The 2-Center Problem in Three Dimensions*

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ABSTRACT

Let P be a set of n points in $\mathbb{R}^3$. The *2-center problem* for P is to find two congruent balls of the minimum radius whose union covers P. We present two randomized algorithms for computing a 2-center of P. The first algorithm runs in $O(n^3 \log^8 n)$ expected time, and the second algorithm runs in $O(n^2 \log^8 n/(1-r^*/r_0)^3)$ expected time, where r^* is the radius of the 2-center of P and r_0 is the radius of the smallest enclosing ball of P. The second algorithm is faster than the first one as long as r^* is not very close to r_0, which is equivalent to the condition of the centers of the two balls in the 2-center of P not being very close to each other.

Categories and Subject Descriptors

F.2.2 [**Analysis of algorithms and problem complexity**]: Non-numerical algorithms and problems—*Geometrical problems and computations*; I.5.3 [**Pattern recognition**]: Clustering—*algorithms*

General Terms

Algorithms, Theory

Keywords

2-center problem, facility location, geometric optimization, intersection of congruent balls, spherical polytopes, multi-dimensional parametric searching

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1. INTRODUCTION

Background. Let $P = \{p_1, \ldots, p_n\}$ be a set of n points in $\mathbb{R}^3$. The *2-center problem* for P is to find two congruent balls of the minimum radius whose union covers P. This is a special case of the general p-center problem in $\mathbb{R}^d$, which calls for covering a set P of n points in $\mathbb{R}^d$ by p congruent balls of the minimum radius. If p is part of the input, the problem is known to be NP-complete [28] even for $d = 2$, so the complexity of the algorithms for solving the p-center problem, for any fixed p, is expected to increase exponentially with p. Agarwal and Procopiuc showed that the p-center problem in $\mathbb{R}^d$ can be solved in $n^{O(p^{1-1/d})}$ time [3], improving upon a naive $n^{O(p)}$-solution. At the other extreme end, the 1-center problem (also known as the smallest enclosing ball problem) is known to be an LP-Type problem, and can thus be solved in $O(n)$ randomized expected time in any fixed dimension, and also in deterministic linear time [16, 26, 27]. Faster approximate solutions to the general p-center problem have also been proposed [3, 5, 6].

If d is not fixed, the 2-center problem in $\mathbb{R}^d$ is NP-Complete [29]. The 2-center problem in $\mathbb{R}^2$ has a relatively rich history, mostly in the past two decades. Hershberger and Suri [22] showed that the decision problem of determining whether P can be covered by two disks of a given radius r, can be solved in $O(n^2 \log n)$ time. This has led to several nearly-quadratic algorithms [4, 19, 23] that solve the optimization problem, the best of which, due to Jaromczyk and Kowaluk [23], runs in $O(n^2 \log n)$ deterministic time. Sharir [33] considerably improved these bounds and obtained a deterministic algorithm with $O(n \log^9 n)$ running time. His algorithm combines several geometric techniques, including parametric searching, searching in monotone matrices, and dynamic maintenance of planar configurations. Chan [13] (following an improvement by Eppstein [20]) improved the running time to $O(n \log^2 n \log^2 \log n)$.

The only earlier work on the 2-center problem in $\mathbb{R}^3$ we are aware of is by Agarwal *et al.* [2], which presents an algorithm with $O(n^{3+\varepsilon})$ running time, for any $\varepsilon > 0$. It uses a rather complicated data structure for dynamically maintaining upper and lower envelopes of bivariate functions.

Our results. We present two randomized algorithms for the 2-center problem in $\mathbb{R}^3$. We first present an algorithm whose expected running time is $O(n^3 \log^8 n)$. It is conceptually a natural generalization of the earlier algorithms for the planar 2-center problem [4, 19, 23]; its implementation however is considerably more involved. The second algorithm runs in $O(n^2 \log^8 n/(1-r^*/r_0)^3)$ expected time, where r^* is the radius of the 2-center and r_0 is the radius of the smallest enclosing ball of P. This is based on some of the ideas in Sharir's planar algorithm [33], but requires several new ideas. As in the previous algorithms, we first present algorithms for the decision problem: given $r > 0$, determine whether P can be covered by two balls of radius r. We then combine it with an adaption of Chan's randomized optimization technique [12] to obtain a solution for the optimization problem. The asymptotic expected running time of the optimization algorithm is the same as that of the decision procedure.

The paper is organized as follows. Section 2 briefly sketches our two solutions. Section 3 presents the near-cubic algorithm, and Section 4 presents the improved algorithm.

2. SKETCHES OF THE SOLUTIONS

The near-cubic algorithm. To solve the decision problem, in the less efficient but conceptually simpler manner, we use a standard point-plane duality, and replace each point $p \in P$ by a dual plane p^*, and each plane h by a dual point h^*, such that the above-below relations between points and planes are preserved. We note that if P can be covered by two balls B_1, B_2 (not necessarily congruent), then there exists a plane h (containing the circle $\partial B_1 \cap \partial B_2$, if they intersect at all, or separating B_1 and B_2 otherwise) separating P into two subsets P_1, P_2, such that $P_1 \subset B_1$ and $P_2 \subset B_2$. We therefore construct the arrangement $\mathcal{A}$ of the set $\{p^* \mid p \in P\}$ of dual planes. It has $O(n^3)$ cells, and each cell τ has the property that, for any point $w \in \tau$, its primal plane w^* separates P into two subsets of points, P_τ^+ and P_τ^-, which are the same for every $w \in \tau$, and depend only on τ. We thus perform a traversal of $\mathcal{A}$, which proceeds from each visited cell to a neighbor cell. When we visit a cell τ, we check whether the subsets P_τ^+ and P_τ^- can be covered by two balls of radius r, respectively. To do so, we maintain dynamically the intersection of the sets $\{B_r(p) \mid p \in P_\tau^+\}$, $\{B_r(p) \mid p \in P_\tau^-\}$, where $B_r(p)$ is the ball of radius r centered at p, and observe that (a) any point in the first (resp., second) intersection can serve as the center of a ball of radius r which contains P_τ^+ (resp., P_τ^-), and (b) no ball of radius r can cover P_τ^+ (resp., P_τ^-) if the corresponding intersection is empty. Moreover, when we cross from a cell τ to a neighbor cell τ', P_τ^+ changes by the insertion or deletion of a single point, and P_τ^- undergoes the opposite change, so each of the sets of balls $\{B_r(p) \mid p \in P_\tau^+\}$, $\{B_r(p) \mid p \in P_\tau^-\}$ changes by the deletion or insertion of a single ball. As we know the sequence of updates in advance, maintaining dynamically the intersection of either of these sets of balls is done in an offline manner. Still, the actual implementation is fairly complicated. It is performed performed using a variant of the multi-dimensional parametric searching technique of Matoušek [25] (see also [11, 18, 30]), which we spell out in some detail in Section 3.

The main algorithm uses a segment tree to represent the sets P_τ^+ (and another segment tree for the sets P_τ^-). Roughly, viewing the traversal of $\mathcal{A}$ as a sequence Σ of cells, each ball $B_r(p)$ has a *life-span* (in P_τ^+), which is a union of contiguous maximal subsequences of cells τ, in which $p \in P_\tau^+$, and a complementary life-span in P_τ^-. We store these (connected portions of the) life-spans as segments in the segment tree. Each leaf of the tree represents a cell τ of $\mathcal{A}$, and the balls stored at the nodes on the path to the leaf are exactly those, whose centers belong to the set P_τ^+ (or P_τ^-). By precomputing the intersection of the balls stored at each node of the tree, we can express each of the intersections $\bigcap\{B_r(p) \mid p \in P_\tau^+\}$ and $\bigcap\{B_r(p) \mid p \in P_\tau^-\}$, for each cell τ, as the intersection of a logarithmic number of precomputed intersections (see also [19]). We show that such an intersection can be tested for emptiness in $O(\log^8 n)$ time. This in turn allows us to execute the decision procedure with a total cost of $O(n^3 \log^8 n)$. We then return to the original optimization problem and apply a variant of Chan's randomization technique [12] to solve the optimization problem by a small number of calls to the decision problem, obtaining an overall algorithm with $O(n^3 \log^8 n)$ *expected* running time.[1]

The improved solution. The above algorithm runs in nearly cubic time because it has to traverse the entire arrangement $\mathcal{A}$, which is of complexity $O(n^3)$. In Section 4 we improve this bound by traversing only portions of $\mathcal{A}$, adapting some of the ideas in Sharir's improved solution for the planar problem [33]. Specifically, Sharir's algorithm solves the decision problem (for a given radius r) in three steps, treating separately three subcases, in which the centers c_1, c_2 of the two covering balls are, respectively, far apart ($|c_1c_2| > 3r$), at medium distance apart ($r < |c_1c_2| \le 3r$) and near each other ($|c_1c_2| \le r$). We base our solution on the techniques used in the first two cases. The first case is easy to generalize to $\mathbb{R}^3$, almost "as is". The generalization of the second case is considerably more involved. More precisely, letting $B_r(p)$ denote the disk of radius r centered at a point p, Sharir's algorithm guesses a constant number of lines l, one of which separates the centers c_1, c_2 of the respective solution disks D_1, D_2, so that the set P_L of the points to the left of l is contained in D_1. We then compute the intersection $K(P_L) = \bigcap_{p \in P_L} B_r(p)$, and intersect each $\partial B_r(p)$, for $p \in P_R = P \setminus P_L$ (the subset of points to the right of l), with $\partial K(P_L)$. It is easily seen that $\partial K(P_L)$ has linear complexity and that each circle $\partial B_r(p)$, for $p \in P_R$, intersects it at two points (at most). This produces $O(n)$ critical points (vertices and intersection points) on $\partial K(P_L)$ and $O(n)$ arcs in between. As argued in [33], it suffices to search these points and arcs for possible locations of the center of D_1 (and dynamically test whether the balls centered at the uncovered points have nonempty intersection).

Generalizing this approach to $\mathbb{R}^3$, we need to guess a separating plane λ, to retrieve the subset $P_L \subseteq P$ of points to the left of λ, to compute $\partial K(P_L)$ (which, fortunately, still has only linear complexity), to intersect $\partial B_r(p)$, for each $p \in P_R$, with $\partial K(P_L)$, and to form the arrangement of the resulting intersection curves. Each cell of this arrangement is a candidate for the location of the center of the left covering ball B_1.

However, the complexity of the resulting arrangement M_K on $\partial K(P_L)$ might potentially be cubic. We therefore compute only a portion M of M_K, which suffices for our purposes, and prove that its complexity is only $O(n^2)$. In addition, we can extend the solution for any positive separation between the centers of the covering balls. Specifically, our decision procedure assumes that $|c_1c_2| \ge \beta r$, for some pre-specified $\beta > 0$, and solves the decision version of the problem in $O((1/\beta^3)n^2 \log^8 n)$ time. We show that $\beta = 2(r_0/r - 1)$, where r_0 is the radius of the smallest enclosing ball of P, is a valid choice for β.

To solve the optimization problem, we conduct a search on the optimal radius r^*, using our decision procedure, starting from small values of r and going up, halving the gap between r and r_0 at each step[2], until the first time we reach a value $r > r^*$. Then we use a variant of Chan's technique [12], combined with our decision procedure, to find the exact value of r^*. The way the search is conducted guarantees that its cost does not exceed the bound $O((1/\beta^3)n^2 \log^8 n)$, for the separation parameter β for r^*. Hence, we obtain a randomized algorithm that solves the 2-center problem for any positive separation of c_1 and c_2, and runs in $O(n^2 \log^8 n/(1 - r^*/r_0)^3)$ expected time.

[1] The earlier algorithm in [2] follows the same general approach, but uses an even more complicated, and slightly less efficient machinery for dynamic emptiness testing of the intersection of congruent balls.

[2] We have to act in this manner to make sure that we do not call the decision procedure with values of r, which are too close to r_0, thereby losing control over the running time.

Due to lack of space, some details are omitted in this extended abstract. Full details can be found in [1].

3. A NEARLY CUBIC ALGORITHM

The decision procedure. In this section we give details of the implementation of our less efficient solution, many of which are applicable in the improved solution. Recall from the description in the introduction that the decision procedure, on a given radius r, constructs two segment trees T^+, T^-, on the life-spans of the balls $B_r(p)$, for $p \in P$ (with respect to the tour of the dual arrangement $\mathcal{A}$). Each leaf is a cell τ of $\mathcal{A}$, and the balls, whose centers belong to P_τ^+ (resp., P_τ^-), are those stored at nodes on the path from the root to τ in T^+ (resp., T^-).

For each node u of T^+, let S_u denote the intersection of all the balls (of radius r) stored at u. We refer to each S_u as a *spherical polytope*; see [7, 8, 9] for (unrelated) studies of spherical polytopes. We compute each S_u in $O(|S_u| \log |S_u|)$ deterministic time, using the algorithm by Brönnimann et al. [10] (see also [17, 31]). Since the arrangement $\mathcal{A}$ consists of $O(n^3)$ cells, standard properties of segment trees imply that the two trees require $O(n^3 \log n)$ storage (see below for more details) and $O(n^3 \log^2 n)$ preproccessing time.

Intersection of spherical polytopes. Let $\mathcal{S} = \{S_1, \ldots, S_t\}$ be the set of $t = O(\log n)$ spherical polytopes stored at the nodes of a path from the root to a leaf of T^+ or of T^-. Each S_i is the intersection of some n_i balls, and $\sum_{i=1}^{t} n_i \le n$. Our current goal is to determine, in polylogarithmic time, whether the intersection K of the spherical polytopes in $\mathcal{S}$ is nonempty. If this is the case for at least one path of T^+ and for the same path in T^- then $r^* \le r$, and otherwise $r^* > r$. We describe this step in detail next, and then show how to adjust this solution to also discriminate between the cases $r^* < r$ and $r^* = r$.

As is well known, each ball b participating in the intersection S_i contributes at most one (connected) face to ∂S_i (see [31]). The vertices and edges of S_i are the intersections of two or three bounding spheres, respectively (for degenerate values of r, one vertex might be incident to four spheres). To simplify the forthcoming analysis, we assume that the points of P are in general position, meaning that no five of them are co-spherical, and that the radii of the balls determined by quadruples of points of P are all distinct. Hence S_i is a planar (or, rather, spherical) map with at most $|S_i|$ faces, which implies that the complexity of ∂S_i is $O(|S_i|)$. Consequently, the storage in each node u of T^+ (or T^-) is $O(|S_u|)$, so, as mentioned above, the total storage for T^+ (and for T^-) is $O(n^3 \log n)$.

We solve this problem by employing a technique similar to the multi-dimensional parametric searching technique of Matoušek [25] (see also [2, 11, 18, 30]). We solve in succession the following three subproblems, $\Pi_0(q)$, where q is a point in the xy-plane, $\Pi_1(l)$, where l is a y-parallel line in the xy-plane, and Π_2, over the entire xy-plane. In the latter problem we wish to to determine whether the xy-projection K^* of K is nonempty. During the execution of the algorithm for solving Π_2, we call recursively the algorithm for solving $\Pi_1(l)$, for certain y-parallel lines $l \subset \mathbb{R}^2$, and we wish to determine whether K^* meets l. If so, then Π_2 is solved directly (with a positive answer). Otherwise, we wish to determine which side of l, within $\mathbb{R}^2$, can meet K^* (since K^* is convex, there can exist at most one such side). The recursion bottoms out at certain points q, on which we run $\Pi_0(q)$ and determine whether K^* contains q. If so, then $\Pi_1(l)$ is solved directly (with a positive answer). Otherwise, we determine which side of q, within l, can meet K^*, and continue the search accordingly.

Our solutions to the subproblems Π_k, $0 \le k \le 2$, are based on generic simulations of the standard point-location machinery of Sarnak and Tarjan [32]. In each of the subproblems, if we find a point in $f \cap K^*$, for the respective point, line, or plane f, we know that $K \ne \emptyset$ and stop right away. If $f \cap K^* = \emptyset$, we want to "prove" it, by returning a small set of *witness balls* $b_1, \ldots, b_y$, where, for each j, b_j is one of the balls that participates in some spherical polytope S_i (so $b_j \supseteq S_i$), so that their intersection $K_0 = \bigcap_{j=1}^{y} b_j$ satisfies $f \cap K_0^* = \emptyset$ (where, as above, K_0^* is the xy-projection of K_0). If $K_0 = \emptyset$ then $K = \emptyset$ too and we stop. Otherwise, K_0 determines the side of f (within $\mathbb{R}^2$ if f is a line, or within the line l if f is a point) that might meet K^*; the opposite side is asserted at this point to be disjoint from K^*. We use this information to perform binary search (or, more precisely, parametric search) to locate K^* within the flat, from which we have recursed into f. The execution of the algorithm for solving Π_2 will therefore either find a point in K or determine that $K = \emptyset$, because it has collected a polylogarithmic number of witness balls, whose intersection, which has to contain K, is found to be empty.

Solving $\Pi_0(q)$ for a point q. Here we have a point $q \in \mathbb{R}^2$ and we wish to determine whether $q \in K^*$. We preprocess each of the t spherical polytopes S_i into a point-location structure. We first partition ∂S_i into its upper portion $\partial S_i{}^+$ and lower portion $\partial S_i{}^-$. We project vertically each of $\partial S_i{}^+$ and $\partial S_i{}^-$ onto the xy-plane and obtain two respective planar maps M_i^+ and M_i^- (see Figure 1). For each face ζ of each map we store the ball b that created it; that is, ζ is the projection of the (unique) face of ∂S_i that lies on ∂b. The xy-projection S_i^* of S_i is equal to both projections of ∂S_i^+, ∂S_i^-, and is bounded by a convex curve E^* that is the concatenation of the xy-projections of certain edges of S_i and of portions of horizontal equators of some of its balls.

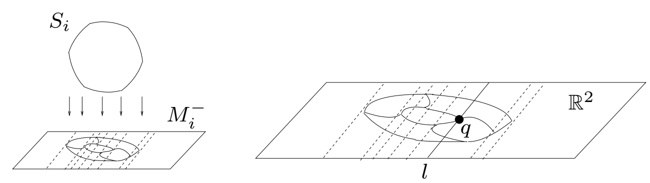

Figure 1. Projecting $\partial S_i{}^-$ vertically onto the xy-plane (left), and the point location structure for the resulting map M_i^- (right).

In order to locate a point in S_i we use the standard point-location algorithm of Sarnak and Tarjan [32]. That is, we divide each planar map into slabs by parallel lines (to the y-axis) through each of the endpoints (and locally x-extremal points) of the arcs obtained by projecting the edges of ∂S_i, including the new equatorial arcs. Using the persistent search structure of [32], the total storage is linear in n_i and the preprocessing cost is $O(n_i \log n_i)$, where n_i is the number of balls forming S_i. To locate a point q_0 in M_i^+ (or in M_i^-), we first find the slab in the x-structure that contains q_0, and then find the two curves between which q_0 lies in the y-structure.[3]

To determine whether $q \in S_i^*$, we locate the face ζ^+ (resp., ζ^-) of the map M_i^+ (resp., M_i^-) that contains q, as just described. Each of these faces can be a 2-face, an edge or a vertex. We therefore retrieve a set $\mathcal{B}_i^+$ (resp., $\mathcal{B}_i^-$) of the one, two, or three or four balls associated (respectively) with the 2-face, edge or vertex containing q. (We omit here the construction of witness balls when the faces ζ^+ and ζ^- are not associated with any ball, that is, $q \notin S_i^*$.)

Let $\mathcal{B}_i$ denote the set $\mathcal{B}_i^+ \cup \mathcal{B}_i^-$. We observe that $q \in S_i^*$ if and only if the z-vertical line λ_q through q intersects S_i. Moreover, we

[3] All these standard details are presented to make more precise the infrastructure used by the higher-dimensional routines Π_1 and Π_2.

have, by construction, $\lambda_q \cap S_i = \lambda_q \cap (\bigcap_i \mathcal{B}_i)$. Hence $q \in S_i^*$ if and only if $s_i := \lambda_q \cap (\bigcap_i \mathcal{B}_i) \neq \emptyset$. We compute the line segment s_i in $O(1)$ time, and repeat this step for each $i = 1, \ldots, t$. We then have $K_0 := \lambda_q \cap K = \bigcap_{i=1}^t s_i$, so it suffices to compute this intersection (in $O(t)$ time) and test whether it is nonempty. If K_0 is nonempty, then we have found a point q' in K. Otherwise, we return the set $\mathcal{B}_0 = \bigcup\{\mathcal{B}_i \mid 1 \leq i \leq t\}$ of up to $8 \log n$ balls as witness balls for the higher-dimensional step (involving the y-parallel line containing q).

The time complexity for solving $\Pi_0(q)$ is $O(\log^2 n)$, since it takes $O(\log n)$ time to compute, for each of the $O(\log n)$ spherical polytopes S_i, the intersection $\lambda_q \cap S_i$.

Solving $\Pi_1(l)$ for a line l. Here we have a y-parallel line $l \subset \mathbb{R}^2$ and we wish to determine whether K^* meets l. We first locate l in each of the planar maps M_i^+ and M_i^- of each S_i, and find the slabs ψ_i^+ and ψ_i^-, which contain l (in some cases l is the bounding line of two slabs ψ_i' and ψ_i'' of M_i^+ or of M_i^-, so we retrieve both slabs). We then simulate the binary search through the y-structure of each of the obtained slabs separately. In each step of the search, within some fixed slab ψ_0, we consider an arc γ of the y-structure, and determine whether K^* meets l above or below γ (within $\mathbb{R}^2$), assuming $K^* \cap l \neq \emptyset$. To this end, we find the intersection point $q_0 = l \cap \gamma$, and run the algorithm for solving $\Pi_0(q_0)$ (see Figure 2(left)). If $q_0 \in K^*$, then we have found a point q' in K, and we immediately stop. Otherwise, we have a set $\mathcal{B}_0$ of up to $8 \log n$ balls returned by the algorithm for solving $\Pi_0(q_0)$. We test whether the xy-projection K_0^* of $\bigcap \mathcal{B}_0$ intersects l. If $K_0^* \cap l = \emptyset$, then (due to the convexity of K) we know which side of l (within $\mathbb{R}^2$) meets K^*, and we return $\mathcal{B}_0$ as a set of witness balls for the higher-dimensional (planar) step. Otherwise (again due to the convexity of K), we know which side of γ, within l, meets K^*, and we continue the search through the y-structure of ψ_0 on this side. We continue the search in this manner, until, for each S_i, we obtain an interval ξ_i of l between two consecutive arcs of the y-structure of ψ_0, which meets K^* (assuming $l \cap K^* \neq \emptyset$). Let Ξ denote the collection of all these intervals. Clearly, $l \cap K^* \subseteq \bigcap \Xi$. We find the lowest endpoint E^- among the top endpoints of the intervals in Ξ and the highest endpoint E^+ among the bottom endpoints of the intervals in Ξ, and test whether E^- is above E^+. If so, we consider the set $\mathcal{B}_1$ of up to $16 \log n$ witness balls returned by the algorithms for solving $\Pi_0(E^-)$ and $\Pi_0(E^+)$. If the xy-projection K_1^* of $\bigcap \mathcal{B}_1$ intersects l, then K^* meets l and we stop immediately, for we have found that K is nonempty. Otherwise, we know which side of l (within $\mathbb{R}^2$) can meet K^*, and we return $\mathcal{B}_1$ as a set of witness balls for the higher (planar) recursive level. If E^- is not above E^+, then $K^* \cap l = \emptyset$ and we return $\mathcal{B}_1$ as a set of witness balls for the higher (planar) recursive level as well.[4]

The time complexity for solving $\Pi_1(l)$ is $O(\log^4 n)$, since for each of the $O(\log n)$ spherical polytopes S_i we run a binary search through the y-structure of at most two slabs of each of the maps M_i^+ and M_i^-, and in each of the binary search steps, we run the algorithm for solving $\Pi_0(q_0)$ for some point q_0. The other substeps take less time.

Solving Π_2. We next consider the main problem Π_2, where we want to determine whether $K^* \neq \emptyset$ (i.e., whether $K \neq \emptyset$). We use parametric searching, in which we run the point location algorithm that we used for solving Π_0, in the following generic manner.

In the first stage of the generic point location, we run a binary search through the slabs of each of the planar maps M_i^+ and M_i^-, for $i = 1, \ldots, t$. In each step of the search through any of the maps, we take a line l_0 delimiting two consecutive slabs of the map, and run the algorithm for solving $\Pi_1(l_0)$, thereby deciding on which side of l_0 to continue the search. At the end of this stage, unless we have already found a point in K or determined that K is empty, we obtain a single slab in each map that contains K^*. Let ψ denote the intersection of these slabs, which must therefore contain K^* (unless K is empty).

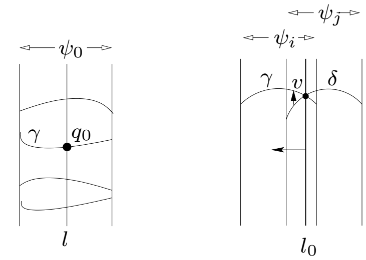

Figure 2. Left: The line l on which we run $\Pi_1(l)$. The point q_0 on which we run $\Pi_0(q_0)$ is the intersection point of l with some arc γ. Right: Comparing $\gamma \cap K^*$ with δ. The outcome of $\Pi_1(l_0)$ determines (a) the side of δ in which the search in ψ_j should continue, and (b) the portion of γ which can still meet K^*.

In the next stage of the generic point location, we consider each map M_i^+ or M_i^- (for simplicity we refer to it just as M_i) separately, and run a binary search through the y-structure of its slab ψ_i that contains ψ. In each step of the search we consider an arc γ of the y-structure, and determine which side of γ (within the slab ψ), can meet K^*, assuming that $\psi \cap K^* \neq \emptyset$; if $\gamma \cap K^* \neq \emptyset$ we will detect it and stop right away. The logic of our search is as follows. We act under the assumption that $\gamma \cap K^* \neq \emptyset$, and try to locate a point of $\gamma \cap K^*$ in each of the other maps. Suppose, to simplify the description, that we managed to locate the entire γ in a single face of each of the other maps M_j^+ and M_j^-. This yields a set $\mathcal{B}$ of $O(t)$ balls, so that a point $v \in \gamma$ lies in K^* if and only if it lies in the xy-projection K_0^* of $\bigcap \mathcal{B}$. We then test whether γ intersects K_0^*. If so, we have found a point in K and stop right away. Suppose then that $K_0^* \cap \gamma = \emptyset$. If $K_0^* \cap \psi = \emptyset$ then K must be empty, because we already know that $K^* \subset \psi$. If $K_0^* \cap \psi \neq \emptyset$, then we know on which side of γ to continue the binary search in ψ_i.

In general, though, γ might split between several cells of a map M_j, where M_j denotes one of the maps M_j^+ or M_j^-. This forces us to narrow the search to a subarc of γ, in the following manner. We run a binary search through the y-structure of the corresponding slab ψ_j of M_j, which contains ψ, and repeat it for each of the maps M_j. In each step of the search, we need to compare γ (or, more precisely, some point in $\gamma \cap K^*$) with some arc δ of ψ_j, which we do as follows. If γ lies, within ψ, completely on one side of δ, we continue the binary search in ψ_j on that side of δ. If γ intersects δ, we pick an intersection point v of γ and δ, pass a y-parallel line $l_0 \subset \mathbb{R}^2$ through v, and run the non-generic version of the algorithm to solve $\Pi_1(l_0)$. (See Figure 2(right).) As before, if $l_0 \cap K^* \neq \emptyset$ we detect this and stop. Otherwise, we know which of the two portions of γ, delimited by v, can intersect K^*. We repeat this step for each of the at most four intersection points of γ and δ (observing that these are elliptic arcs), and obtain a connected portion γ' of γ, delimited by two consecutive intersection points, whose relative interior lies completely above or below δ, so that $\gamma \cap K^*$, if nonempty, lies in γ'. This allows us to resolve the

[4] With some care, the number of witness balls can be significantly reduced. We do not go into this improvement, because handling the witness balls is an inexpensive step, whose cost is subsumed by the cost of the other steps of the algorithms.

generic comparison with δ, and continue the binary search through ψ_j.

When these searches terminate, we end up with a 2-face in each M_j, in which $\gamma \cap K^*$ lies, and we reach the scenario described in a preceding paragraph. As explained there, we can now either determine that $K \neq \emptyset$, or that $K = \emptyset$, or else we know which side of γ, within ψ_i (or, rather, within ψ) can contain K^*, and we continue the binary search through ψ_i on that side.

When the binary search through ψ_i terminates, we have a 2-face ζ_i of M_i, where K^* must lie, and we retrieve the ball b_i corresponding to ζ_i. We repeat this step to each of the maps M_i^+ and M_i^- of each of the t spherical polytopes S_i, and obtain a set $\mathcal{B}_1$ of $2t$ balls. In addition, the searches through the maps M_i^+ and M_i^- may have trimmed ψ to a narrower strip ψ', and have produced a set $\mathcal{B}_2$ of witness balls, so that the xy-projection of their intersection lies inside ψ'. $\mathcal{B}_2$ may consist of a total of $O(t^3 \log^2 n)$ witness balls, as is easy to verify. In addition, the second-level searches produce an additional collection $\mathcal{B}_2'$, consisting of balls corresponding to faces of the maps M_j^+ and M_j^-, in which the second-level searches have ended; their overall number is $O(t^2 \log n)$. Put $K_2 = \bigcap(\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_2')$. Hence $K \neq \emptyset$ if and only if $K_2 \neq \emptyset$.

Finally, consider the running time of the procedure. It is easily seen to be subsumed by the cost of the second stage, where we run a binary search through the y-structure of each slab ψ_i, and in each of its steps we have to run another binary search over the corresponding slabs ψ_j of all the other maps. Since each comparison that we make involves the execution of the algorithm to solve $\Pi_1(l_0)$, for an appropriate line l_0, the time complexity for this stage is $O(\log^8 n)$.

Hence, the overall cost of the decision procedure is $O(n^3 \log^8 n)$. Note that the above procedure determines whether $r^* \leq r$ or $r^* > r$.

Detecting equality $r^* = r$. As described so far, the decision procedure discriminates between the cases $r^* \leq r$ and $r^* > r$. To detect the case $r^* = r$, we note that if both $K(P_\tau^+)$ and $K(P_\tau^-)$ are non-degenerate (i.e., contain more than one point), then we can shrink the covering balls of P and find two balls of radius smaller than r that still cover P (that is, $r^* < r$). Hence, if for all separations of P to P_τ^+ and P_τ^- (i.e., for all cells of $\mathcal{A}$), that admit a solution for the decision procedure (with the given r), at least one of the intersections $K(P_\tau^+)$ or $K(P_\tau^-)$ is degenerate, then P cannot be covered by two balls of radius smaller than r. It follows that if at least one such separation exists then $r = r^*$. This affects the execution of the decision procedure in an obvious way: Stop when obtaining a solution for which both $K(P_\tau^+)$ and $K(P_\tau^-)$ are nonempty and non-degenerate, and report that $r^* < r$. Otherwise, test all cells τ. If at least one degenerate solution is found, report that $r^* = r$, and otherwise $r^* > r$. To detect whether the intersection K of t spherical polytopes is degenerate, we modify the algorithm for detecting whether K is nonempty, as follows. Each step in the decision procedure which detects that $K \neq \emptyset$ obtains a specific point w that belongs to K. Moreover, w belongs to the intersection K_1 of polylogarithmically many witness balls, and does not lie on the boundary of any other ball. This is because each of the procedures Π_0, Π_1, or Π_2 locate the xy-projection w^* of w (which, for Π_1 and Π_2 is a generic, unknown point in K) in each of the maps $M_i^+, M_i^-, i = 1, \ldots, t$, and the collection of the witness balls gathered during the various steps of the searches contains all the balls that participate in the corresponding spherical polytopes S_i on whose boundary w can lie. Thus, when we terminate with a point $w \in K$, we find, among the polylogarithmically many witness balls, the at most four balls whose boundaries contain w (recall our general position assumption), and test whether their intersection is the singleton $\{w\}$. It is easily checked that this is equivalent to the condition that K is degenerate.

Solving the optimization problem. We now combine our decision procedure with the randomized optimization technique of Chan [12], to obtain an algorithm for the optimization problem, which runs in $O(n^3 \log^8 n)$ *expected* time. Our application of Chan's technique, described next, is somewhat non-standard, because each recursive step has also to handle global data, which it inherits from its ancestors.

Chan's technique, in its "purely recursive" form, takes an optimization problem that has to compute an optimum value $w(P)$ on an input set P. The technique replaces P by several subsets $P_1, \ldots, P_s$, such that $w(P) = \min\{w(P_1), \ldots, w(P_s)\}$, and $|P_i| \leq \alpha|P|$ for each i (here $\alpha < 1$ and s are constants). It then processes the subproblems P_i in a *random* order, and computes $\min_i w(P_i)$ by comparing each $w(P_i)$ to the minimum w collected so far, and by replacing w by $w(P_i)$ if the latter is smaller.[5] Comparisons are performed by the decision procedure, and updates of w are computed recursively. The crux of this technique is that the expected number of recursive calls (in a single recursive step) is only $O(\log s)$, and this (combined with some additional enhancements, which we omit here) suffices to make the expected cost of the whole procedure asymptotically the same as the cost of the decision procedure, for *any* values of s and α. Technically, if the cost $D(n)$ of the decision procedure is $\Omega(n^\gamma)$, where γ is some fixed positive constant, the expected running time is $O(D(n))$ provided that

$$(\ln s + 1)\alpha^\gamma < 1. \tag{1}$$

However, even when (1) does not hold "as is", Chan's technique enforces it by compressing l levels of the recursion into a single level, for l sufficiently large, so its expected cost is still $O(D(n))$. See [12] for details.

To apply Chan's technique to our decision procedure, we pass to the dual space, where each point $p \in P$ is mapped to a plane p^*, as done in the decision procedure. We obtain the set $P^* = \{p^* \mid p \in P\}$ of dual planes, and we consider its arrangement $\mathcal{A} = \mathcal{A}(P^*)$, where each cell τ in $\mathcal{A}$ represents an equivalence class of planes in the original space, which separate P into the same two subsets of points P_τ^+, P_τ^-.

To decompose the optimization problem into subproblems, as required by Chan's technique, we construct a $(1/\varrho)$-*cutting* for $\mathcal{A}(P^*)$, for a specific constant value of ϱ, that we will fix later (see [14, 15, 24] for details concerning cuttings), and obtain $O(\varrho^3)$ simplices, such that the interior of each of them is intersected by at most n/ϱ planes of P^*. Each simplex Δ_i corresponds to one subproblem and contains some (possibly only portions of) cells $\tau_1, \ldots, \tau_k$ of the arrangement $\mathcal{A}$. All these subproblems have in common the sets $(P^*)^+_{\Delta_i}$, $(P^*)^-_{\Delta_i}$, consisting, respectively, of all the planes that pass fully above Δ_i and those that pass fully below Δ_i. (These sets are dual to respective subsets $P^+_{\Delta_i}, P^-_{\Delta_i}$ of P, where $P^+_{\Delta_i}$ is contained in all the sets $P^+_{\tau_j}$, for the cells τ_j, that meet Δ_i, and symmetrically for $P^-_{\Delta_i}$.) Note that most of the dual planes belong to $(P^*)^+_{\Delta_i} \cup (P^*)^-_{\Delta_i}$; the "undecided" planes are those that cross the interior of Δ_i, and their number is at most n/ϱ. We denote the set of these planes as $(P^*)^0_{\Delta_i}$ (and the set of their primal points as $P^0_{\Delta_i}$).

To apply Chan's technique, we construct two segment trees on the arrangement of $(P^*)^0_{\Delta_i}$, as described above. Consider one of these segment trees, T^+, that maintains the set of balls $\mathcal{B}^+ =$

[5] So the value of w keeps shrinking.

$\{B_r(p) \mid p \in P^+_{\tau_j}\}$. Each cell τ_j in Δ_i is represented by a leaf of T^+. Each ball is represented as a collection of disjoint life-spans, with respect to a fixed tour of the cells of $\mathcal{A}((P^*)^0_{\Delta_i})$, which are stored as segments in T^+, as described earlier. In addition, we compute the intersection of the balls centered at the points of $P^+_{\Delta_i}$, in $O(n \log n)$ time, and store it at the root of T^+. Note that, as we go down the recursion, we keep adding planes to $(P^*)^+_{\Delta_i}$, that is, points to $P^+_{\Delta_i}$, and the actual set $P^+_{\Delta_i}$ of points dual to the planes above the current Δ_i is the union of logarithmically many subsets, each obtained at one of the ancestor levels of the recursion, including the current step. As we go down the recursion, Chan's technique keeps 'shrinking' the radius of the balls. Hence, each time we have to solve a decision subproblem, we compute the intersection of the balls centered at the points of $P^+_{\Delta_i}$ from scratch. We build a second segment tree T^- that maintains the balls of $\mathcal{B}^- = \{B_r(p) \mid p \in P^-_{\tau_j}\}$, in a fully analogous manner. The running time so far (of the decision procedure) is $O(n \log n + m^3 \log^2 m)$, where m is the number of planes in $(P^*)^0_{\Delta_i}$ and n is the size of the initial input set P.

To solve the decision procedure for a given subproblem associated with a simplex Δ_i, we test, by going over all the root-to-leaf paths in T^+ and T^-, whether there exists a cell τ (overlapping Δ_i), for which the intersections of the spherical polytopes on the two respective paths in T^+ and T^- are nonempty (and, if nonempty, whether one of them is degenerate). The overall cost of this step, iterating over the $O(m^3)$ cells of $\mathcal{A}((P^*)^0_{\Delta_i})$ and applying the procedure for intersecting spherical polytopes, as described above, is $O(m^3 \log^8 n)$.

The solution at the bottom of the recursion, where only $O(1)$ points of P are "undecided" can be performed in $O(n)$ time in a straightforward manner, which we omit here.

As stated earlier, the (expected) cost of applying Chan's technique to a decision procedure is asymptotically the same as that of the decision procedure. Here, however, we need to overcome some subtle issues, involving the "global" parameter n (appearing in the global sets $P^+_{\Delta_i}, P^-_{\Delta_i}$, which are passed down the recursion), which is involved in the recursive subproblems. What "saves" us is the fact that the dependence of the running time of the decision procedure on n is only $O(n \log n)$. This allows us to obtain the following recurrence for the expected running time $T(m, n)$ of a recursive step in Chan's procedure, involving m "local" points and n "global" points.

$$T(m,n) \le \begin{cases} \log(c\varrho^3) T(m/\varrho, n) & \\ +O(m^3 \log^8 n + n \log n), & \text{for } m \ge \varrho, \\ & \\ O(n), & \text{for } m < \varrho, \end{cases} \quad (2)$$

where c is an appropriate absolute constant (so that $c\varrho^3$ bounds the number of cells of the cutting), and ϱ is chosen to be a sufficiently large constant. It is fairly routine (and we omit the details) to show that the recurrence (2) yields the overall bound $O(n^3 \log^8 n)$ on the expected cost of the initial problem; i.e., $T(n, n) = O(n^3 \log^8 n)$.

We thus obtain the following intermediate result.

THEOREM 1. *Let P be a set of n points in $\mathbb{R}^3$. A 2-center for P can be computed in $O(n^3 \log^8 n)$ randomized expected time.*

4. AN IMPROVED ALGORITHM

An improved decision procedure Γ. Consider the decision problem, where we are given a radius r and a parameter $\beta > 0$, and have to determine whether P can be covered by two balls of radius r, such that the distance between their centers c_1, c_2 is at least βr. By this we mean that there is no placement of two balls of radius r, which cover P, such that the distance between their centers is smaller than βr; see Figure 3(left).

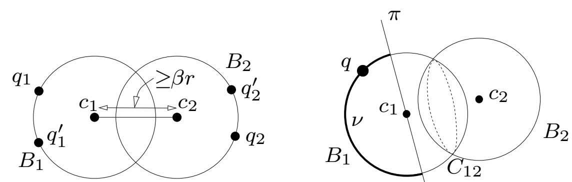

Figure 3. Left: The points q_1, q_1', q_2, q_2' prevent $|c_1c_2|$ from getting smaller. Right: The plane π passes through c_1 and is disjoint from C_{12}. The hemisphere ν delimited by π, which lies on the side of π not containing C_{12}, must contain a point q of P.

This assumption is easily seen to imply the following property: Let C_{12} denote the intersection circle of ∂B_1 and ∂B_2. Then any hemisphere ν of ∂B_1, such that (a) the plane π through c_1 delimiting ν is disjoint from C_{12}, and (b) ν and C_{12} lie on different sides of π, must contain a point q of P, for otherwise we could have brought B_1 and B_2 closer together by moving c_1 in the normal direction of π, into the halfspace containing c_2 (and C_{12}). See Figure 3(right).

The procedure tests for the existence of a solution under the following two complementary assumptions: (i) $|c_1c_2| \ge 3r$, and (ii) $\beta r \le |c_1c_2| < 3r$, for the specified $\beta > 0$.

We first assume that the distance between c_1 and c_2 is at least $3r$. We fix a set D of $O(1)$ orientations (on the unit sphere $\mathbb{S}^2$) with the property that for each orientation $u \in \mathbb{S}^2$ there exists $v \in D$, such that the angle between u and v is at most 1°, say. For each $v \in D$ we rotate the coordinate frame so that v becomes the x-axis. In one of these orientations the line connecting c_1 and c_2, with c_1 lying to the left of c_2, will form an angle of at most one degree with the x-axis. In this case $x(c_2) - x(c_1)$ will be at least $|c_1c_2| \cos 1° \ge 3r \cos 1° \approx 2.99r$, so we can separate B_1 from B_2 by a plane parallel to the yz-plane.

We sort the points by their x-coordinates (for each fixed choice of v), and try all $O(n)$ partitions of P into subsets P_L, P_R, so that all the points of P_L lie to the left of all points of P_R. For each such partition, we compute the smallest enclosing balls of P_L and of P_R, in linear (deterministic or randomized expected) time [16, 26, 27]. If both radii are at most r, we have found a solution to the decision problem. With this implementation, the cost of handling this case is $O(n^2)$, which is subsumed in the cost of handling the following main case.

We now focus on the second case, $\beta r \le |c_1c_2| < 3r$, which is more difficult. The algorithm is composed of many components, and we describe each of them in detail.

Guessing orientations and separating planes. We again choose a set D of canonical orientations, so that the maximum angular deviation of any direction u from its closest direction in D is an appropriate multiple α of β. The connection between α and β is given by the following reasoning. Fix a direction $v \in D$ so that the angle between the orientation of c_1c_2 and v is at most α. Rotate the coordinate frame so that v becomes the x-axis. As above, let C_{12} denote the intersection circle of ∂B_1 and ∂B_2 (assuming that the balls intersect). Let v_1 be the leftmost point of C_{12} (in the x-direction); see Figure 4(left). If B_1 and B_2 are disjoint (which only happens when $|c_1c_2| > 2r$) we define v_1 to be the leftmost point of B_2. To determine the value of α, we note that (in complete analogy with Sharir's algorithm in the plane [33]) our procedure will try to find a yz-parallel plane, which separates c_1 from v_1. For this, we want to ensure that $x(v_1) - x(c_1) > \beta r/4$, say, to leave enough room for guessing such a separating plane. Let θ denote the angle $\sphericalangle v_1 c_1 c_2$ (see Figure 4(right)). Using the triangle inequality on angles, the angle between $\overrightarrow{c_1 v_1}$ and the x-axis is at most $\theta + \alpha$, so $x(v_1) - x(c_1) \geq r\cos(\theta + \alpha)$. Hence, to ensure the above separation, we need to choose α, such that $\cos(\theta + \alpha) > \beta/4$. Since $|c_1 c_2| \geq \beta r$, we have $\cos\theta \geq \beta/2$. Hence, it suffices to choose α, such that

$$\alpha \leq \cos^{-1}\frac{\beta}{4} - \cos^{-1}\frac{\beta}{2} = \sin^{-1}\frac{\beta}{2} - \sin^{-1}\frac{\beta}{4} = \Theta(\beta).$$

With this constraint on α, the size of D is $\Theta\left(1/\alpha^2\right) = \Theta\left(1/\beta^2\right)$.

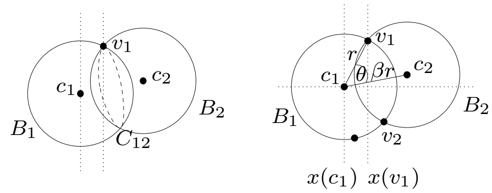

Figure 4. Left: The case $\beta r \leq |c_1 c_2| < 3r$. Right: $x(v_1) - x(c_1) \geq r\cos(\theta + \alpha)$.

Since the difference between the largest and smallest x-coordinates of points of P is at most $5r$, we can find a plane λ, parallel to the yz-plane, which separates v_1 from c_1, by drawing $O(1/\beta)$ yz-parallel planes, with horizontal separation of $\beta r/4$. Thus, the total number of guesses that we make (an orientation in D and a separating plane) is $O(1/\beta^3)$. The following description pertains to a correct guess, in which the properties that we require are satisfied.

Reducing to a 2-dimensional search. By the property noted above, the left hemisphere ν_{λ_0} of ∂B_1, delimited by the yz-parallel plane λ_0 through c_1, must pass through at least one point q of P (see Figure 5(left)).

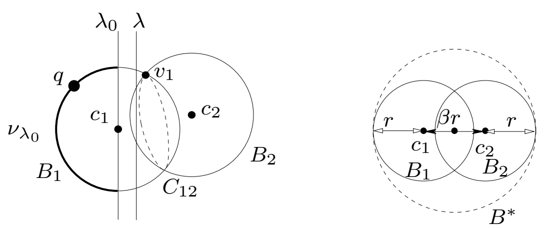

Figure 5. Left: The separating plane λ and its parallel copy λ_0 through c_1. The hemisphere ν_{λ_0} of ∂B_1 to the left of λ_0 must contain a point q of P. Right: The smallest enclosing ball B^* of $B_1 \cup B_2$.

Let P_L denote the subset of points of P lying to the left of λ. Then P_L must be fully contained in B_1 and contain q. We compute the intersection $K(P_L) = \bigcap\{B_r(p) \mid p \in P_L\}$ in $O(n\log n)$ time [10]. If $K(P_L)$ is empty, then P_L cannot be covered by a ball of radius r and we determine that the currently assumed configuration does not yield a positive solution for the decision problem. Otherwise, since $P_L \subseteq B_1$, c_1 must lie in $K(P_L)$. Moreover, since $q \in P_L$ lies on the *left* portion of ∂B_1, c_1 must lie on the *right* portion of the boundary of $K(P_L)$. Finally, since c_1 lies to the left of λ, only the portion σ_L of the right part of $\partial K(P_L)$ to the left of λ has to be considered. If $K(P_L)$ is disjoint from λ then σ_L is just the right portion of $\partial K(P_L)$. Otherwise, σ_L has a "hole", bounded by $\partial K(P_L) \cap \lambda$, which is a convex piecewise-circular curve, being the boundary of the intersection of the disks $B_r(p) \cap \lambda$, for $p \in P_L$.

For each $p \in P_R = P \setminus P_L$, we intersect $\partial B_r(p)$ with σ_L and obtain a curve γ_p on σ_L; this curve bounds the portion of the unique face of $\partial K(P_L \cup \{p\})$ within σ_L. Let M denote the arrangement formed on σ_L by the curves γ_p, for $p \in P_R$, and by the arcs of σ_L. Apriori, M might have cubic complexity, if many of the $O(n^2)$ pairs of curves γ_a, γ_b, for $a, b \in P_R$, traverse a linear number of common faces of σ_L, and intersect each other on many of these faces, in an overall linear number of points. Equivalently, the "danger" is that the intersection circle C_{ab} of a corresponding pair of spheres $\partial B_r(a), \partial B_r(b)$, for $a, b \in P_R$, could intersect a linear number of faces of σ_L (and each of these intersections is also an intersection point of γ_a and γ_b).

Complexity of M. In the assumed configuration, we claim that this cubic behavior is impossible — C_{ab} can meet only a constant number of faces of σ_L. Consequently, the overall complexity of M is only quadratic. This crucial claim follows from the observation that, for C_{ab} to intersect many faces of σ_L, it must have many short arcs, each delimited by two points on σ_L and lying outside $K(P_L)$. The main geometric insight, which rules out this possibility, and leads to our improved algorithm, is given in the following lemma.

LEMMA 2. *Let λ be a yz-parallel plane, which separates v_1 from c_1. Let $P_L \subseteq P$ be the subset of points of P to the left of λ, and let $P_R = P \setminus P_L$. Let C_{ab} denote the intersection circle of $\partial B_r(a), \partial B_r(b)$, for some pair of points $a, b \in P_R$, and let $q \in P_L$. If the arc $\omega = C_{ab} \setminus B_r(q)$ is smaller than a semicircle of C_{ab}, then at least one of its endpoints must lie to the right of λ.*

PROOF. The situation is depicted in Figure 6. To slightly simplify the analysis, and without loss of generality, assume that $r = 1$. Let h be the plane passing through a, b and q. Let c_{ab} denote the midpoint of ab, and let w denote the center of the circumscribing circle Q of $\triangle qab$. Denote the distance $|ab|$ by $2x$, and the radius of Q by y (so $|wp_1| = |wp_2| = |wq| = y$). Note that c_{ab} and w lie in h and that $y \geq x$. Observe that c_{ab} is the center of the intersection circle C_{ab} of $\partial B_r(a)$ and $\partial B_r(b)$. See Figure 6(a).

The intersection points z, z' of C_{ab} and $\partial B_r(q)$ are the intersection points of the three spheres $\partial B_r(a)$, $\partial B_r(b)$, and $\partial B_r(q)$. They lie on the line ℓ passing through w and orthogonal to h, at equal distances $\sqrt{1-y^2}$ from w. See Figure 6(b). (If $y > 1$ then z and z' do not exist, in which case C_{ab} does not intersect $\partial B_r(q)$; in what follows we assume that $y \leq 1$.) Hence, within C_{ab}, zz' is a chord of length $2\sqrt{1-y^2}$. In the assumed setup, z and z' delimit a short arc ω of C_{ab}, which lies outside $B_r(q)$, so points on the arc are (equally) closer to a and b than to q.

Hence, the projection of the arc ω onto h is a small interval ww', which lies on the bisector of ab in the direction that gets away from q; that is, it lies on the Voronoi edge of ab in the diagram $\mathrm{Vor}(\{a, b, q\})$ within h. See Figure 6(c). Moreover, c_{ab} also lies on the bisector, but it has to lie on the other side of w, or else the smaller arc ω would have to lie inside $B_r(q)$. That is, c_{ab} has to be closer to q than to a and b. Since λ separates a and b from q, it also separates c_{ab} from q. Moreover, the preceding arguments are easily seen to imply that wq crosses ab (as in Figure 6(a)), which implies that λ also separates q and w, so w has to lie to the right of λ. Since z and z' lie on two sides of w on the line ℓ, at least one of them has to lie on the same side of λ as w (i.e., to the right of λ). This completes the proof. $\square$

Let $a, b \in P_R$ and consider those arcs of C_{ab} which lie outside $K(P_L)$ but their endpoints lie on σ_L. Clearly, all these arcs are

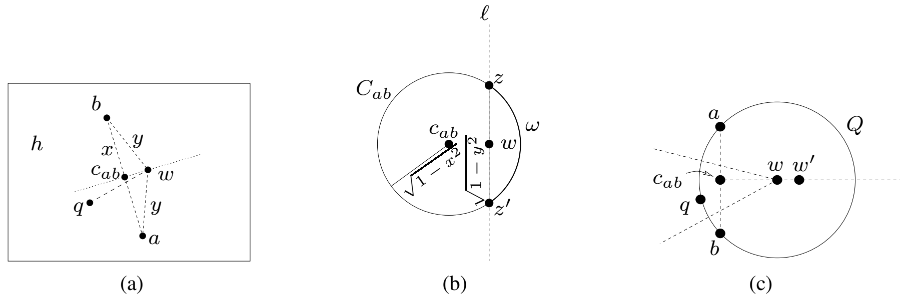

Figure 6. The setup in Lemma 2: (a) the setup within the plane h; (b) the setup within C_{ab}; (c) ww' lies on the bisector of ab in the direction that gets away from q.

pairwise disjoint. At most one such arc can be larger than a semicircle. Let ω be an arc of this kind which is smaller than a semicircle, and let $q \in P_L$ be such that one endpoint of ω lies on $\partial B_r(q)$. Then $\omega' = C_{ab} \setminus B_r(q)$ is contained in ω and therefore is also smaller than a semicircle. By Lemma 2, exactly one endpoint of ω' lies to the right of λ (the other endpoint lies on σ_L). Note that C_{ab} cannot have more than two such short arcs lying outside $K(P_L)$, since, due to the convexity of C_{ab}, only two arcs of C_{ab} can have their two endpoints lying on opposite sides of λ. Hence the number of arcs of C_{ab} under consideration is at most 3, implying that γ_a and γ_b intersect at most three times, and thus the complexity of M is $O(n^2)$, as asserted.

Constructing and searching M. We compute M in $O(n^2 \log n)$ time (see the full version [1] for the straightforward details). We next perform a traversal of the cells of M in a manner similar to the one used in the preceding section, via a tour, which proceeds from each visited cell to an adjacent one. For each cell τ that we visit, we place the center c_1 of B_1 in τ, and maintain dynamically the subset P_τ^+ of points of P not covered by B_1. (Here, unlike the algorithm of Section 3, the complementary set P_τ^- is automatically covered by B_1 and there is no need to test it.) As before, when we move from one cell τ to an adjacent cell τ_1, $P_{\tau_1}^+$ gains one point or loses one point. This implies that this tour generates only $O(n^2)$ connected life-spans of the points of P, where a life-span of a point p is a maximal connected interval of the tour, in which p belongs to P_τ^+. We can thus use a segment tree T_M to store these life-spans, as before. Each leaf u of T_M represents a cell τ of M, and the balls not containing τ are those with life-spans that are stored at the nodes on the path from the root to u. Arguing exactly as in Section 3, we can compute T_M in overall $O(n^2 \log^2 n)$ time, and the total storage used by T_M is $O(n^2 \log n)$.

As in the previous algorithm, we next test, for each leaf u of T_M, whether the spherical polytopes along the path from the root to u have non-empty intersection. We do this using the parametric search technique described in Section 3, which takes $O(\log^8 n)$ time for each path, for a total of $O(n^2 \log^8 n)$. More precisely, as above, we also need to distinguish between $r = r^*$ and $r > r^*$. We therefore stop only when both the intersection along the path and the cell of σ_L corresponding to u are non-degenerate, and then report that $r^* < r$. Otherwise, we continue running the above procedure over all paths of T_M, and repeat it for each of the $O(1/\beta^3)$ combinations of an orientation v and a separating plane λ. If we find at least one (degenerate) solution, we report that $r^* = r$, and otherwise conclude that $r^* > r$. Hence, the cost of handling Case 2, and thus also the overall cost of the decision procedure, is $O((1/\beta^3)n^2 \log^8 n)$.

Solving the optimization problem. We now combine the decision procedure Γ described above with the randomized optimization technique of Chan [12] (as briefly described in Section 3), to obtain a solution for the optimization problem.

The decision procedure Γ, on a specified radius r, relies on an apriori knowledge of a lower bound β for the separation ratio $|c_1 c_2|/r$. To supply such a β, let r_0 denote the radius of the smallest enclosing ball of P, and observe that if there exist two balls B_1, B_2 of radius r covering P then the smallest ball B^* enclosing $B_1 \cup B_2$ must be at least as large as the smallest enclosing ball of P, so its radius must be at least r_0. Since this radius is $(1+\beta/2)r$ (see Figure 5(right)), we have $(1+\beta/2)r \geq r_0$ or $\beta \geq 2(r_0/r - 1)$. The running time is thus

$$O\left(\frac{1}{\beta^3} n^2 \log^8 n\right) = O\left(\frac{1}{(1-r/r_0)^3} n^2 \log^8 n\right).$$

We consider the interval $(0, r_0)$, which contains r^*, and run an "exponential search" through it, calling Γ with the values $r_i = r_0\left(1 - 1/2^i\right)$, for $i = 1, 2, \ldots$, in order, until the first time we reach a value $r' = r_i \geq r^*$. Note that $1 - r'/r_0 = 1/2^i$ and $1/2^i < 1 - r^*/r_0 < 1/2^{i-1}$, so our lower bound estimates for the separation ratio β at r' and at r^* differ by at most a factor of 2, so the cost of running Γ at r' is asymptotically the same as at r^*. Moreover, since the (constants of proportionality in the) running time bounds on the executions of Γ at $r_1, \ldots, r_i$ form a geometric sequence, the overall cost of the exponential search is also asymptotically the same as the cost of running Γ at r^*. We then run Chan's technique, with r' as the initial minimum radius obtained so far. Hence, from now on, each call to Γ made by Chan's technique will cost asymptotically no more than the cost of calling Γ with r' (which is asymptotically the same as calling Γ with r^*).

Combining Chan's technique with the decision procedure Γ. To apply Chan's technique with our decision procedure, we use the same cutting-based decomposition as in Section 3. Consider the application of Γ to a subproblem represented by a simplex Δ_i of the cutting. The presence of "global" points (those dual to planes passing above or below Δ_i) forces us, as in Section 3, to modify each of the two cases considered in Γ.

Case 1: $|c_1 c_2| \geq 3r$. The set $P_{\Delta_i}^-$ of points dual to the planes of $(P^*)_{\Delta_i}^-$ is fully contained in one of the solution balls, say B_1, and the set $P_{\Delta_i}^+$ of points dual to the planes of $(P^*)_{\Delta_i}^+$ is fully contained in the other ball B_2. The task at hand is to decide how to divide the set $P^0_{\Delta_i}$ of the dual points of the planes in $(P^*)^0_{\Delta_i}$ between B_1 and B_2. As is done in Case 1 of the "pure" decision procedure, we rotate the coordinate axes, in a constant number of ways. For each orientation we sort only the points in $P^0_{\Delta_i}$ by their x-coordinates, consider all $O(m)$ partitions of $P^0_{\Delta_i}$ into a left subset P_L and a right subset P_R, and, for each partition, compute the smallest enclosing balls of $P_L \cup P^+_{\Delta_i}$ and of $P_R \cup P^-_{\Delta_i}$. If the radii of these balls are both at most r, we have a solution to the decision procedure. The overall cost of running this step is $O(mn)$, which is subsumed in the cost of handling the second case.

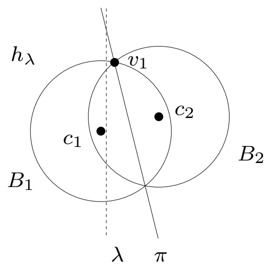

Figure 7. h_λ does not contain any point of $P^+_{\Delta_i}$.

Case 2: $\beta r \le |c_1c_2| < 3r$. (Relevant only when $\beta < 3$.) We again rotate the coordinate axes, in $O\left(1/\beta^2\right)$ ways (in the same manner as in Case 2 of the "pure" decision procedure), and draw $O(1/\beta)$ yz-parallel planes, such that, at the correct orientation, one of these planes, λ, separates c_1 from v_1 (if there is a solution for r). We assume, without loss of generality, that $P^-_{\Delta_i} \subseteq B_1$, and that $P^+_{\Delta_i} \subseteq B_2$. Recall also that the points in the left halfspace h_λ bounded by λ are all contained in B_1. Moreover, the plane π containing the intersection circle C_{12} is dual to a point π^*, which has to separate $(P^*)^+_{\Delta_i}$ from $(P^*)^-_{\Delta_i}$. Hence, all the points of $P^+_{\Delta_i}$ have to lie on the other side of π, and in B_2, which is easily seen to imply that none of them can lie in h_λ. See Figure 7. We thus verify that $P^+_{\Delta_i} \cap h_\lambda = \emptyset$, aborting otherwise the guess of λ. (Note that, in contrast, points of $P^-_{\Delta_i}$ can also lie to the right of λ.)

We now have a subset $P_L \subseteq P^0_{\Delta_i}$ of $O(m)$ points to the left of λ, which are assumed, together with the points of $P^-_{\Delta_i}$, to be contained in B_1. Note however that, for Lemma 2 to hold, we have to define σ_L only in terms of the points to the left of λ. Therefore, we compute the surface $\sigma'_L = \partial K(P_L \cup (P^-_{\Delta_i} \cap h_\lambda)) \cap h_\lambda$ and search on it for a placement of the center c_1 of B_1. However, since the remaining points of $P^-_{\Delta_i}$ are also assumed to belong to B_1, we need to consider only the portion of σ'_L inside $\bigcap\{B_r(p) \mid p \in P^-_{\Delta_i} \setminus h_\lambda\}$. Let σ''_L denote this portion. It is easy to compute σ''_L in $O(n \log n)$ time (details omitted). So far, the cost of the decision procedure also depends (cheaply — see below) on the initial input size n, but the saving in this setup comes from the fact that it suffices to intersect the $O(m)$ spheres $\partial B_r(p)$, for $p \in P^0_{\Delta_i} \setminus h_\lambda$, with σ''_L to obtain the map M, since only the points of $P^0_{\Delta_i}$ are "undecided". (The points of $P^+_{\Delta_i}$ are always placed in B_2 — see below.)

Note that σ''_L need not to be connected, but we can nevertheless visit all the cells of M in a single connected tour (possibly with repetitions), by "hopping" from one connected component of σ''_L to another via the (connected) network of edges of σ'_L. As is easy to check, the overall length of the tour is $O(m^2 + mn) = O(mn)$. We thus build a segment tree $T = T_M$ to maintain the subset P^+_τ of points of P not covered by B_1. Building T and searching through its paths is done as in Case 2 of the pure decision procedure, except for several easy modifications, such as precomputing $K(P^+_{\Delta_i})$, storing this spherical polytope at the root of the segment tree, and including it in the intersections along each path of the tree. We omit the further details, which can be found in [1].

Running time. For each cell of M we run the procedure described in Section 3 for determining whether the intersection of the corresponding spherical polytopes is non-degenerate. Therefore, solving each subproblem requires $O(mn \log^8 n)$ time. The $O(mn \log n)$ time required to build M, the $O(n \log n)$ time required to construct the intersection of the balls in $\{B_r(p) \mid p \in P^+_{\Delta_i}\}$, and the $O(mn)$ cost of Case 1, are all subsumed in that cost. Repeating this for each of the $O(1/\beta^3)$ guesses of an orientation and a separating plane, results in $O\left((1/\beta^3)mn \log^8 n\right)$ time. When the recursion bottoms out, we handle it the same way as in Section 3.

The expected cost of solving the optimization problem using Chan's technique is asymptotically the same as that of the decision procedure. To see this, we need to overcome some subtleties in the application of the original analysis of Chan, similar to those in the less efficient algorithm. Arguing similarly, we obtain the following recurrence for the maximum expected cost $T(m, n)$ of solving a recursive subproblem involving m "local" points, where n is the number of initial input points in P.

$$T(m,n) \le \begin{cases} \log(c\varrho^3)T(m/\varrho, n) & \\ +O\left((1/\beta^3)mn \log^8 n\right), & \text{for } m \ge \varrho, \\ & \\ O(n), & \text{for } m < \varrho, \end{cases} \tag{3}$$

where c is an appropriate absolute constant (as in Section 3), ϱ is the parameter of the cutting, chosen to be a sufficiently large constant, and $\beta = 2\,(r_0/r' - 1)$, where r' is the value of r at which the initial exponential search is terminated.

It can be shown, rather easily (and we omit the details), that the recurrence (3) yields the overall bound $O\left((1/\beta^3)n^2 \log^8 n\right)$ on the expected cost of the initial problem; i.e.,

$$T(n,n) = O\left((1/\beta^3)n^2 \log^8 n\right).$$

We thus finally obtain our main result:

THEOREM 3. *Let P be a set of n points in $\mathbb{R}^3$. A 2-center for P can be computed in $O((n^2 \log^8 n/(1 - r^*/r_0)^3)$ randomized expected time, where r^* is the radius of the 2-center for P and r_0 is the radius of the smallest enclosing ball of P.*

5. CONCLUSIONS

In this paper we presented two algorithms for computing the 2-center of a set of points in $\mathbb{R}^3$. The first algorithm takes near-cubic time, and the second one takes near-quadratic time provided that the two centers are not too close to each other. An obvious open problem is to design an algorithm for the 2-center problem that runs in near-quadratic time on all point sets in $\mathbb{R}^3$. Another interesting question is whether the 2-center problem in $\mathbb{R}^3$ is n^2-*hard* (see [21] for details), which would suggest that a near-quadratic algorithm is (almost) the best possible for this problem.

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