crosses another critical surface σ', another visible feature, at a different location, undergoes a discrete change, and similarly for a third critical surface σ''. So near the intersection point $\sigma \cap \sigma' \cap \sigma''$, all eight combinations of old and new features are possible, and this is what makes the size of aspect graphs so large. The real issue is to encode in a more compact form all these different views, so that searching in each of them can still be made efficient.

Guarding a 3-dimensional scene. This is a natural extension of the extensively studied topic of "art gallery" problems, where some 2-dimensional scene has to be guarded by the minimal number of guards, so that each point in the scene is visible from at least one guard. Many instances of this problem are known to be NP-hard, already in the plane [108], and the situation does not get easier in 3-space.

One specific topic that has attracted some attanetion is that of guarding a polyhedral terrain by *watchtowers*. The goal is to erect a small set of watchtowers (vertical segments whose bottom endpoints lie on the terrain), so that collectively their top endpoints guard the entire terrain. The problem is interesting because a single watchtower can watch the entire terrain if it is sufficiently high. The goal is to optimize the solution, either by specifying the number of watchtowers and aiming at minimizing their (common) height, or by specifying their height and aiming at minimizing their number. The latter problem is NP-hard (already for height zero) [42], while the former can be solved in polynomial time for any fixed number k of watchtowers. For a single watchtower, the problem can be solved in $O(n \log n)$ time [112] (see also [97]). For two watchtowers, a recent work of Agarwal *et al.* [6] gives an algorithm with running time $O(n^{11/3+\varepsilon})$ for the case where the watchtowers can be erected only at vertices of the terrain.

7.6 Transversal Theory

Let $\mathcal{F}$ be a family of convex sets in $\mathbb{R}^3$. A line ℓ is a *transversal* of $\mathcal{F}$ if it intersects every member of $\mathcal{F}$. The set of all line transversals of $\mathcal{F}$ is called the *transversal space* of $\mathcal{F}$, and is denotd by $T^3(\mathcal{F})$. There has been considerable study of various topological, combinatorial, and other structural properties of spaces of line transversals. We will survey here only some of them, and ignore others, such as

Helly-type and Hadwiger-type theorems for line transversals. Also, the study of line transversals is a special topic of the broader theory of geometric transversals, which will not address either. See the survey papers [56, 90, 110].

Concerning the topological and combinatorial structure of $T^3(\mathcal{F})$, we note that there are two equivalence relations between line transversals of $\mathcal{F}$: *isotopy* (lying in the same connected component of $T^3(\mathcal{F})$), and inducing a common *geometric permutation* (meeting the objects of $\mathcal{F}$ in the same linear order). Two isotopic transversal lines induce the same geometric permutation [55], but the converse does not hold in general (although it does hold in the plane) [56].

Isotopy. The *combinatorial complexity* of $T^3(\mathcal{F})$ is defined as the total number of topological faces, of all dimensions, on the boundary of $T^3(\mathcal{F})$. A standard reduction exhibits $T^3(\mathcal{F})$ as a region (a "sandwich" region) in $\mathbb{R}^4$ enclosed between the lower envelope and upper envelope of certain surface patches describing upper and lower tangents to each member of $\mathcal{F}$; see, for example, [5, 72] for a description of this reduction.

Any upper bound on the maximal *combinatorial complexity* of $T^3(\mathcal{F})$ serves as a natural upper bound on the maximal number of connected components of $T^3(\mathcal{F})$.

In the planar case, the complexity of $T^2(\mathcal{F})$, when the elements of $\mathcal{F}$ are pairwise disjoint, is $O(n)$. In the 3-dimensional case, the complexity of $T^3(\mathcal{F})$ depends on the description complexity of the sets in $\mathcal{F}$. When the sets in $\mathcal{F}$ are semi-algebraic of *constant description complexity* (for example, if the sets in $\mathcal{F}$ are balls), the complexity of $T^3(\mathcal{F})$ is $O(n^{3+\varepsilon})$, as follows from the general result of Koltun and Sharir [72] on sandwich regions of trivariate functions. Indeed, similar to the analysis in Section 2.14.2, fix an input object $s \in \mathcal{F}$. We consider the representation of lines in space by equations of the form $y = ax + b$, $z = cx + d$, and parametrize such a line ℓ by the quadruple (a, b, c, d). When we translate ℓ up and down, a, b, and c remain fixed and d varies. We translate ℓ in this way until it becomes tangent to s, and denote the resulting upper and lower tangent lines by $(a, b, c, U_s(a, b, c))$ and $(a, b, c, L_s(a, b, c))$, respectively. (In general, the functions U_s and L_s are only partially defined, with a common domain of definition.) Then ℓ intersects s if and only if

$$L_s(a, b, c) \le d \le U_s(a, b, c).$$

Hence if ℓ is a transversal of $\mathcal{F}$, we must have

$$\max_{s\in\mathcal{F}} L_s(a, b, c) \le d \le \min_{s\in\mathcal{F}} U_s(a, b, c).$$

In other words, $T^3(\mathcal{F})$ is the sandwich region between the upper envelope of the trivariate functions L_s and the lower envelope of the functions U_s. Hence, by [72], the complexity of $T^3(\mathcal{F})$ is $O(n^{3+\varepsilon})$.

This general bound was preceded by several works that considered special cases, and obtained slightly improved bounds. Pellegrini and Shor [91] showed that if S is a set of triangles in $\mathbb{R}^3$, then the space of line transversals of S has $n^3 2^{O(\sqrt{\log n})}$ complexity. The bound was slightly improved by Agarwal [3] to $O(n^3 \log n)$. He reduced the problem to bounding the complexity of a family of cells in an arrangement of $O(n)$ hyperplanes in $\mathbb{R}^5$.

There are also almost matching lower bounds, showing that the complexity of $T^3(\mathcal{F})$ can be $\Omega(n^3)$, see [5, 89]. However, for the number of connected components the best known lower bounds are $\Omega(n^2)$, or $\Omega(n^{d-1})$ in the d-dimensional case [103]. Narrowing this gap, even for restricted families of objects, is an intriguing open problem.

In the d-dimensional case, $d \geq 4$, where the sets of $\mathcal{F}$ are semi-algebraic of constant description complexity, the complexity of $T^d(\mathcal{F})$, defined in an analogous manner to the 3-dimensional case, is $O(n^{2d-2+\varepsilon})$, for any $\varepsilon > 0$, which actually bounds the overall complexity of the arrangement of surface patches described above. Extending the result of Koltun and Sharir [72] to higher dimensions is an important open problem.

These are the best general known bounds on the number of connected components, but there are some improved bounds in restricted cases: Aronov and Smorodinsky [15] proved that when restricting the transversals to pass through the origin the transversal space has $\Theta(n^{d-1})$ components. Brönnimann *et al.* [24] gave a complete description of the transversal space of n segments in $\mathbb{R}^3$. In this case the transversal space consists of $\Theta(n)$ connected components.

Geometric permutations. Recall that a geometric permutation is the order in which a transversal line ℓ meets the objects of $\mathcal{F}$ (assuming them to be pairwise disjoint, this order is well defined, up to a reversal if ℓ is not oriented). The study of geometric permutations was initiated by Katchalski et al. [67]. One of the most interesting questions in this field is to prove tight asymptotic bounds on the maximal number $g_d(n)$ of geometric permutations that a family of n arbitrary pairwise disjoint convex sets in $\mathbb{R}^3$ can admit. The known bounds on $g_d(n)$ in the general case are:

$g_2(n) = 2n - 2$; see [49].

$g_d(n) = O(n^{2d-2}), d \geq 3$; see [109].

$g_d(n) = \Omega(n^{d-1}), d \geq 3$; see [68, 103].

Improved bounds are known in several special cases: Smorodinsky *et al.* [103] give a tight bound $\Theta(n^{d-1})$ for the case of pairwise disjoint balls in $\mathbb{R}^d$, which has been extended by Katz and Varadarajan [70] for families of α-*fat* convex objects in $\mathbb{R}^d$, where the constant of proportionality depends on α. If the sets in $\mathcal{F}$ have constant description complexity, then in $\mathbb{R}^3$ they induce at most $O(n^{3+\varepsilon})$ geometric permutations, as follows from the preceding discussion.

The case of balls was further studied by Zhou and Suri [111], who considered families $\mathcal{B}$ of n pairwise disjoint balls in $\mathbb{R}^d$ with a bounded *radial ratio* γ, namely the ratio between the largest radius of a ball in the family and the smallest one. Zhou and Suri show that, in this case, $\mathcal{B}$ admits at most $O(\gamma^{\log \gamma})$ geometric permutations.

If the balls all have the same radius, the number of geometric permutations is at most two for $n \geq 9$, and at most three for $3 \leq n \leq 8$. This has been recently shown by Cheong *et al.* [38], following several papers that obtained weaker bounds [103, 16, 69, 63].

Isotopy and geometric permutations. As mentioned, a simple continuity argument shows that two isotopic transversal lines induce the same geometric permutation. The converse is true in the plane, but an example, due to Goodman *et al.* [56], shows that the same geometric permutation can be attained by arbitrarily many connected components of $T^3(\mathcal{F})$, for any $d \geq 3$.

It is therefore natural to seek conditions on the family $\mathcal{F}$ that ensure that these two notions of equivalence coincide. For example, do these equivalence relations coincide for families of pairwise disjoint balls in $\mathbb{R}^d$? in $\mathbb{R}^3$?

The study of $T^d(\mathcal{F})$ can be reduced to the study of the space of orientations of line transversals to $\mathcal{F}$, as a subset of $\mathbb{S}^{d-1}$. Holmsen *et al.* [62] and Greenstein [57] show that, for families $\mathcal{B}$ of balls,

the number of components of $T^3(\mathcal{B})$ is equal to the number of components of the orientation space.

Holmsen *et al.* [62] have studied the case of pairwise disjoint *unit* balls, and have shown that, for any triple of pairwise disjoint unit balls B_1, B_2, B_3 in $\mathbb{R}^3$, the space $D^3(B_1, B_2, B_3) \subseteq \mathbb{S}^2$ of orientations of line transversals to the triple that meet them in the fixed order (B_1, B_2, B_3) is *spherically convex*. This implies, among other consequences, that each geometric permutation of any family $\mathcal{B}$ of pairwise disjoint arbitrary balls in $\mathbb{R}^3$ induces a single connected component of $T^d(\mathcal{B})$, or, equivalently, all transversals with the same geometric permutation are isotopic to each other.

Greenstein [57] gives an example that shows that the key technical step in the analysis of [62] fails for arbitrary balls. Nevertheless, it is still conceivable (and is open at the moment) that the property itself does hold for arbitrary balls.

The work of Greenstein [57] sheds more light on the topological structure of the transversal space of a family $\mathcal{B}$ of n pairwise disjoint balls in $\mathbb{R}^3$, where the ratio between the radius of the largest ball in the family and the radius of the smallest ball is γ. Specifically, Greenstein shows that in this case $T^3(\mathcal{B})$ consists of $O(\gamma n \lambda_{12}(n) + \gamma^2 n^2)$ connected components (where $\lambda_{12}(n)$ is the maximal length of a Davenport-Schinzel sequence of order 12 on n symbols). This work is related to a long standing conjecture of Danzer [43], whose 3-dimensional version has recently been settled in the affirmative by Holmsen *et al.* [62]. The analysis proceeds by fixing a pair of balls B_i, B_j, whose centers are "too close", and by bounding the number of common tangents to B_i, B_j and two other balls that correspond to (potentially) non-convex features of the orientation space. The latter task is accomplished by a careful analysis of a specific representation of the tangent lines to a fixed pair of balls.

7.7 Open Problems

Although the manifold of all non-oriented lines in 3-space has been well studied [61], less seems to be known about the manifold of oriented lines that we have used in this chapter, and which seems to be computationally of significant advantage. It is known that this manifold is topologically equivalent to the oriented Grassmannian manifold $\tilde{M}_{4,2}(\mathbb{R})$, which happens to be the same[4] as $S^2 \times S^2$.

In general, in spite of the progress reviewed above, many questions about lines in space are still open. These questions can be gathered from the reviews of the various topics given above. Here is a short list of some of these questions. The reader can easily add many other open problems to the list.

Problem 7.7.1. *(a) Tighten the bound on the number of joints in a line arrangement in 3-space.*

(b) Is there a general algorithm for ray shooting amid n triangles in space that uses near-linear storage and answers queries in $O(n^\gamma)$ time, for $\gamma < 3/4$? Similarly, is there a general algorithm that answers queries in logarithmic (or polylogarithmic) time but uses $O(n^\gamma)$ storage, with $\gamma < 4$?

(c) Is there a subquadratic bound on the number of cuts that eliminate all cycles in a set of lines in general position?

[4] A geometric proof can be given by associating to every pair of unit vectors u, v (placed at the origin of 3-space) the oriented line ℓ that passes through the tip of the vector $(u \times v)/(1 + u \cdot v)$ and has the direction of the vector $u + v$. When $u = -v$ the line ℓ is by definition the line at infinity on the plane normal to v and oriented relative to v according to the right-hand rule. It is easy to check that this mapping is continuous, one-to-one, and generates all oriented lines of 3-space.

(d) Tighten the bound on the maximum number of geometric permutations in a set of n arbitrary pairwise disjoint convex objects in $\mathbb{R}^3$. Obtaining a (nearly) cubic upper bound would already constitute a major development.

(e) Is there a constant upper bound on the number of connected components of the space of line transversals to pairwise disjoint balls in space (of arbitrary radii)?

(f) Here is a nice (and probably very hard) visibility problem that has been around for nearly a decade: Given a collection $\mathcal{T}$ of n pairwise disjoint triangles in 3-space, report all pairs of mutually visible triangles in $\mathcal{T}$; a pair (Δ_1, Δ_2) is mutually visible if there exists a segment that connects a point on Δ_1 to a point on Δ_2 and does not meet any other triangle of $\mathcal{T}$. The goal is to solve this problem in $O(n^\gamma)$ time, for some $\gamma < 4$.

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Chapter 8

Geometric Coloring Problems: Sphere Packings and Frequency Allocation

8.1 Multiple Packings and Coverings

The notion of multiple packings and coverings was introduced independently by Davenport and László Fejes Tóth. Given a system $\mathcal{S}$ of subsets of an underlying set X, we say that they form a *k-fold packing (covering)* if every point of X belongs to *at most (at least)* k members of $\mathcal{S}$. A 1-fold packing (covering) is simply called a *packing (covering)*. Clearly, the union of k packings (coverings) is always a k-fold packing (covering). Today there is a vast literature on this subject [13], [15].

Many results are concerned with the determination of the maximum density $\delta^k(C)$ of a k-fold packing (minimum density $\theta^k(C)$ of a k-fold covering) with congruent copies of a fixed convex body C. The same question was studied for multiple *lattice packings (coverings)*, giving rise to the parameter $\delta_L^k(C)$ ($\theta_L^k(C)$). Because of the strongly combinatorial flavor of the definitions, it is not surprising that combinatorial methods have played an important role in these investigations. For instance, Erdős and Rogers [11] used the "probabilistic method" to show that $\mathbb{R}^d$ can be covered with congruent copies (actually, with translates) of a convex body so that no point is covered more than $e(d\ln d+\ln\ln d+4d)$ times (see [22], and [17] for another combinatorial proof based on Lovász' Local Lemma). Note that this easily implies that

$$k\theta_d \le \theta^k(C) \le k\theta(C)$$

for a constant $\theta_d > 0$ depending on d (where $\theta(C)$ is shorthand for $\theta^1(C)$).

To establish almost tight density bounds, at least for lattice arrangements, it would be sufficient to show that any k-fold packing (covering) splits into roughly k packings (coverings), or into about k/l disjoint l-fold packings (coverings) for some $l < k$. The initial results were promising. Blundon [3] and Heppes [19] proved that for unit disks $C = B^2$, we have

$$\theta_L^2(C) = 2\theta_L(C), \quad \delta_L^k(C) = k\delta_L(C) \ \text{ for } k \le 4,$$

and these results were extended to arbitrary centrally symmetric convex bodies in the plane by Dumir and Hans-Gill [9] and by G. Fejes Tóth [12], [14]. In fact, there was a simple reason for this

phenomenon: It turned out that every 3-fold lattice packing of the plane can be decomposed into 3 packings, and every 4-fold lattice packing into *two* 2-fold ones. This simple scheme breaks down for larger values of k.

The situation becomes slightly more complicated if we do not restrict our attention to *lattice* arrangements. It is easy to see that any 2-fold packing of homothetic copies of a plane convex body splits into 4 packings [23]. Furthermore, any k-fold packing $\mathcal{C}$ with not too "elongated" convex sets splits into at most $9\lambda k$ packings, where

$$\lambda := \max_{C \in \mathcal{C}} \frac{(\text{circumradius}(C))^2 \pi}{\text{area}(C)}.$$

(Here the constant factor 9λ can be easily improved.)

One would expect that similar results hold for coverings rather than packings. However, in this respect we face considerable difficulties. For any k, it is easy to construct a k-fold covering of the plane with not too elongated convex sets (of different shapes but of roughly the same size) that cannot be decomposed even to *two* coverings [23]. The problem is far from being trivial even for coverings with congruent disks. In an unpublished manuscript, Mani-Levitska and Pach have shown that every 33-fold covering of the plane with congruent disks splits into two coverings [21]. Another positive result was established in [24].

Theorem 8.1.1. *For any centrally symmetric convex polygon P, there exists a constant $k = k(P)$ such that every k-fold covering of the plane with translates of P can be decomposed into two coverings.*

At first glance, one may believe that approximating a disk by centrally symmetric polygons, the last theorem implies that any sufficiently thick covering with congruent disks is decomposable. The trouble is that, as we approximate a disk with polygons P, the value $k(P)$ tends to infinity. Nevertheless, it follows from Theorem 1.1 that if $k = k(\varepsilon)$ is sufficiently large, then any k-fold covering with disks of radius 1 splits into a covering and an "almost covering" in the sense that it becomes a covering if we replace each of its members by a concentric disk whose radius is $1 + \varepsilon$.

Recently, Tardos and Tóth [28] have managed to extend Theorem 8.1.1 to any (not necessarily centrally symmetric) convex polygon P. Here the assumption that P is convex cannot be dropped.

Surprisingly, the analogous decomposition result is false for multiple coverings with balls in *three* and higher dimensions.

Theorem 8.1.2 ([21]). *For any k, there exists a k-fold covering of $\mathbb{R}^3$ with unit balls that cannot be decomposed into two coverings.*

Somewhat paradoxically, it is the very heavily covered points that create problems. Pach [23], [1] (p. 68) noticed that by the Lovász Local Lemma we obtain

Theorem 8.1.3. *Any k-fold covering of $\mathbb{R}^3$ with unit balls, no $c2^{(k-1)/3}$ of which have a point in common, can be decomposed into two coverings. (Here c is a positive constant.)*

Similar theorems hold in $\mathbb{R}^d$ ($d > 3$), except that the value $2^{(k-1)/3}$ must be replaced by $2^{(k-1)/d}$.

8.2 Cover-Decomposable Families and Hypergraph Colorings

These questions can be reformulated in a slightly more general combinatorial setting. A family $\mathcal{F}$ of sets in $\mathbb{R}^d$ is called *cover-decomposable* if there exists a positive integer $k = k(\mathcal{F})$ such that any k-fold covering of $\mathbf{R}^d$ with members from $\mathcal{F}$ can be decomposed into two coverings.

In particular, Theorem 8.1.1 above can be rephrased as follows. The family consisting of all translates of a given centrally symmetric convex polygon P in the plane is cover-decomposable. (P is considered to be an *open* region.)

Note that Theorem 8.1.1 has an equivalent "dual" form. Given a system $\mathcal{S}$ of translates of P, let $C(\mathcal{S})$ denote the set of centers of all members of $\mathcal{S}$. Clearly, $\mathcal{S}$ forms a k-fold covering of the plane if and only if every translate of P contains at least k elements of $C(\mathcal{S})$. The fact that the family of translates of P is cover-decomposable can be expressed by saying that there exists a positive integer k satisfying the following condition: any set C of points in the plane such that $|P' \cap C| \geq k$ for all translates P' of P can be partitioned into two disjoint subsets C_1 and C_2 with

$$|C_1 \cap P'| \neq \emptyset \text{ and } |C_2 \cap P'| \neq \emptyset \text{ for every translate } P' \text{ of } P.$$

We can think of C_1 and C_2 as "color classes."

This latter condition, in turn, can reformulated as follows. Let $H(C)$ denote the (infinite) hypergraph whose vertex set is C and whose (hyper)edges are precisely those subsets of C that can be obtained by taking the intersection of C by a translate of P. By assumption, every hyperedge of $H(C)$ is of size at least k. The fact that C can be split into two color classes C_1 and C_2 with the above properties is equivalent to saying that $H(C)$ is *two-colorable*.

Definition 8.2.1. *A hypergraph is* two-colorable *if its vertices can be colored by two colors such that no edge is monochromatic.*

A hypergraph is called two-edge-colorable *if its edges can be colored by two colors such that every vertex is contained in edges of both colors.*

Obviously, a hypergraph H is two-edge colorable if and only if its *dual hypergraph* H^* is two-colorable. (By definition, the vertex set and the edge set of H^* are the edge set and the vertex set of H, respectively, with the containment relation reversed.)

Summarizing, Theorem 8.1.1 can be rephrased in two equivalent forms. For any centrally symmetric convex polygon P in the plane, there is a $k = k(P)$ such that

1. any k-fold covering of $\mathbb{R}^2$ with translates of P (regarded as an infinite hypergraph on the vertex set $\mathbb{R}^2$) is two-edge-colorable;
2. for any set of points $C \subset \mathbb{R}^2$ with the property that each translate of P covers at least k elements of C, the hypergraph $H(C)$ whose edges are the intersections of C with all translates of P is two-colorable.

Here we outline a geometric construction showing that certain families of sets in the plane are not cover-decomposable.

Let T_k denote a rooted k-ary tree of depth $k-1$. That is, T_k has $1+k+k^2+k^3+\ldots+k^{k-1} = \frac{k^k-1}{k-1}$ vertices. The only vertex at level 0 is the root v_0. For $0 \leq i < k-1$, each vertex at level i has precisely k children. The k^{k-1} vertices at level $k-1$ are all leaves.

Definition 8.2.2. *For any rooted tree T, let $H(T)$ denote the hypergraph on the vertex set $V(T)$, whose hyperedges are all sets of the following two types:*

1. Sibling *hyperedges: for each vertex $v \in V(T)$ that is not a leaf, take the set of all children of v;*

2. Descendent *hyperedges: for each leaf $v \in V(T)$, take all vertices along the unique path from the root to v.*

Obviously, $H_k := H(T_k)$ is a k-uniform hypergraph with the following property. No matter how we color the vertices of H_k by two colors, red and blue, say, at least one of the edges will be monochromatic. In other words, H_k is not two-colorable. Indeed, assume without loss of generality that the root v_0 is red. The children of the root form a sibling hyperedge $S(v_0)$. If all points of $S(v_0)$ are blue, we are done. Otherwise, pick a red point $v_1 \in S(v_0)$. Similarly, there is nothing to prove if all points of $S(v_1)$ are blue. Otherwise, there is a red point $v_2 \in S(v_1)$. Proceeding like this, we either find a sibling hyperedge $S(v_i)$, all of whose elements are blue, or we construct a red descendent hyperedge $\{v_0, v_1, \ldots, v_{k-1}\}$.

Definition 8.2.3. *Given any hypergraph H, a* planar realization *of H is defined as a pair $(P, \mathcal{S})$, where P is a set of points in the plane and $\mathcal{S}$ is a system of sets in the plane such that the hypergraph obtained by taking the intersections of the members of $\mathcal{S}$ with P is isomorphic to H.*

A realization of the dual hypergraph of H is called a dual realization *of H.*

It was shown in [26] that for any rooted tree T, the hypergraph $H(T)$ defined above has both a planar and a dual realization, in which the members of $\mathcal{S}$ are (1) open strips or (2) open disks of different radii. In particular, the hypergraph $H_k = H(T_k)$ permits such realizations for every positive k. Thus, the two dual realization results immediately imply the following theorems.

Theorem 8.2.4 ([26]). *The family of open strips in the plane is not cover-decomposable.*

Theorem 8.2.5 ([26]). *The family of open disks in the plane is not cover-decomposable.*

Theorem 8.2.5 settles Problem 3 of Section 2.1 in Research Problems in Discrete Geometry [5].

Theorem 8.2.5 as well as the fact that for every k, the hypergraph $H_k = H(T_k)$ permits a planar realization, immediately implies

Theorem 8.2.6 ([21]). *The family of open unit balls in $\mathbb{R}^d$ is not cover-decomposable, for any $d \geq 3$.*

It is also true that the hypergraph $H_k = H(T_k)$ permits a dual realization in the plane with axis-parallel rectangles, for every positive k. Therefore, we have

Theorem 8.2.7 ([26]). *The family of open axis-parallel rectangles in the plane is not cover-decomposable.*

Problem 8.2.8. *Does H_k permit a planar realization with axis-parallel rectangles, for every k?*

For illustration, we prove only Theorem 8.2.4; the other proofs are slightly trickier, but follow the same scheme.

Recall that a *dual* realization of a hypergraph H is a planar realization of its dual H^*. That is, given a tree T, a dual realization of $H(T)$ is a pair $(P, \mathcal{S})$, where P is a set of points in the plane representing the (sibling and descendent) hyperedges of $H(T)$, and $\mathcal{S}$ is a system of regions

representing the vertices of T such that a region $S \in \mathcal{S}$ covers a point $p \in P$ if and only if the vertex corresponding to S is contained in the hyperedge corresponding to p.

Lemma 8.2.9. *For any rooted tree T, the hypergraph $H(T)$ permits a dual representation with strips.*

Proof. We prove the statement by induction on the number of vertices of T. The statement is trivial if T has only one vertex. Suppose that T has n vertices and that the statement has been proved for all rooted trees with fewer than n vertices. Let v_0 be the root of T, and let $v_0 v_1 \dots v_m$ be a path of maximum length starting at v_0. Let $U = \{u_1, u_2, \dots u_k\}$ denote the set of children of v_{m-1}. Clearly, each element of U is a leaf of T, one of them is v_m, and U is a sibling hyperedge of $H(T)$. Let T' denote the tree obtained by deleting from T all elements of U. The vertex v_{m-1} is then a leaf of T'.

By the induction hypothesis, $H(T')$ permits a dual realization $(P, \mathcal{S})$ with (open) strips. We can assume without loss of generality that no element of P lies on the boundary of any strip in $\mathcal{S}$, otherwise, we could slightly decrease the widths of some strips without changing the containment relation.

Let $p \in P$ be the point corresponding to the descendent hyperedge $\{v_0, v_1, \dots, v_{m-1}\}$ of $H(T')$. Take a short segment σ whose one endpoint is p and which does not intersect the boundary of any strip in $\mathcal{S}$. Then, for any point $x \in \sigma$ and for any strip $S \in \mathcal{S}$, we have $x \in S$ if and only if $p \in S$. Let $\sigma_1, \sigma_2, \dots, \sigma_k$ be pairwise disjoint subsegments of σ.

For any $1 \le i \le k$, choose a (very thin) strip S^i, almost parallel to σ, such that

1. $S^i \cap \sigma = \sigma_i$,
2. $S^i \cap P = \emptyset$,
3. all strips S^i have a point q in common.

Add $S^1, S^2, \dots, S^k$ to $\mathcal{S}$. These strips will represent the vertices $u_1, u_2, \dots u_k \in V(T)$, respectively. For any $1 \le i \le k$, take a point $p_i \in \sigma_i$, these points will represent the descendent hyperedges of $H(T)$, corresponding to the paths connecting v_0 to $u_1, u_2, \dots u_k$, respectively. Finally, delete p and add q to the set P; the latter point represents the sibling hyperedge $U = \{u_1, u_2, \dots u_k\}$. We have obtained a dual realization of $H(T)$, so we are done. □

8.3 Frequency Allocation and Conflict-Free Coloring

Motivated by a frequency assignment problem in cellular telephone networks, Even, Lotker, Ron, and Smorodinsky [10] studied the following question. Given a set P of n points in the plane, what is the smallest number of colors in a coloring of the elements of P with the property that any closed disk D with $D \cap P \neq \emptyset$ has an element whose color is not assigned to any other element of $D \cap P$. We refer to such a coloring as a *conflict-free* coloring of P with respect to disks.

In the specific application, the points correspond to *base stations* interconnected by a fixed backbone network. Each *client* continuously scans frequencies in search of a base station within its (circular) range with good reception. Once such a base station is found, the client establishes a radio link with it, using a frequency not shared by any other station within its range. Therefore, a

conflict-free coloring of the points corresponds to an assignment of frequencies to the base stations, which enables every client to connect to a base station without interfering with the others.

Theorem 8.3.1 ([10]). *Any set of n points in the plane has a conflict-free coloring with respect to disks, using $O(\log n)$ colors, and the order of magnitude of this bound cannot be improved.*

Proof. Assume for simplicity that the points are in general position, that is, no four lie on a circle. Define the *Delaunay graph* $G(P)$ of the point set P by connecting a pair of its elements with a straight-line segment if there is disk containing both of them but no other point in P. It is well known that these edges define a triangulation of the convex hull of P. In particular, $G(P)$ is a planar graph and so it is four-colorable. Take an at least $\frac{n}{4}$-element independent set $I_1 \subset P$ in this graph, and color each point of I_1 by color 1. Repeat the same procedure for the set $P_1 := P \setminus I_1$. That is, take an independent set of at least $\frac{|P_1|}{4}$ vertices in the Delaunay graph $G(P_1)$, and color all of its elements with color 2. Set $P_2 := P_1 \setminus I_2$, and continue until no vertices are left. Since in each step we color at least a positive fraction (one quarter) of the uncolored vertices, the algorithm will terminate in $O(\log n)$ steps.

It remains to verify that the resulting coloring is conflict-free. Take any disk D that contains at least one element of P. Pick a point in $P \cap D$ whose color j is the largest. We claim that the color j occurs only once in D. Suppose, in order to obtain a contradiction, that there are at least two points in D with color j. By continuously shrinking D, we can obtain a disk $D' \subset D$ that contains precisely two points of color j. These points must have been connected by an edge in the Delaunay graph $G(P_{j-1})$. Therefore, it is impossible that both of them belonged to the independent set of vertices I_j n this graph. Hence, at least one of them was not colored j, a contradiction.

It is easy to check that his algorithm can be implemented in polynomial time. □

The logarithmic bound in Theorem 8.3.1 cannot be improved. To see this, consider a set P of n points on a line. Clearly, any disk contains an interval of this line and vice versa, for any interval there is a disk whose intersection with the line is precisely this interval. Consider an interval I_1 (or a disk) that contains all elements of P. If we start with a conflict-free coloring, I_1 must contain a point $p_1 \in P$ of unique color. This point cuts I_1 into two smaller intervals at least one of which contains at least $\frac{n-1}{2}$ points. Denote this interval by I_2, and choose a point of unique color in it. We can repeat this procedure at least $\log_2 n$ times, and in each step we discover a point whose color has not been encountered before.

In fact, a stronger statement is true.

Theorem 8.3.2 ([25]). *Any set of n points requires at least constant times $\log n$ colors for a conflict-free coloring with respect to disks.*

This is a very general method for constructing conflict-free colorings of point sets with respect to various families of geometric figures. For instance, let P be a set of points in general position in three-dimensional space, and suppose that we want to color them with as few colors as possible so that in every *half-space* H that contains at least one element of P, there is a point whose color is not assigned to any other element of $P \cap D$. As before, we can define a "Delaunay-type" graph $G(P)$ by connecting two points of P with an edge if there is a half-space containing both of them but no other element of P. Notice that the edges of $G(P)$ are precisely the edges of the *convex hull* of P in $\mathbb{R}^3$. In particular, they form a planar graph, and the rest of the argument can be repeated *verbatim*.

Corollary 8.3.3 ([10]). *Any set of n points in $\mathbb{R}^3$ has a conflict-free coloring with respect to half-spaces, using $O(\log n)$ colors, and the order of magnitude of this bound cannot be improved.*

The last result has an immediate corollary via duality, which is also interesting from the point of view of applications for cellular networks. Let now the base stations be represented by disks (by their reception ranges). Each base station uses a fixed frequency. If a client falls into the reception range of several stations, to avoid interference, he wants to establish connection with a station whose frequency is not used by the others. In this model, we wish to assign frequencies to the ranges, that is, we wish to color the disks rather than the points.

Corollary 8.3.4 ([10]). *Any system $\mathcal{D}$ of n disks in the plane can be colored by $O(\log n)$ colors with the property that for each point p of the plane that is covered by at least one member of $\mathcal{D}$, there is a disk $D_p \in \mathcal{D}$ whose color is "unique," that is, whose color differs from the color of any other disk that contains p.*

Proof. To each point $p = (a, b)$ in $\mathbb{R}^2$, assign the plane p^* in $\mathbb{R}^3$, whose equation is $z = -2ax - 2by + a^2 + b^2$. To each disk D of radius r, centered at $(x, y) \in \mathbb{R}^2$ assign the point $D^* = (x, y, r^2 - x^2 - y^2) \in \mathbb{R}^3$. It is easy to verify that a point p lies in D if and only if the point D^* lies in the half-space above p^*. Thus, Corollary 8.3.4 follows directly from Corollary 8.3.3. □

Har-Peled and Smorodinsky [18] generalized Corollary 8.3.4 to any system of *pseudo-disks*, that is, simply connected regions whose boundary curves have no tangencies and have at most two crossings per pair. They use the probabilistic approach of Clarkson and Shor [8] combined with a lemma of Kedem et al. [20], according to which the boundary of the union of any system $\mathcal{D}$ of n pseudo-disks in the plane consists of at most $O(n)$ "simple" arcs, where an arc is called simple if it belongs to the boundary of a member of $\mathcal{D}$. Smorodinsky [27] designed a quadratic time algorithm for constructing a coloring of n pseudo-disks with $O(\log n)$ colors satisfying the properties in Corollary 8.3.4.

As we have seen before, the independent sets in the Delaunay graph associated with a family of geometric figures play an important role in constructing conflict-free colorings. This raises some interesting questions for *axis-parallel* rectangles in the plane [10].

Given a set P of n points in general position in the plane, define their Delaunay graph $G_{\text{rect}}(P)$ with respect to axis-parallel rectangles, as a graph with vertex set P, whose two elements $p, q \in P$ are connected by an edge if and only there is axis-parallel rectangle containing p and q, which does not contain any other element of P. It was shown in [25] that $G_{\text{rect}}(P)$ always has an independent set of size at least $\Omega\left(\sqrt{n \log n / \log\log n}\right)$. On the other hand, it was recently shown by Chen, Pach, and Szegedy that this bound cannot be improved to linear. They have constructed n-element sets P such that the size of the largest independent set in $G_{\text{rect}}(P)$ is $O(n/\log n)$.

Corollary 8.3.5 ([25]). *Any set of n points in general position in the plane permits a conflict-free coloring using $O(\sqrt{n \log\log n / \log n})$ colors, with respect to the family of all axis-parallel rectangles.*

8.4 Online conflict-free coloring

Several recent papers have addressed *online conflict-free coloring*. Here we maintain a set P of input points. Initially, P is empty, and we repeatedly insert points into P, one point at a time. We denote by $P(t)$ the set P after the t-th point has been inserted. Each time we insert a point p, we need to assign a color $c(p)$ to it, which is a positive integer. Once the color has been assigned to p, it cannot be changed in the future. The coloring should remain conflict-free at all times, with respect to a given set of ranges. That is, as in the static case, for any range I that contains points of $P(t)$, there is a color that appears exactly once in I.

As it turns out, the online version is considerably harder to analyze, even for points on a line, where the ranges are *intervals*. As in the static case, it is easy to establish an $\Omega(\log n)$ lower bound on the number of required colors, where n is the final size of P. A matching upper bound, for points on a line, can be obtained with a randomized coloring algorithm, so that the expected number of colors is $O(\log n)$, but a deterministic solution with such a worst-case bound is not known.

Here is a sketch of the randomized algorithm. Imagine that we have an infinite sequence of available colors, which we identify with the integers $\mathbb{N}$. Each color i is independently assigned a random *label* λ_i from among three fixed labels a, b, c. For each point q in the current set P, let $c(q)$ denote its color. For each integer $j \le c(q)$ we assign to q a label $L_j(q) \in \{a, b, c\}$, in a manner to be described shortly. These labels maintain the following invariant: For each i, let $P_{\ge i}$ denote the subset of all the points currently in P which have been colored by colors $\ge i$. Then, for each color (integer) j, and for each pair of consecutive points $q_1, q_2 \in P_{\ge j}$, we have $L_j(q_1) \ne L_j(q_2)$. That is, the j-th labels are a valid 3-coloring of the path graph that connects all the points that have colors $\ge j$ in their x-order.

When a new point p is inserted, we check, for each color $i \in \mathbb{N}$ in increasing order, whether p can be given color i, and color p with the first legal color. Suppose we have reached color j, and wish to determine whether p can be colored with j. We first give p its j-th label $L_j(p)$, which is a label different from those of its two neighbors (if p is the leftmost or rightmost current point of $P_{\ge j}$, there is more than one choice of label, and any of them will do). If $L_j(p) = \lambda_j$, we give p color j and stop. Otherwise, we proceed to the next potential color $j + 1$ and repeat the process. Note that the process terminates with probability 1.

Correctness is straightforward: Consider any step during the insertion process, and any interval I that contains points of the current set. Let j be the highest color assigned to points of the current set in I. We claim that I contains only one point with color j. Indeed, suppose to the contrary that I contains more than one such point, and let p be the last inserted point from among these points. Then, at the time of its insertion, p has at least one neighbor q in $P_{\ge j}$ that lies in I. By construction, $L_j(q) \ne L_j(p) = \lambda_j$. But this is a contradiction, because only points whose j-th label is equal to λ_j receive color j, so q has failed this test and must have been given a larger color, contrary to assumption.

Efficiency is also easy: We claim that, for each j, among the points of $P_{\ge j}$ at least a third receive color j in expectation. Indeed, consider a newly inserted point p that reaches $P_{\ge j}$. Since the random choice of λ_j is independent of the insertion order (see a comment below), the probability that $L_j(p) = \lambda_j$ is exactly $1/3$, so, conditioned on having reached $P_{\ge j}$, p gets color j with probability $1/3$. This is easily seen to imply that the expected number of points that get color j is $\frac{1}{3}\left(\frac{2}{3}\right)^{j-1} n$. From this, using Markov's inequality, it follows that the number of colors used is $O(\log n)$ with high probability.

We comment that this algorithm assumes an *oblivious adversary*, meaning that the insertion order does not depend in any way on the random choices of the labels λ_j. A non-oblivious adversary can force the algorithm to generate significantly more colors.

This algorithm follows recent work by Bar-Noy *et al.* [2] (who present a more general method), and it has also been discovered independently by S. Olonetsky in an unpublished work. The first paper to address online conflict-free coloring is by Fiat *et al.* [16]. Subsequent work on this problem also appear in Chen [6], Chen *et al.* [7], and Bar-Noy *et al.* [2].

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Chapter 9

From Sam Loyd to László Fejes Tóth: The 15 Puzzle and Motion Planning

9.1 Sam Loyd and the Fifteen Puzzle

Sam Loyd (1841–1911) was one of the greatest puzzle designers of all times. Was he a mathematician? Certainly not, but he could have become a great one. At the age of fourteen, with two of his brothers he joined a chess club in New York and soon he became obsessed with the game. His first chess problem was published in the New York Saturday Courier on 14 April, 1855. One year later, one of his problems won the first prize in a competition run by the New York Clipper, and next year he became the problem editor of Chess Monthly. He began to study engineering, but he abandoned his studies after he discovered that he can make a living by writing newspapers columns, inventing and selling games and puzzles, working as a plumbing contractor and running music stores. His 1878 collection "Chess Strategy," based on his columns in Scientific American is a classic. He invented many surprising new themes and methods, e.g., the so-called *retrograde analysis*, which clearly required not only intuition but disciplined sharp mathematical thinking. He produced over ten thousand puzzles, many of which involved sophisticated mathematical ideas. After Sam Loyd's death, his son published "Sam Loyd's Cyclopaedia of 5000 Puzzles, Tricks and Conundrums," another classic, which is now freely available on the internet [29].

However, Loyd's most famous puzzle was the Fifteen Puzzle (formerly known as "Fourteen-Fifteen Puzzle") that conquered the world in two waves just like Rubik's cube did a hundred years later. The puzzle consists of fifteen moveable unit square blocks, numbered from 1 to 15, arranged in a four-by-four square box. The goal is to bring the squares in the standard position when the they are numbered consecutively. "People became infatuated with the puzzle and ludicrous tales are told of shopkeepers who neglected to open their stores; of a distinguished clergyman who stood under a street lamp all through a wintry night trying to recall the way he had performed the feat. The mysterious feature of the puzzle is that none seem able to remember the sequence of moves whereby they feel sure they have succeeded in solving the puzzle. Pilots are said to have wrecked their ships, and engineers rush their trains past stations. A famous Baltimore editor tells how he went for his noon lunch and was discovered by frantic staff long past midnight pushing little pieces of pie around on a plate! Farmers are known to have deserted their ploughs..." Who wrote this?

Sam Loyd, of course, who was also great advertising expert and self-promoter! But the truth is that at many places playing the puzzle during office hours was strictly prohibited, just like today using e-mail for personal purposes.

Loyd offered a thousand dollars for anyone who can solve the puzzle starting from the position that can be obtained from the standard one by swapping the last two squares. Of course, he knew that his money was safe, and the proof clearly required a precise mathematical argument. In 1978, Herstein and Kaplansky [26] wrote that "No really easy proof seems to be known," but this is not quite true! (Come up with such a proof! [27], [39], [5])

In his obituary in The Times, it was written that "Loyd had a real gift - such as that shown in the 'Curiosa Mathematica' of the Rev. C. L. Dodgson ... for the fantastic in mathematical science, and had he devoted himself to making use of it, might have earned fame as an investigator in the vast and poetic region of pure mathematics, a worthy follower of Cayley and Sylvester."

In 1984, in their FOCS paper [28], Kornhauser, Miller, and Spirakis generalized the Fifteen Puzzle to arbitrary graphs. Consider a graph G of n vertices, and put $k < n$ numbered "coins" ("chips" or "pebbles") on distinct vertices. A *move* consists of shifting a coin from one vertex to a neighboring one. Obviously, one can ask two general algorithmic problems.

1. Given an *initial* and a *target position* of the k coins, is it possible to reach the second position from the first one?
2. If the answer to the first question is yes, design an algorithm for finding the shortest sequence of moves (or a reasonable short one).

According to Kornhauser et al., these problems are relevant to memory management in totally distributed computing systems, where we want to coordinate the transfer of indivisible packets of data between the devices without ever exceeding the capacity of any device. In the above model, each device has unit capacity and each packet occupies a unit memory. The problem can also be studied in the framework of general motion planning problems or "piano movers' problems" [37], [38].

Kornhauser et al. proved that reachability can be decided, that is, the answer to first question can be found, in polynomial time. If the answer is yes, then there is sequence of $O(n^3)$ moves that solves the problem, and this bound is asymptotically best possible.

However, if we want to find the *shortest* number of moves, that is, we want to answer question 2, the problem becomes *NP-hard*. Moreover, as was shown by Ratner and Warmuth [35], it remains NP-hard even if we restrict the problem to the case when G is the $\sqrt{n} \times \sqrt{n}$ grid and $k = n - 1$, which is the direct generalization of the Fifteen Puzzle.

Papadimitriou et al. [33] has further generalized the question by introducing unnumbered movable *obstacles* that can also reside on the vertices. The coins ("robots") are not allowed to collide with one another and with any of the obstacles. In any step we may move either a robot or an obstacle across a single edge. The problem is already interesting in the case when there is a *single* robot and several obstacles. Our goal is to bring the robot from a starting position (vertex) to a target one, while the final position of obstacles is irrelevant. It was shown that to decide whether this can be achieved by making at most M moves is NP-complete, even when restricted to a planar graph [33].

Auletta et al. [4] found a linear time algorithm for solving the reachability problem on a tree

with k robots and no obstacles.

In certain applications, objects are indistinguishable, therefore the chips (coins) are unlabelled; for instance, a modular robotic system consists of a number of identical modules (robots), each of which having identical capabilities [18], [19]. In this variant, the problem is always easier and feasible (in every nontrivial setting).

In some other variants, there is no reason to restrict the movement of the chips to a graph; any collision-free movement in the plane or in a region is permitted.

9.2 Unlabelled Coins in Graphs and Grids

In this section, we use a different notion of *moves*, which is borrowed from the motion planning model. Given a connected graph G, a move from a vertex v_1 to a vertex v_2 is defined as shifting a chip from v_1 to v_2 along a path in G so that no intermediate vertices are occupied.

Let V and V' be two n-element subsets of $V(G)$. Imagine that we place a chip at each element of V and we want to move them into the positions of V' (V and V' may have common elements). A move is called a *target move* if it moves a chip to a final target position belonging to V. Otherwise it is a *non-target* move.

Theorem 9.2.1. *In G one can get from any n-element initial configuration V to any n-element final configuration V' using at most n moves, so that no chip moves twice. Moreover, for the case of a tree T with r vertices, there is a $O(r)$-time algorithm which performs the optimal (minimum) number of moves.*

Proof. It is sufficient to prove the theorem for trees. We argue by induction on the number of chips. Take the smallest tree T containing V and V', and consider an arbitrary leaf l of T. Assume first that the leaf l belongs to V: say $l = v$. If v also belongs to V', the result trivially follows by induction, so assume that this is not the case. Choose a path P in T, connecting v to an element v' of V' such that no internal point of P belongs to V'. Apply the induction hypothesis to $V \setminus \{v\}$ and $V' \setminus \{v'\}$ to obtain a sequence of at most $n-1$ moves, and add a final (unobstructed) move from v to v'.

The remaining case when the leaf l belongs to V' is symmetric: say $l = v'$; choose a path P in T, connecting v' to an element v of V such that no internal point of P belongs to V. Move first v to v' and append the sequence of at most $n-1$ moves obtained from the induction hypothesis applied to $V \setminus \{v\}$ and $V' \setminus \{v'\}$.

We further refine this algorithm so as to minimize the number of moves. We call a vertex that is both a start and target position an *obstacle*. We have four types of vertices: free vertices, chip-only vertices, target-only vertices, and obstacles. Denote by c (resp. t) the number of chip-only (resp. target-only) vertices, and by o the number of obstacles. We have $c + o = o + t = n$. We call a tree *balanced* if it contains an equal number of chip-only and target-only vertices. Clearly, the initial tree T is balanced. If there exists an obstacle whose removal from T breaks T into balanced subtrees, we keep this obstacle fixed and proceed recursively (by induction) on the subtrees. If no obstacle removal breaks T into balanced subtrees, then all obstacles must move (each at least once), hence the number of moves necessary is at least $o + c = n$, and the algorithm in the first part of our proof can be used to obtain an optimal schedule.

The observation above, together with postorder traversal keeping additional information for every node, is the basis of the linear time algorithm which we omit due to lack of space. □

Corollary 9.2.2 ([15]). *In the infinite rectangular grid, we can get from any starting position to any ending position of the same size n in at most n moves.*

Using the intractability of the *Set Cover* problem, one it is not hard to see that to the determine the *smallest* number of moves necessary to solve the above *Graph Reconfiguration* (GR) problem is NP-hard.

Theorem 9.2.3 ([15]). *The Graph Reconfiguration problem is NP-complete. Moreover, assuming $P \neq NP$, there is an absolute constant $\varepsilon_1 > 0$ such that no polynomial-time algorithm has approximation guarantee at most $1 + \varepsilon_1$. That is, the problem is APX-hard.*

Proof. The decision version of the problem is clearly in NP, so we only have to prove its NP-hardness. We reduce the *set cover* problem SC to GR. An instance of the set cover problem consists of a family $\mathcal{F}$ of subsets of a finite set U. The problem is to decide whether there is a set cover of size k for $\mathcal{F}$, i.e., a subset $\mathcal{F}' \subseteq \mathcal{F}$, with $|\mathcal{F}'| \leq k$, such that every element in U belongs to at least one member of $\mathcal{F}'$. SC is known to be NP-complete [23].

Consider an instance of SC represented by a bipartite graph $(B \cup C, E)$, where $U = C$, $\mathcal{F} = B$, and edges describe the membership relation. Construct the undirected graph G shown in Fig. 9.1, with $|A| = |C|$. The chips are $S = A \cup B$ and the targets are $T = B \cup C$. Clearly, G can be constructed

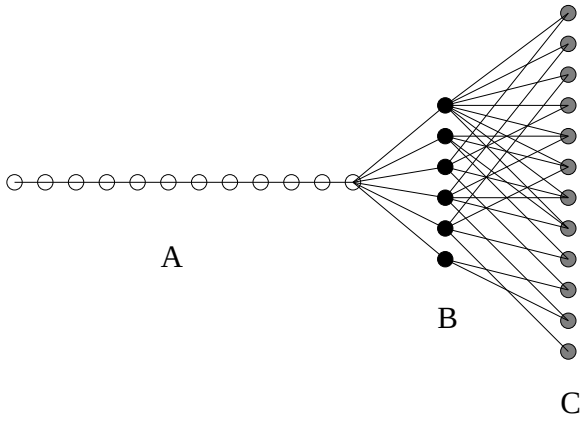

FIGURE 9.1. The "broom" graph G corresponding to a set cover instance with $|U| = 12$, and $|\mathcal{F}| = 6$. The vertices occupied only by chips are white, those occupied by both chips and targets are black, and those occupied only by targets are shaded. An optimal reconfiguration takes 15 moves (an optimal set cover has size 3).

in polynomial time. The reduction is complete once we establish the following claim, whose proof is left to the reader.

Claim 9.2.4. *There is a set cover consisting of at most q sets if and only if reconfiguration in G can be done using at most $|A| + q$ moves.*

To prove the approximation hardness, we use the same reduction and the fact that 3-SC, the set cover problem in which the size of each set in $\mathcal{F}$ is bounded from above by 3 is APX-hard [34], [2]. □

Theorem 9.2.5 ([15]). *There is a 3-approximation algorithm for estimating the minimum number of moves needed to solve the Graph Reconfiguration problem.*

The algorithm is based on Bar-Yehuda's local ratio algorithm [8].

It is interesting to note that, if we count as a move every *edge* traversed by a chip (as we did in Section 9.1), we can easily minimize the number of moves in *polynomial time*, as follows. Construct a complete weighted bipartite graph $B = (V \cup V', F)$ with bipartition V: the vertices containing chips and V': the vertices containing targets (with obstacles in both sides of the bipartition). The weight of an edge in F is equal to the length of the shortest path connecting the endpoints of the edge in G. Now apply an algorithm for Minimum Weight Perfect Matching in B, and move accordingly: if the path a chip c_1 would take to reach its destination has another chip c_2 on it, have the two chips switch destinations and continue moving c_2. One can check that the number of moves does not exceed the weight of the perfect matching. On the other hand, the optimum solution must move chips to targets and cannot do better than the total length of the shortest paths in a minimum matching.

9.3 László Fejes Tóth and Sliding Coins

Consider a set (system) of n pairwise disjoint, unlabelled, congruent disks ("coins") in the plane that need to be brought from a given start (initial) configuration S into a desired goal (target) configuration T. In one *move*, we are allowed to take one coin and slide it to another position without colliding with the other coins.

It is easy to see that for the class of congruent disks this problem is always feasible. More generally, it is also feasible for the class of all convex objects.

This follows from an old result (Theorem 9.3.1) that appears in the work of Fejes Tóth and Heppes [22], but it can be traced back to de Bruijn [13]; the algorithmic aspects of the problem have been studied by Guibas and Yao [25]. We refer to this set of motion rules (moves) as the *sliding model*.

Theorem 9.3.1. *Any set of n convex objects in the plane can be separated via translations all parallel to any given fixed direction, with each object moving once only. If the topmost and bottommost points of each object are given (or can be computed in $O(n \log n)$ time), an ordering of the moves can be computed in $O(n \log n)$ time.*

The following simple *universal* algorithm that can be adapted to any set of n convex objects performs $2n$ moves for reconfiguration of n disks. In the first step (n moves), in decreasing order of the x-coordinates of their centers, slide the disks initially along a horizontal direction, one by one to the far right. Note that no collisions can occur. In the second step (n moves), bring the disks "back" to target positions in increasing order of the x-coordinates of their centers. (General convex objects need rotations and translations in the second step). Already for the class of all (not necessarily congruent) disks, one cannot do much better in terms of the number of moves. For the

class of congruent segments (as objects), it is easy to construct examples that require $2n-1$ moves for reconfiguration.

As in the previous section, we call a move a *target move* if it slides a disk to a final target position. Otherwise, it is a *non-target* move. Our *lower* bounds use the following argument: if no target disk coincides with a start disk (so each disk must move), a schedule with x non-target moves consists of at least $n+x$ moves.

In order to obtain an *upper* bound for the number of moves required for the reconfiguration in the worst case, we need to prove a geometric lemma of independent interest: there always exists a line bisecting the set of centers of the start disks into two almost equal parts such that the strip of width 6 around it contains only a small number of disks. A slightly weaker statement guaranteeing the existence of a bisecting line that cuts through few disks was given by Alon *et. al* [3]. We include our almost identical proof, for completeness.

We say that a line ℓ bisects an n-element point set into two *almost equal* parts if the number of points on one side of ℓ differs from the number of points on the side by at most one.

Lemma 9.3.2. *Let S be a set of n pairwise disjoint unit (radius) disks in the plane. Then there exists a line ℓ bisecting the set of centers of the disks into two almost equal parts such that the parallel strip of width 6 around ℓ contains entirely at most $O(\sqrt{n\log n})$ disks.*

Proof. Set $m = c_2\sqrt{n\log n}$ where $c_2 > 0$ is a suitable large constant to be chosen later. Assume for contradiction that the strip of width $w = 6$ around each line bisecting the set of centers of S into two almost equal parts contains at least m disks. Set $k = \lceil \sqrt{n/\log n} \rceil$ and consider the k bisecting lines that form angles $i\theta$ with the positive direction of the x-axis (in counterclockwise order), where $i = 0, \ldots, k-1$, and $\theta = \pi/k$.

Let A_i be the set of disks contained (entirely) in the i-strip of width $w = 6$ around the ith bisecting line, $i = 0, \ldots, k-1$. Clearly, we have

$$n \geq |A_0 \cup \ldots A_{k-1}| \geq \sum_{i=0}^{k-1} |A_i| - \sum_{0\leq i<j\leq k-1} |A_i \cap A_j| \tag{9.3.1}$$

by the inclusion-exclusion formula. By our assumption, $\sum_{i=0}^{k-1} |A_i| \geq km$. The summand $|A_i \cap A_j|$ counts the number of disks contained in the intersection of the strips i and j. This intersection is a rhombus whose area is

$$F_{ij} = \frac{w^2}{\sin(j-i)\theta}.$$

Since the disks are pairwise disjoint,

$$|A_i \cap A_j| \leq \frac{F_{ij}}{\pi}.$$

We thus have

$$\sum_{0\leq i<j\leq k-1} |A_i \cap A_j| = O\left(\sum_{0\leq i<j\leq k-1} \frac{1}{\sin(j-i)\theta}\right).$$

The identity $\sin\alpha = \sin(\pi-\alpha)$ yields

$$\sum_{0\leq i<j\leq k-1} \frac{1}{\sin(j-i)\theta} \leq k \sum_{i=1}^{\lfloor k/2\rfloor} \frac{1}{\sin i\theta}.$$

For $1 \leq i \leq k/2$

$$\frac{1}{\sin i\theta} = \frac{1}{\sin \frac{i\pi}{k}} = O\left(\frac{k}{i}\right).$$

Consequently the second sum in Equation (9.3.1) is bounded as follows:

$$\sum_{0 \leq i < j \leq k-1} |A_i \cap A_j| = O\left(k^2 \sum_{i=1}^{\lfloor k/2 \rfloor} \frac{1}{i}\right) = O(k^2 \log k).$$

Let $c_1 > 0$ be an absolute constant such that $\sum_{0 \leq i < j \leq k-1} |A_i \cap A_j| \leq c_1 \cdot k^2 \log k$. Since $\log k \leq (\log n)/2$ for $n \geq 16$, and using the above estimates, Equation (9.3.1) can be rewritten as

$$n \geq mk - c_1 \cdot k^2 \log k \geq c_2 \sqrt{n \log n} \sqrt{\frac{n}{\log n}} - 2c_1 \frac{n}{\log n} \frac{\log n}{2} = (c_2 - c_1)n.$$

Take now $c_2 = c_1 + 2$, and obtain $n \geq 2n$ which is a contradiction. □

At first glance one would conjecture that Lemma 9.3.2 holds with $O(\sqrt{n})$ in the place of $O\left(\sqrt{n \log n}\right)$, but surprisingly this is not the case!

Theorem 9.3.3 ([9]). *Given a pair of start and target configurations S and T, consisting of n congruent disks each, $\frac{3n}{2} + O(\sqrt{n \log n})$ moves always suffice for transforming the start configuration into the target configuration. The entire motion can be computed in $O(n^{3/2}/(\log n)^{1/2})$ time. On the other hand, there exist pairs of configurations that require $\left(1 + \frac{1}{15}\right) n - O(\sqrt{n})$ moves for this task.*

Proof. We start with the upper bound. Let S' and T' be the centers of the start disks and target disks, respectively, and let ℓ be the line guaranteed by Lemma 9.3.2. Without loss of generality we can assume that ℓ is vertical. Denote by $s_1 = \lfloor n/2 \rfloor$ and $s_2 = \lceil n/2 \rceil$ the number of centers of start disks to the left and to the right of ℓ. Let $m = O(\sqrt{n \log n})$ be the number of start disks contained in the vertical strip around ℓ. Denote by t_1 and t_2 the number of centers of target disks to the left and to the right of ℓ, respectively. By symmetry we can assume that $t_1 \leq n/2 \leq t_2$.

Let R be a region containing all start and target disks (e.g., the smallest axis-aligned rectangle that contains all disks). The algorithm has three steps. All moves in the region R are taken along horizontal lines, i.e., perpendicularly to the line ℓ.

Step 1 Slide to the far right all start disks whose centers are to the right of ℓ and the (other) start disks in the strip, one by one, in decreasing order of their x-coordinates (with ties broken arbitrarily). At this point all $t_2 \geq n/2$ target disks whose centers are right of ℓ are free.

Step 2 Using all the $s_1' \leq n/2$ remaining disks whose centers are to the left of ℓ, in increasing order of their x-coordinates, we fill free target positions to the right of ℓ, in increasing order of their x-coordinates: each disk slides first to the left, then to the right on a wide arc and to the left again in the end. Note that $s_1' \leq n/2 \leq t_2$. Now all the target positions whose centers are to the left of ℓ are free.

Step 3 Move to place the far away disks: first continue to fill target positions whose centers are to the right of ℓ, in increasing order of their x-coordinates. When we are done, we fill target positions whose centers are to the left of ℓ, in decreasing order of their x-coordinates. Note that at this point all target positions to the left of ℓ are "free."

The only non-target moves are those done in STEP 1 and their number is $n/2 + O(\sqrt{n \log n})$, so the total number of moves is $3n/2 + O(\sqrt{n \log n})$.

Algorithm. A trivial implementation of the algorithm examines all $k = \lceil \sqrt{n/\log n} \rceil$ strip directions each in $O(n)$ time, in order to find a suitable one, as described in the proof of Lemma 9.3.2. After that, $O(n \log n)$ time is spent for this direction for sorting and performing the moves. The resulting time complexity is $O(n^{3/2}(\log n)^{-1/2})$.

Lower bound. The target configuration consists of a set of n densely packed unit (radius) disks contained, for example, in a square of side length $\approx 2\sqrt{n}$. The disks in the start configuration enclose the target positions in a ring-like structure with long "legs." Its design is more complicated and uses "rigidity" considerations as described below.

Following László Fejes Tóth, we say that a packing $\mathcal{C}$ of unit (radius) disks in the plane is *stable* if each disk is kept fixed by its neighbors [12]. More precisely, $\mathcal{C}$ is stable if none of its elements can be translated by any small distance in any direction without colliding with the others. It is easy to see that any stable system of (unit) disks in the plane has infinitely many elements. Refuting a conjecture of Fejes Tóth, K. Böröczky [11] showed that there exist stable systems of unit disks with arbitrarily small density.

The main building block used in Böröczky's construction was a one-way infinite "bridge" made up of disks, which can be defined as follows. In Fig. 9.2, the initial section of such a one-way infinite bridge appears on the left of the vertical line ℓ. Fix an x-y rectilinear coordinate system in the plane.

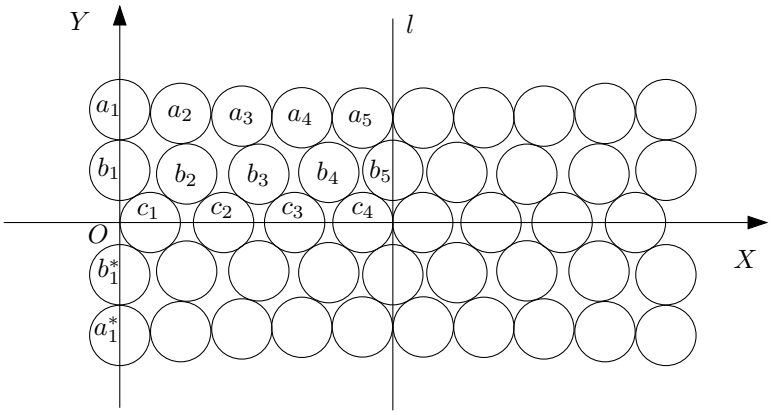

FIGURE 9.2. A double bridge and its vertical line of symmetry ℓ. The part left of ℓ forms the initial section of a one-way infinite bridge.

Let us start with five unit disks centered at

$$a_1 = (0, 2 + \sqrt{3}), b_1 = (0, \sqrt{3}), c_1 = (1, 0), b_1^* = -b_1, \quad a_1^* = -a_1,$$

that serve as an "abutment." The bridge will be symmetric about the x-axis, so it is sufficient to describe the part of the packing in the upper half-plane. The set of centers of the disks is denoted by C.

Take a strictly convex function $f(x)$ defined for all $x \geq 0$ such that $f(0) = 2 + \sqrt{3}$ and $\lim_{x \to \infty} f(x) = 2\sqrt{3}$. Starting with a_1, choose a series of points $a_2, a_3, a_4, \ldots$ belonging to the graph

of f such that the distance between any two consecutive points satisfies

$$|a_i - a_{i+1}| = 2 \quad (i = 1, 2, 3, \ldots).$$

All unit disks around these points belong to the packing, so that $a_i \in C$ for every i. These points will uniquely determine all other elements of C, according to the following rules.

Let b_2 be the point at distance 2 from both c_1 and a_2, which lies to the right of the line c_1a_2. Once b_2 is defined, let c_2 be the point on the x-axis, different from c_1, whose distance from b_2 is 2. In general, if b_i and c_i have already been defined, let b_{i+1} denote the point at distance 2 from both c_i and a_{i+1}, lying on the right-hand side of their connecting line, and let $c_{i+1} \neq c_i$ be the (other) point of the x-axis at distance 2 from b_{i+1}. Let C, the set of centers of the disks forming the bridge, consist of all points a_i, b_i, c_i $(i = 1, 2, 3, \ldots)$ and their reflections about the x-axis. Note that the points $c_i \in C$ lie on the x-axis, so they are identical with their reflections.

We need four properties of this construction, whose simple trigonometric proofs can be found in [11]:

1. the distance between any two points in C is at least 2;
2. all unit disks around a_i, b_i, c_i $(i = 2, 3, 4, \ldots)$ are kept fixed by their neighbors;
3. all points $b_2, b_3, b_4, \ldots$ lie strictly below the line $y = \sqrt{3}$;
4. the x-coordinate of c_i is smaller than that of a_{i+1} $(i = 1, 2, 3, \ldots)$.

It is not hard to see that the difference between the x-coordinates of c_i and a_{i+1} tends to zero as i tends to infinity.

Next, we slightly modify the above construction. Take a small positive ε and replace $f(x)$ by the strictly convex function

$$f_\varepsilon(x) := (1 + \varepsilon)f(x) - \varepsilon f(0)$$

whose asymptote is the line $y = 2\sqrt{3} - (2 - \sqrt{3})\varepsilon$. Clearly, $f_\varepsilon(0) = f(0)$. If we carry out the same construction as above, nothing changes before we first find a point a_i that lies below the line $y = 2\sqrt{3}$. However, if ε is sufficiently large, sooner or later we get stuck: the construction cannot be continued forever without violating any of the conditions listed above. Let k be the first integer for which such an event occurs, involving a_k, a_{k+1}, b_k, or c_k. By varying $\varepsilon > 0$, it can be shown by a simple case analysis that the construction can be realized up to level k so that the difference between the x-coordinates of b_k and a_k is 1. It follows that the disk around a_k is tangent to the vertical line ℓ passing through b_k. Remove the rightmost disk centered at c_k from the set. Thus from the above condition, by taking the union of the part of C built so far together with its reflection about ℓ, we obtain the following:

Lemma 9.3.4. *There exist arbitrarily long finite packings ("double-bridges") consisting of five rows of unit disks, symmetric about the coordinate axes, in which all but eight disks are kept fixed by their neighbors. These eight exceptional disks are at the two abutments of the double-bridge and their y-coordinates are $\pm\sqrt{3}, \pm(2 + \sqrt{3})$.*

Notice that three such bridges can be connected at a "junction" depicted in Fig. 9.3 so that the angles between their "long" half-axes of symmetries (corresponding to the positive x-axis) are

$\frac{2\pi}{3}$. Consequently, using six double-bridges connected by six junctions one can enclose an arbitrarily large hexagonal region H. Let us attach a one-way infinite bridge to each of the unused sides of the junctions. As Böröczky pointed out, the resulting packing is stable.

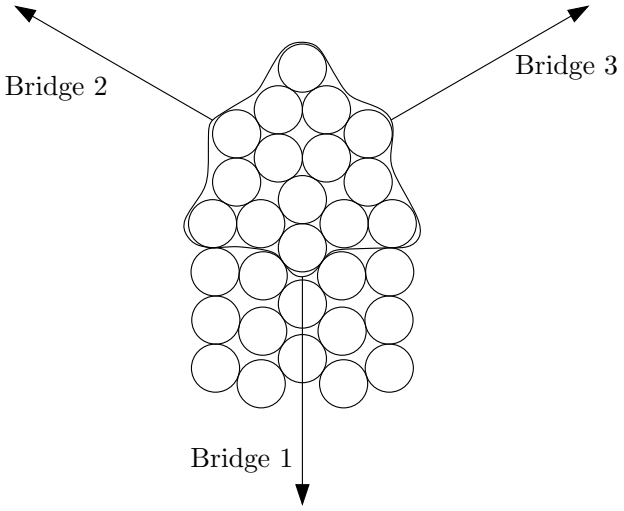

FIGURE 9.3. Junction of type 1.

Let us refer to the disks in the start (resp. target) configuration as white (resp. black) disks. Now fix a large n, and take n white disks. Use $O(\sqrt{n})$ of them to build six junctions connected by six double-bridges (as described above) to enclose a hexagonal region that can accommodate the n nonoverlapping black disks. See also Fig. 9.4. Divide the remaining white disks into six roughly equal groups, each of size $\frac{n}{6} - O(\sqrt{n})$, and rearrange each group to form the initial section of a one-way infinite bridge attached to the unused sides ("ports") of the junctions. Notice that the number of necessary moves is at least $\left(1 + \frac{1}{30}\right) n - O(\sqrt{n})$. To see this, it is enough to observe, that in order to fill the first target, we have to break up the hexagonal ring around the black disks. That is, we have to move at least one element of the six double-bridges enclosing H. However, with the exception of the at most $6 \times 5 = 30$ white disks at the far ends of the truncated one-way infinite bridges, every white disk is fixed by its neighbors. Each of these bridges consists of five rows of disks of "length" roughly $\frac{n}{30}$, where the length of a bridge is the number of disks along its side. Therefore, before we could move any element of the ring around H, we must start at a far end and move a sequence of roughly $\frac{n}{30}$ white adjacent disks.

Instead of enclosing the n black disks by a hexagon, we can construct a triangular ring T around them, consisting of three double-bridges (see Fig. 9.4). To achieve this, we have to build a junction of three sides establishing a connection between the abutments of three bridges such that the angles between their half-axes of symmetry are $\frac{5\pi}{6}$, $\frac{5\pi}{6}$, and $\frac{\pi}{3}$. Such a junction is shown on Fig. 9.5. The convex hull of the disk centers (for the disks in the junction) is a pentagon symmetric with respect to a vertical line passing through the top vertex. Four out of the five centers along each of the three sides of the pentagon connected to bridges are collinear. The disk centers on the other two sides form two slightly concave chains. The number of necessary moves is at least $\left(1 + \frac{1}{15}\right) n - O(\sqrt{n})$ for this second construction. This completes the proof of Theorem 9.3.3. □

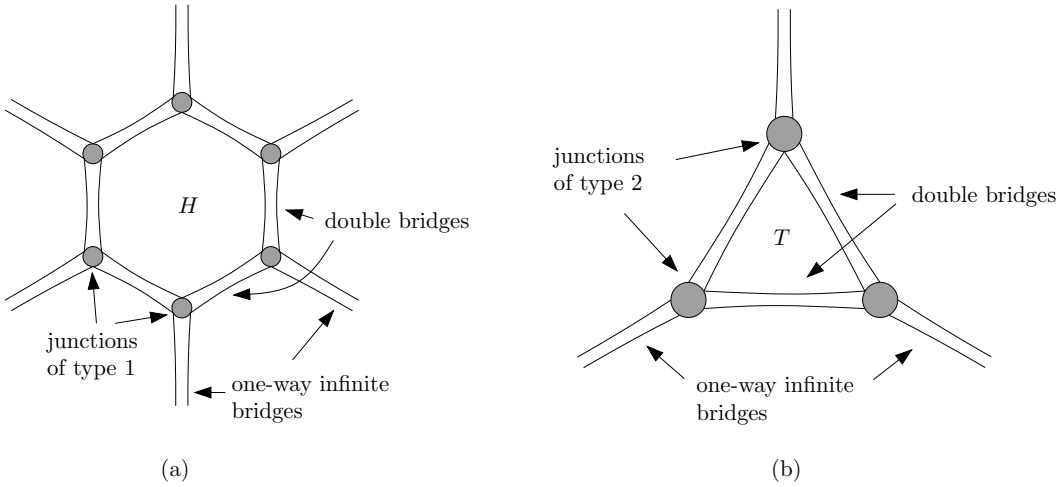

FIGURE 9.4. Two start configurations based on hexagonal and triangular rings.

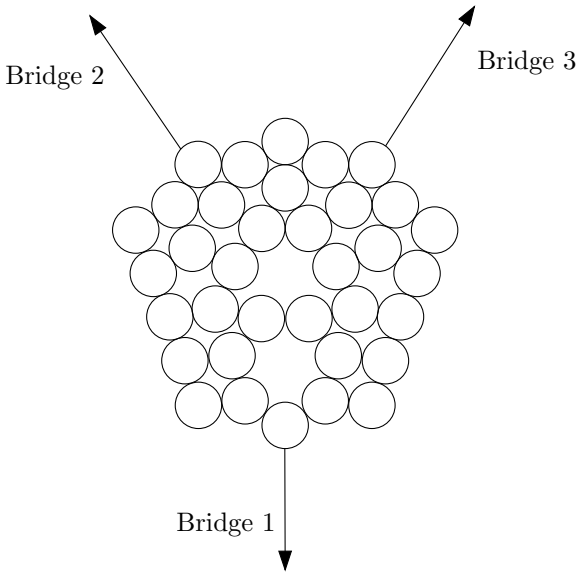

FIGURE 9.5. Junction of type 2.

László Fejes Tóth's (1915–2005) name appeared and ideas were used at two crucial points in this argument. In fact, it is not a coincidence! He played a pioneering role in creating the field that is called today *Discrete and Computational Geometry*. In the 1940's he started to explore the structural properties of 2- and 3-dimensional packings and coverings with disks, spheres, and other convex sets. Apart from some scattered results by Thue and Kershner, all previous work in this area concerned lattice packings and lattice coverings. Lattices play a fundamental role in number theory and in the geometry of numbers, but restricting investigations to lattice arrangements appeared to be an artificial exercise from a geometric point of view. Fejes Tóth's idea was that the solution of many extremal problems would remain the same if this restriction were dropped. For example, he showed that a densest packing of congruent copies of a centrally symmetric convex set in the plane is necessarily latticelike. In a similar spirit he proved that most regular polytopes can be obtained as the unique solution to a suitable "natural" optimization problem, in the same way that the regular hexagonal honeycomb is, in a certain well defined sense, the most economical structure to house bees. This area of geometry can be regarded as the "genetics" of symmetric objects.

In 1953, Fejes Tóth summarized his results in the seminal book, *Lagerungen in der Ebene auf der Kugel und im Raum* [21] that appeared in the prestigious series of Springer-Verlag: Die Grundlehren der Mathematischen Wissenschaften. It is hard to overestimate the impact of this monograph. As C. A. Rogers wrote, "until recently, the theory of packing and covering was not sufficiently well developed to justify the publication of a book devoted exclusively to it. After the publication of L. Fejes Tóth's book in 1953, there would be no need for a second work on the subject..."

László Fejes Tóth was a high school teacher for many years. Perhaps that is why he loved simply stated problems that have an aesthetic appeal and can be explained to a layman. In fact, he is a master of raising such questions, and he knows exactly who the right people are to tell them to. Because of these special abilities, his straightforward, modest style, and his warm personality, he became one of the most influential discrete geometers of the second half of the twentieth century. He passed away in 2005, at the age of 90.

9.4 Pushing Squares Around

A modular *metamorphic* system consists of a number of identical modules that can connect to, disconnect from, and relocate relative to adjacent modules (see, for example, [16], [30], [32], [36], [42]). While individual modules are not capable of moving by themselves, the entire system may be able to reconfigure or move to a new position, when its members repeatedly change their positions relative to their neighbors, by rotating or sliding around other modules [14], [30], [41], or by expansion and contraction [36]. It is usually assumed that the entire system must remain connected during reconfiguration.

The *motion planning* problem for such a system is that of computing a sequence of module motions that brings the system from a given initial configuration I into a desired goal configuration F. Depending on the existence of such a sequence of motions, we say that the problem is *feasible* or respectively, *infeasible*.

For instance, in [16], [32] upper and lower bounds on the number of moves needed to change I to F are discussed for a system of hexagonal modules. In [31] it is shown that in such a system any two connected configurations are mutually reachable as long as they do not contain a certain prohibited pattern. Demaine et. al. [17] have considered a family of one-player games, involving the

movement of coins from one configuration to another. Moves are restricted so that a coin can only be placed in a free position adjacent to at least two other coins, in contrast to our motion rules that are required to maintain overall connectedness throughout the reconfiguration process.

Consider a plane that is partitioned into a rectangular integer grid of square cells indexed by their center coordinates in the underlying x-y coordinate system. Of the eight *adjacent* cells of cell $c = c_{x,y}$ in the E $(+x)$, W $(-x)$, N $(+y)$, S $(-y)$, NE, SE, NW and SW directions, the four in the E, W, N and S directions are said to be *side-adjacent* to c, while the other four in the NE, SE, NW and SW directions are said to be *corner-adjacent* to c. We denote by $N(c)$ (resp. $NE(c)$) the cell side-adjacent to c in the N direction (resp. the cell corner-adjacent to c in the NE direction). Similar notation is used to denote the cells side-adjacent or corner-adjacent to c in the other axis and diagonal directions.

At any time each cell may be empty or occupied by a module. The *reconfiguration* of a metamorphic system consisting of n modules is a sequence of configurations (distributions) of the modules in the grid at discrete time steps $t = 0, 1, 2, \ldots$, see below. Let V_t be the configuration of the modules at time t, where we often identify V_t with the set of cells occupied by the modules or with the set of their centers. We are only interested in configurations that are connected, i.e., for each t, the graph $G_t = (V_t, E_t)$ must be connected, where for any t, E_t is the set of edges connecting pairs of cells in V_t that are side-adjacent. V_t *yields* V_{t+1} when one module m moves from its current location to new location in step t. In this paper we restrict ourselves to sequential reconfiguration, in which only one module moves at each discrete time step, as explained above. Note that, according to the above definition, the pattern (set of cells or set of integer points) V_t uniquely determines the edge set E_t so that the graph G_t can be characterized by its vertex set V_t. The union of all closed squares belonging to V_t is a connected point set of area $|V_t|$, which will be denoted by $S(V_t)$.

The following two generic motion rules (Figure 9.6) define the *rectangular model*. These are to be understood as possible in all axis parallel orientations, in fact generating 16 rules, eight for rotation and eight for sliding.

- *Rotation*: A module m side-adjacent to a stationary module f rotates through an angle of 90° around f either clockwise or counterclockwise. Figure 9.6(a) shows a clockwise NE rotation. For rotation to take place, both the target cell and the cell at the corresponding corner of f that m passes through (NW in the given example) have to be empty.
- *Sliding*: Let f_1 and f_2 be stationary cells that are side-adjacent. A module m that is side-adjacent to f_1 and adjacent to f_2 slides along the sides of f_1 and f_2 into the cell that is adjacent to f_1 and side-adjacent to f_2. Figure 9.6(b) shows a sliding move in the E direction. For sliding to take place, the target cell has to be empty.

In order to ensure motion precision, each move is guided by one or two modules that are stationary during the same step. The two motion rules of this model also appear in [19], [20]. A somewhat similar model is presented in [14].

Theorem 9.4.1 ([18]). *The set of motion rules of the rectangular model guarantees the feasibility of motion planning for any pair of connected configurations V and V' having the same number of modules. That is, following the above rules, V and V' can always be transformed into each other so that all intermediate configurations are connected.*

A set of modules that form a straight-line chain in the grid is called a *straight chain*. It is easy to construct examples so that neither sliding nor rotation alone can reconfigure them to straight

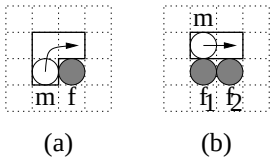

FIGURE 9.6. (a) Clockwise NE rotation and (b) sliding in the E direction. Fixed modules are shaded. The cells in which the moves take place are outlined in the figure.

chains. However, it was proved in [18] that the motion rules of the rectangular model (rotation and sliding, Figure 9.6) are sufficient to guarantee reachability, while maintaining the system connected at each discrete time step. Moreover, this can be achieved by a $O(n^3)$ time algorithm. We conjecture that there exists a quadratic time algorithm for this task.

At this point we cannot resist mentioning a similar question. A configuration consisting of unit cubes of integer coordinates in d-space is called an *animal* if the boundary of their union is homeomorphic to a $(d-1)$-sphere. It is easy to see that in the plane, any animal can be transformed into any other by adding or removing one square at a time so that all intermediate configurations are animals. The corresponding statement in higher dimensions is not known to be true. There exist, however, relatively small animals in 3-space with the property that no cube can be removed from them without violating the condition. This is in sharp contrast to the situation in the plane.

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