

Nondegenerate Spheres in Three Dimensions *

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Abstract

Let P be a set of n points in \mathbb{R}^3 , and $k \leq n$ an integer. A sphere σ is k -rich with respect to P if $|\sigma \cap P| \geq k$, and is η -nondegenerate, for a fixed fraction $0 < \eta < 1$, if no circle $\gamma \subset \sigma$ contains more than $\eta|\sigma \cap P|$ points of P .

We improve the previous bound given in [1] on the number of k -rich η -nondegenerate spheres in 3-space with respect to any set of n points in \mathbb{R}^3 , from $O(n^4/k^5 + n^3/k^3)$, which holds for all $0 < \eta < 1/2$, to¹ $O^*(n^4/k^{11/2} + n^2/k^2)$, which holds for all $0 < \eta < 1$ (in both bounds, the constants of proportionality depend on η). The new bound implies the improved upper bound $O^*(n^{58/27}) \approx O(n^{2.1482})$ on the number of mutually similar triangles spanned by n points in \mathbb{R}^3 ; the previous bound was $O(n^{13/6}) \approx O(n^{2.1667})$ [1].

1 Introduction

1.1 Nondegenerate hyperplanes and spheres

The concept of *degeneracy* of a hyperplane was introduced by Elekes and Tóth [10]. Given a finite point set $P \subset \mathbb{R}^d$ and a constant $0 < \eta < 1$, a hyperplane π in \mathbb{R}^d is said to be η -degenerate (with respect to P), if there exists some lower-dimensional affine subspace $\pi' \subset \pi$ such that

$$|\pi' \cap P| \geq \eta|\pi \cap P|.$$

If no such affine subspace π' exists, then π is said to be η -nondegenerate. A hyperplane π is called k -rich (with respect to P) if $|\pi \cap P| \geq k$. Elekes and Tóth [10] showed that for any dimension d there exists a constant $\eta_d < 1$ which depends on d , so that, for any $\eta < \eta_d$ and for any set of n points in \mathbb{R}^d , the number of η -nondegenerate k -rich hyperplanes is

$$O\left(\frac{n^d}{k^{d+1}} + \frac{n^{d-1}}{k^{d-1}}\right), \quad (1.1)$$

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¹The $O^*(\cdot)$ notation hides subpolynomial factors.

with the multiplicative constant depending on η and d , and that this bound is asymptotically best possible. We have $\eta_3 = 1$, $\eta_4 = 1/2$, and as d increases, η_d gets smaller and smaller. It is not known whether the bound applies to all $0 < \eta < 1$ in any dimension.

The motivation for studying nondegenerate hyperplanes comes from incidence problems between points and hyperplanes in three and higher dimensions. These problems face the major difficulty that without imposing any restrictions on the points and/or hyperplanes, the incidence graph can be the complete bipartite graph: This can happen already in three dimensions, if we place all the points on a common line and make all the planes pass through that line. We note, however, that in such constructions, the planes (or hyperplanes in higher dimensions) are highly degenerate. As Elekes and Tóth [10] showed, by restricting the hyperplanes to be nondegenerate, one can obtain better nontrivial bounds for point-hyperplane incidences.

In this paper we consider a related problem involving incidences between points and *spheres*. Given a finite point set $P \subset \mathbb{R}^d$ and a constant $0 < \eta < 1$, a $(d-1)$ -sphere $\sigma \subset \mathbb{R}^d$ is called η -*degenerate* (with respect to P) if there exists some $(d-2)$ -subsphere $\sigma' \subset \sigma$ such that

$$|\sigma' \cap P| \geq \eta |\sigma \cap P|.$$

Otherwise, σ is called η -*nondegenerate*. The notion of k -*richness* (with respect to P) is defined as in the case of hyperplanes.

Point-sphere incidence problems arise in several problems involving distances between points. For example, the study of Aronov et al. [4] on the number of distinct distances determined by n points in \mathbb{R}^3 involves incidences between the points and a certain collection of spheres, and faces the issue of degeneracy of these spheres. Another study by the authors and others [1] considers the problem of bounding the number of similar triangles (or simplices) determined by n points in d dimensions. Here too the problem is reduced to incidences between points and spheres, and handling degenerate and nondegenerate spheres is a major step in the analysis.

By lifting \mathbb{R}^d to the standard paraboloid in \mathbb{R}^{d+1} (see, e.g., [8]), every sphere is transformed into a hyperplane, and the incidence relation, as well as degeneracy and nondegeneracy, are preserved. It thus follows that for any $\eta < \eta_{d+1}$, the number of k -rich η -nondegenerate $(d-1)$ -spheres, with respect to a set of n points in \mathbb{R}^d is

$$O\left(\frac{n^{d+1}}{k^{d+2}} + \frac{n^d}{k^d}\right). \quad (1.2)$$

See [1] for more details.

It has been conjectured (see [1]) that this bound is not tight. A supporting evidence comes from the fact that in the plane, where the spheres are circles, which are clearly nondegenerate (if they are k -rich and k is sufficiently large), we have an upper bound of $O^*(n^3/k^{11/2} + n^2/k^3 + n/k)$ on the number of k -rich circles [2, 5, 12], which is significantly better than the $O(n^3/k^4 + n^2/k^2)$ bound implied by lifting the circles into nondegenerate planes in \mathbb{R}^3 . We recall that this improved bound holds for circles in any dimension; see [3].

At any rate, the bound for spheres in \mathbb{R}^d should lie in between those for hyperplanes in \mathbb{R}^d and for hyperplanes in \mathbb{R}^{d+1} . The second threshold follows from the lifting transform just discussed, and the first threshold follows from noting that an *inversion* of \mathbb{R}^d takes a collection of hyperplanes in \mathbb{R}^d into a collection of spheres (all passing through a common point), while preserving incidences, richness, degeneracy, and nondegeneracy. In particular, we obtain the lower bound (the best known, as far as we are aware)

$$\Omega\left(\frac{n^d}{k^{d+1}} + \frac{n^{d-1}}{k^{d-1}}\right)$$

on the number of k -rich η -nondegenerate spheres in \mathbb{R}^d , by taking the Elekes-Tóth lower bound construction for nondegenerate hyperplanes in \mathbb{R}^d , and by inverting space.

Narrowing this gap between the upper and lower bounds is an interesting problem in its own right. Moreover, any improvement of the upper bound will immediately improve the bounds on the problems studied in the papers [1, 4] mentioned above.

1.2 Our results

In this paper we improve the upper bound on the number of rich nondegenerate spheres in 3-space. We show:

Theorem 1.1. *For any constant $\eta < 1$, for any set P of n points in \mathbb{R}^3 , and for any $k \leq n$, the number of k -rich η -nondegenerate spheres is*

$$O^* \left(\frac{n^4}{k^{11/2}} + \frac{n^2}{k^2} \right),$$

with the multiplicative constant depending on η .

Clearly, this is an improvement over the previous bound of $O(n^4/k^5 + n^3/k^3)$. Using this bound, we get an improved upper bound on the number of mutually similar triangles spanned by n points in \mathbb{R}^3 .

Theorem 1.2. *Let P be a set of n points in \mathbb{R}^3 , and let Δ be a triangle. Then the number of triples of points of P that span a triangle similar to Δ is $O^*(n^{58/27}) = O(n^{2.1482})$.*

The previous bound was $O(n^{13/6}) = O(n^{2.1667})$ [1].

Discussion. There are three conventional ways to present incidence bounds:

- (a) An upper bound on the number of incidences between n points and m objects. For example, the above mentioned Elekes-Tóth bound [10], when formulated this way, asserts that the number of incidences between n points in \mathbb{R}^d and m η_d -nondegenerate hyperplanes is $O(n^{d/(d+1)}m^{d/(d+1)} + nm^{(d-2)/(d-1)})$.
- (b) An upper bound on the number of objects incident to at least k out of n points. This, for example, is the formulation of the Elekes-Tóth bound, which says that the number of k -rich η_d -nondegenerate hyperplanes spanned by a set of n points in \mathbb{R}^d is $O(n^d/k^{d+1} + n^{d-1}/k^{d-1})$.
- (c) An upper bound on the number of incidences between n points and objects incident to at least k of these points. For example, the Elekes-Tóth bound can be reformulated to say that the number of incidences between n points in \mathbb{R}^d and any family of k -rich η_d -nondegenerate hyperplanes is $O(n^d/k^d + n^{d-1}/k^{d-2})$.

In practically all known instances, all three alternatives are equivalent and any one of them can be derived from any other. Theorem 1.1, when stated in the first form (a), reads as follows.

Theorem 1.3. *For any $\eta < 1$, the number of incidences I between n points in \mathbb{R}^3 and m η -nondegenerate spheres is*

$$I = O^* \left(n^{8/11}m^{9/11} + nm^{1/2} \right),$$

with the multiplicative constant depending on η .

We will prove Theorem 1.3 rather than Theorem 1.1. Theorem 1.1 follows by noting that if each of the m spheres is also k -rich then $I \geq mk$. Solving the resulting inequality $mk = O^*(n^{8/11}m^{9/11} + nm^{1/2})$ for m yields the bound of Theorem 1.1.

We use in the proof the previous bound for the number of incidences between n point in \mathbb{R}^3 and m nondegenerate spheres, and strengthen it by *cutting*. The bound is

$$I = O\left(n^{4/5}m^{4/5} + nm^{2/3}\right), \quad (1.3)$$

which is the alternative form (a) of the Elekes-Tóth bound (1.2) in four dimensions (where the given spheres appear as hyperplanes).

2 Proof of Theorem 1.3

Let P be a set of n points in \mathbb{R}^3 , and let \mathcal{S} be a set of m η -degenerate spheres in \mathbb{R}^3 , for some fixed positive $\eta < 1$. Let $j > 0$ be some integer. We say that a sphere σ is j -bad with respect to P if σ contains some circle $\gamma \subset \sigma$, which contains at least j points of P , i.e., $|\gamma \cap P| \geq j$. If no such circle exists, then σ is said to be j -good. Note that an η -nondegenerate sphere incident to exactly k points is, by definition, ηk -good.

Let $I(P, \mathcal{S})$ denote the number of incidences between P and \mathcal{S} . Denote by $I_\eta(n, m)$, or $I(n, m)$ for short, the maximum of $I(P, \mathcal{S})$ over all sets P of n points and \mathcal{S} of m η -nondegenerate spheres in \mathbb{R}^3 . Note that each sphere $\sigma \in \mathcal{S}$ contains some noncoplanar quadruple of points, and each such quadruple uniquely determines the sphere containing it, so we have $m = |\mathcal{S}| \leq \binom{n}{4}$. In fact, this bound decreases as η decreases, but it remains $O(n^4)$, with the constant of proportionality decreasing with η . Therefore, $I(n, m)$ is defined only for values of n and m satisfying (the appropriate variant of) this relationship.

The next lemma establishes a recurrence relation on $I(n, m)$.

Lemma 2.1. *For any $0 < \eta < 1$, for any two positive integers n, m satisfying $m = O(n^4)$ as above, for any integer $1 < j < n$, and for any number $1 < r < \min\{m, n^{1/3}\}$, we have*

$$I(n, m) = O\left(r^3 \beta(r) \log^3 r \cdot I\left(\frac{n}{r^3}, \frac{m}{r}\right) + mjr^2 \beta(r) \log^3 r + \frac{n^4 \log n}{j^{11/2}} + \frac{n^3}{j^3} + \frac{n^2}{j}\right), \quad (2.4)$$

where the constant of proportionality depends on η , and where $\beta(r) = 2^{O(\alpha^2(r))}$ and $\alpha(r)$ is the extremely slowly growing inverse Ackermann function.

Proof. Let P be a set of n points in \mathbb{R}^3 , and let \mathcal{S} be a set of m η -nondegenerate spheres with respect to P . We partition \mathcal{S} into two subsets consisting respectively of the j -good and the j -bad spheres, and bound the incidences of P with each of these subsets separately.

We bound the number of incidences with the j -bad spheres as follows. The number of j -rich circles with respect to P is $O((n^3 \log n)/j^{11/2} + n^2/j^3 + n/j)$ [2, 3, 5, 12].² Each circle γ can be incident to several spheres of \mathcal{S} . Since each such sphere σ is η -nondegenerate, we have $|\sigma \cap P| > \frac{1}{\eta}|\gamma \cap P|$, or $|\sigma \cap P| < \frac{1}{1-\eta}|(\sigma \setminus \gamma) \cap P|$. Any point of P not on γ can lie in at most one of these spheres, so

$$\sum_{\sigma \in \mathcal{S} | \sigma \supset \gamma} |\sigma \cap P| < \sum_{\sigma \in \mathcal{S} | \sigma \supset \gamma} \frac{1}{1-\eta} |(\sigma \setminus \gamma) \cap P| < \frac{n}{1-\eta}.$$

²This is the form (b) of the bound $O(n^{6/11}m^{9/11} \log^{2/11}(n^3/m) + n^{2/3}m^{2/3} + n + m)$ on the number of incidences between n points and m circles in 3-space. This follows from the best known bound, with the same asymptotic value, for the incidence problem in the plane, as derived by Marcus and Tardos [12]; the previous bounds [2, 3, 5] were slightly weaker.

Hence, the total number of incidences between the spheres containing γ and P is $O(n)$. Multiplying by the number of circles, we get that the number of incidences between P and the j -bad spheres is

$$I_{\text{bad}} = O\left(\frac{n^4 \log n}{j^{11/2}} + \frac{n^3}{j^3} + \frac{n^2}{j}\right). \quad (2.5)$$

To bound the number of incidences with the j -good spheres, we construct a $\frac{1}{r}$ -cutting of \mathcal{S} . We will use the simpler version of cutting, in which we sample spheres of \mathcal{S} by choosing each sphere in the sample independently at random with probability $\frac{Cr}{n} \log r$, for an appropriate sufficiently large constant C , and then construct the vertical decomposition of the arrangement of the random sample (see, e.g., [6, 7]). As follows from the analysis of [7], the result is a set of an expected number of $O(r^3 \beta(r) \log^3 r)$ relatively open cells, each of constant description complexity, such that, with high probability, each cell is *crossed* by (i.e., is intersected by but not contained in) at most m/r spheres of \mathcal{S} . We will assume that our cutting does satisfy these properties, i.e., it consists of $O(r^3 \beta(r) \log^3 r)$ cells, and each cell is crossed by at most m/r spheres of \mathcal{S} . We may also assume that each cell contains at most n/r^3 points of P . To enforce this, we take each cell that contains tn/r^3 points, for any $t > 1$, and partition it into $t + 1$ subcells in some generic way, such that each subcell contains at most n/r^3 points. The number of cells in this refined cutting is still $O(r^3 \beta(r) \log^3 r)$.

For each cell τ separately, we bound the number of incidences between the points of $P \cap \tau$ and the spheres that cross τ , and then sum up the bounds over all cells. A cell τ has at most n/r^3 points and is crossed by at most m/r spheres. If a sphere contains more than j/η points of $P \cap \tau$, then, since it is a j -good sphere, it is locally η -nondegenerate in τ (i.e., with respect to the points of $P \cap \tau$). The number of incidences between such spheres and the points of $P \cap \tau$ is at most $I\left(\frac{n}{r^3}, \frac{m}{r}\right)$. The number of incidences with the other spheres in τ is $O(mj/r)$. The total, summed over all cells, is

$$\sum_{\tau} O\left(I\left(\frac{n}{r^3}, \frac{m}{r}\right) + \frac{mj}{r}\right) = O\left(r^3 \beta(r) \log^3 r \cdot I\left(\frac{n}{r^3}, \frac{m}{r}\right) + mjr^2 \beta(r) \log^3 r\right).$$

It remains to bound, for each cell τ , the number of incidences between the points of $P \cap \tau$ and the spheres of \mathcal{S} that contain τ , which we do separately for cells of dimensions 2, 1 and 0. (Cells of dimension 3 are “exempt” from this analysis.) Any two-dimensional cell can be contained by at most one sphere, so these cells contribute a total of $O(n)$ incidences of this kind. For one-dimensional cells, we can restrict our attention only to these cells which are (portions of) edges in the arrangement of the sample spheres. (By choosing a generic coordinate frame, the other edges of the vertical decomposition will not contain points of P .) Each of the m spheres intersects the $O(r \log r)$ spheres of the sample along $O(r \log r)$ circles, and each such circle contains at most j points of P . Thus, the number of incidences contributed by the one-dimensional cells is $O(mjr \log r)$. As for zero-dimensional cells, which are simply the vertices of the arrangement of the sample spheres (as above, we may ignore the additional vertices created by the vertical decomposition), each of the m spheres σ contains $O(r^2 \log^2 r)$ such vertices, which are intersection points of the $O(r \log r)$ circles of intersection of σ with the sample spheres. Hence, the total contribution of these cells is at most $O(mr^2 \log^2 r)$ incidences. Thus, the total number of incidences, over all cells τ , between the points of $P \cap \tau$ and the spheres of \mathcal{S} that contain τ is $O(n + mjr \log r + mr^2 \log^2 r)$, which is asymptotically smaller than $O(n + mjr^2 \beta(r) \log^3 r)$, and so, the total number of incidences with the good spheres is

$$I_{\text{good}} = O\left(r^3 \beta(r) \log^3 r \cdot I\left(\frac{n}{r^3}, \frac{m}{r}\right) + mjr^2 \beta(r) \log^3 r + n\right). \quad (2.6)$$

Summing both inequalities (2.5) and (2.6), we obtain the bound asserted in the lemma. \square

Next, we simplify the recurrence by getting rid of j . We have, on one hand, the term $mjr^2\beta(r)\log^3 r$, which increases with j , and, on the other hand, the terms

$$\frac{n^4 \log n}{j^{11/2}} + \frac{n^3}{j^3} + \frac{n^2}{j},$$

which decrease with j , so we choose j so as to balance between them. By comparing the increasing term with each of the decreasing terms, this leads to the choice

$$j = \frac{n^{8/13}}{m^{2/13}r^{4/13}} + \frac{n^{3/4}}{m^{1/4}r^{1/2}} + \frac{n}{m^{1/2}r},$$

which implies

$$\begin{aligned} I(n, m) &= O\left(r^3 l(r) I\left(\frac{n}{r^3}, \frac{m}{r}\right)\right. \\ &\quad \left.+ l(r) \left(n^{8/13} m^{11/13} r^{22/13} \log\left(\frac{n^4}{m}\right) + n^{3/4} m^{3/4} r^{3/2} + nm^{1/2} r\right)\right), \end{aligned} \quad (2.7)$$

where $l(r) = \beta(r)\log^3 r$.

Our next and final step is to solve this recurrence, and derive the bound of Theorem 1.3.

Theorem 2.2 (Cf. Theorem 1.3). *For any arbitrarily small $\varepsilon > 0$, there exists a constant A_ε , which depends on ε and η , such that we have*

$$I_\eta(n, m) \leq A_\varepsilon \left(n^{8/11+\varepsilon} m^{9/11+\varepsilon} + nm^{1/2}\right). \quad (2.8)$$

Proof. We use induction on m . Our end conditions are as follows

1. If m is smaller than some constant, say c , then $I \leq nm \leq cn$. Clearly, in this case (2.8) is also satisfied, if we choose A_ε sufficiently large.
2. If m is large enough relative to n , namely, if $m > Cn^4$, for some appropriate constant C depending on η , then $I = O(m) = O(n^4)$ (recall that only the case $m = O(n^4)$ can arise, so in the present situation $m = \Theta(n^4)$). This can be shown, e.g., using the previous bound of $I = O(n^{4/5}m^{4/5} + nm^{2/3})$. In this case we also have $I = O(n^{8/11+\varepsilon}m^{9/11+\varepsilon} + nm^{1/2})$ as in (2.8).

We now deal with the case where n and m are large, and $c < m < Cn^4$, in which case, we apply the induction hypothesis and rewrite (2.7) as

$$\begin{aligned} I(n, m) &\leq B \left(r^3 l(r) A_\varepsilon \left(\left(\frac{n}{r^3}\right)^{8/11+\varepsilon} \left(\frac{m}{r}\right)^{9/11+\varepsilon} + \left(\frac{n}{r^3}\right) \left(\frac{m}{r}\right)^{1/2} \right) \right. \\ &\quad \left. + l(r) \left(n^{8/13} m^{11/13} r^{22/13} \log\left(\frac{n^4}{m}\right) + n^{3/4} m^{3/4} r^{3/2} + nm^{1/2} r \right) \right) \\ &= B \cdot l(r) A_\varepsilon \left(\frac{n^{8/11+\varepsilon} m^{9/11+\varepsilon}}{r^{4\varepsilon}} + \frac{nm^{1/2}}{r^{1/2}} \right) \\ &\quad + B \cdot l(r) \left(n^{8/13} m^{11/13} r^{22/13} \log\left(\frac{n^4}{m}\right) + n^{3/4} m^{3/4} r^{3/2} + nm^{1/2} r \right), \end{aligned}$$

for an appropriate absolute constant B . We now choose r sufficiently large so that $\frac{B \cdot l(r)}{r^{4\varepsilon}} < \frac{1}{2}$, and then choose A_ε sufficiently large so that the last terms (in the second line of the bound) are at most $\frac{1}{2}A_\varepsilon(n^{8/11+\varepsilon}m^{9/11+\varepsilon} + nm^{1/2})$. This can be done, because the terms $n^{8/13}m^{11/13}r^{22/13} \log\left(\frac{n^4}{m}\right)$ and $n^{3/4}m^{3/4}r^{3/2}$ are dominated by the terms $n^{8/11+\varepsilon}m^{9/11+\varepsilon} + nm^{1/2}$, for $m < Cn^4$. Thus,

$$I(n, m) \leq A_\varepsilon \left(n^{8/11+\varepsilon}m^{9/11+\varepsilon} + nm^{1/2} \right),$$

establishing the induction step and thereby completing the proof of the theorem, and consequently also the proof of Theorem 1.3. \square

Remark: The reader may be left wondering where did the exponents $8/11$ and $9/11$ “pop-up” from, since they do not appear explicitly in (2.7). The answer is that a solution of (2.7) with a leading term $n^\alpha m^\beta$ must satisfy the two inequalities $3\alpha + \beta > 3$, and $\alpha + 4\beta > 4$. The first inequality is needed to control the homogeneous part of the recurrence, i.e., the term $r^3 l(r) \cdot I\left(\frac{n}{r^3}, \frac{m}{r}\right)$, and the second inequality is needed to ensure that the term $n^{8/13}m^{11/13}$ is dominated by $n^\alpha m^\beta$ when $m = O(n^4)$, as can be easily verified. The exponents $8/11 + \varepsilon$ and $9/11 + \varepsilon$, for arbitrarily small ε , are, in a sense, the best solution of these inequalities.

3 Applications of the new bound

As already noted, nondegenerate spheres were used in some previous works. In [4], they were used in the analysis of distinct distances in \mathbb{R}^3 , and in [1], we used them to bound the number of mutually similar simplices in d dimensions. The improved bound of Theorem 1.1, once plugged into these analyses, gives improved results for the respective problems. In this section we apply Theorem 1.1 to the analysis of [1] of similar triangles in \mathbb{R}^3 and prove Theorem 1.2. Recall that the previous bound, established in [1], on the number of similar triangles spanned by n points in \mathbb{R}^3 is $O(n^{13/6}) = O(n^{2.1667})$. With the use of Theorem 1.1, the bound becomes $O^*(n^{58/27}) = O^*(n^{2.1482})$, as we prove next.

Proof of Theorem 1.2. Fix a parameter $k < n$, and denote by $M(n, k)$ the maximum possible number of k -rich nondegenerate spheres spanned by n points in \mathbb{R}^3 (for some fixed constant degeneracy factor η). As shown in [1], the number T of triangles spanned by n points in \mathbb{R}^3 and similar to a fixed triangle is upper-bounded by

$$T = O\left(n^2 k^{1/3} + M(n, k)k^{4/3}\right),$$

for any choice of k . The optimal choice is of course the one that minimizes the expression on the right hand side. In [1] we used the old bound $M(n, k) = O(n^4/k^5 + n^3/k^3)$, for which the optimal k is $k = n^{1/2}$, and the resulting bound is $T = O(n^{13/6})$. If we use the new bound, then we get

$$T = O^*\left(n^2 k^{1/3} + \frac{n^4}{k^{25/6}} + \frac{n^2}{k^{2/3}}\right) = O^*\left(n^2 k^{1/3} + \frac{n^4}{k^{25/6}}\right).$$

We choose $k = n^{4/9}$, and get

$$T = O^*(n^{58/27}),$$

as claimed. \square

Theorem 1.1 can also be applied to yield a better result, over the one achieved in [4], for the number, denoted by $g_3(n)$, of distinct distances determined by n points in \mathbb{R}^3 , but this result has already been surpassed by Solymosi and Vu [13]. Specifically, the old bound in [4] was $g_3(n) = \Omega^*(n^{77/141}) = \Omega(n^{0.546})$. With the new bound we get $g_3(n) = \Omega^*(n^{5/9}) = \Omega(n^{0.555})$, but the Solymosi-Vu bound [13] is $g_3(n) = \Omega(n^{0.5644})$.³ (The upper bound on the minimum number of distinct distances is $O(n^{2/3})$.)

4 Conclusion

We have shown an improved bound on the number of incidences between points and nondegenerate spheres in \mathbb{R}^3 , and demonstrated its usefulness by improving the bound on the number of mutually similar triangles in \mathbb{R}^3 . We believe that our main results (Theorems 1.1 and 1.3) can be improved still further. We also believe that our results can be extended to higher-dimensional nondegenerate spheres, and that they shall find applications in various other problems of geometric incidences and repeated subconfigurations.

It is interesting to compare our results to the recent work of Elekes and Szabó [9], in which they define the *combinatorial dimension* of a set of vertices in a bipartite graph. They show that a set of n points and a family of m hyperplanes in \mathbb{R}^d , such that the set of points has combinatorial dimension k in the point-hyperplane incidence graph, has at most $I = O^*(n^{d(k-1)/(dk-1)}m^{k(d-1)/(dk-1)} + n + m)$ incidences. For $d = 4$ and $k = 3$, one gets $I = O^*(n^{8/11}m^{9/11} + n + m)$, very similar to our bound. Recall that in our setting of n points and m spheres in \mathbb{R}^3 , by lifting \mathbb{R}^3 to \mathbb{R}^4 , one can think of them as points and hyperplanes in \mathbb{R}^4 with all the points lying on a three-dimensional paraboloid, thus having dimension 3. In spite of the similarity between the two results, we note that the three-dimensionality of the points in the point-sphere setting does not appear to conform with Elekes's and Szabó's definition of combinatorial dimension.

Indeed, in their setup, we consider the bipartite incidence graph $H \subseteq P \times \mathcal{S}$, and say that a subset $Q \subseteq P$ has combinatorial dimension k , if, after discarding at most b spheres from \mathcal{S} , for some fixed constant b , the following holds: For each sphere σ in the remaining set \mathcal{S}' , the points of $Q \cap \sigma$ have combinatorial dimension $k - 1$, with respect to the spheres in $\mathcal{S}' \setminus \{\sigma\}$. To have combinatorial dimension 0, Q should have at most b points. Thus, for P to have combinatorial dimension 3, $P \cap \sigma$, for most spheres $\sigma \in \mathcal{S}$, should have combinatorial dimension 2, which in turn means that for most pairs σ, σ' of spheres in \mathcal{S} , $P \cap \sigma \cap \sigma'$ should have combinatorial dimension 1, which in turn means that for most triples $\sigma, \sigma', \sigma''$ of spheres, $|P \cap \sigma \cap \sigma' \cap \sigma''|$ should be at most b . Unfortunately, this does not have to hold, since $\sigma \cap \sigma' \cap \sigma''$ could be a circle containing many points of P .

Yet, it is possible that there is some indirect connection between Elekes's and Szabó's combinatorial dimension, and point-nondegenerate sphere incidences. In particular, consider a set P of n points in \mathbb{R}^d and a set \mathcal{S} of m nondegenerate spheres in \mathbb{R}^d . If P had combinatorial dimension d with respect to \mathcal{S} , then, after lifting the spheres to nondegenerate hyperplanes in \mathbb{R}^{d-1} , the analysis in [9] would yield the bound $O^*(n^{(d^2-1)/(d^2+d-1)}m^{d^2/(d^2+d-1)} + n + m)$ on the number of incidences between P and \mathcal{S} . Combining this "lead" with the bound we have for $d = 3$ in Theorem 1.3, we tend to conjecture that the number of these incidences is $I = O^*(n^{(d^2-1)/(d^2+d-1)}m^{d^2/(d^2+d-1)} + nm^{(d-2)/(d-1)})$.

³In their paper [13], they specify a slightly weaker bound of $g_3(n) = \Omega(n^{0.5643})$. Their analysis depends on the respective two-dimensional lower bound, i.e. on the number of distinct distances between n points in the plane. This lower bound was later improved by Katz and Tardos [11], which consequently improved also the lower bound in three dimensions. In more detail, Solymosi and Vu [13] showed that $g_3(n) = \Omega^*(n^{\omega_3})$, where $\omega_3 \geq 3/(3 + 2/\omega_2)$, and ω_2 is any number such that $g_2(n) = \Omega^*(n^{\omega_2})$. As of today, the best estimate for ω_2 , due to Katz and Tardos [11], is $\omega_2 \geq 0.8641$, which implies $\omega_3 \geq 0.5644$. Solymosi and Vu [13] used the weaker bound $\omega_2 \geq 0.8635$, which only implies $\omega_3 \geq 0.5643$.

References

- [1] P. K. Agarwal, R. Apfelbaum, G. Purdy and M. Sharir, Similar simplices in a d -dimensional point set, *Proc. 23rd Annu. ACM Sympos. Comput. Geom.* (2007), 232–238.
- [2] P. K. Agarwal, E. Nevo, J. Pach, R. Pinchasi, M. Sharir and S. Smorodinsky, Lenses in arrangements of pseudocircles and their applications, *J. ACM* 51 (2004), 139–186.
- [3] B. Aronov, V. Koltun and M. Sharir, Incidences between points and circles in three and higher dimensions, *Discrete Comput. Geom.* 33 (2005), 185–206.
- [4] B. Aronov, J. Pach, M. Sharir and G. Tardos, Distinct distances in three and higher dimensions, *Combinat. Probab. Comput.* 13 (2004), 283–293.
- [5] B. Aronov and M. Sharir, Cutting circles into pseudo-segments and improved bounds for incidences, *Discrete Comput. Geom.* 28 (2002), 475–490.
- [6] B. Chazelle, Cuttings, in *Handbook of Data Structures and Applications* (D. Mehta and S. Sahni, eds.), chap. 25, Chapman and Hall/CRC Press, Boca Raton, FL, 2005.
- [7] K. Clarkson, H. Edelsbrunner, L. Guibas, M. Sharir and E. Welzl, Combinatorial complexity bounds for arrangements of curves and spheres, *Discrete Comput. Geom.* 5 (1990), 99–160.
- [8] H. Edelsbrunner, *Algorithms in Combinatorial Geometry*, Springer Verlag, Heidelberg, 1987.
- [9] G. Elekes and E. Szabó, How to find groups? (And how to use them in Erdős geometry?), accepted to *Combinatorica*.
- [10] G. Elekes and C. D. Tóth, Incidences of not too degenerate hyperplanes, *Proc. 21st Annu. ACM Sympos. Comput. Geom.* (2005), 16–21.
- [11] N. H. Katz and G. Tardos, A new entropy inequality for the Erdős distance problem, in *Towards a Theory of Geometric Graphs* (J. Pach, ed.), Contemporary Mathematics 342, AMS, Providence, RI, 2004, 119–126.
- [12] A. Marcus and G. Tardos, Intersection reverse sequences and geometric applications, *J. Combinat. Theory Ser. A* 113 (2006), 675–691.
- [13] J. Solymosi and V. Vu, Near optimal bounds for the Erdős distinct distances problem in high dimensions, *Combinatorica* 28 (2006), 113–125.