On Cell Complexities in Hyperplane Arrangements*

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Abstract

We derive improved bounds on the complexity of many cells in arrangements of hyperplanes in higher dimensions, and use these bounds to obtain a very simple proof of a bound, due to [2], on the sum of squares of cell complexities in such an arrangement.

1 Complexity of Many Cells

The main result of the paper, which improves upon previous bounds given in [2], is:

Theorem 1.1 *The complexity of m distinct cells in an arrangement of n hyperplanes in d dimensions, for $d \geq 4$, is $O(m^{1/2} n^{d/2} \log^{(\lfloor d/2 \rfloor - 2)/2} n)$ with the implied constant of proportionality depending on d.*

Proof: The proof proceeds by induction on d. The base case $d = 4$ depends on a sharper bound that is known for $d = 3$ and will be cited below.

Let H be a collection of n hyperplanes in d-space. We will assume that the planes are in *general position*, meaning that any k planes meet in a $d-k$-flat, if $k = 1, \ldots, d$, and not at all if $k > d$. It is not difficult to see that worst-case cell complexity can always be achieved by planes in general position. Let P be a set of m points, not lying on any hyperplane. Denote by $K_j^{(d)}(P, H)$ the number of j-faces bounding the cells of $\mathcal{A}(H)$ that contain points of P. We will mainly be concerned with the case $j = \lceil d/2 \rceil$, because, as follows from the Dehn-Sommerville relations (see, e.g., [3]), the total number of faces, of all dimensions, of a cell (which is a simple d-polytope) is at most proportional to the number of its $\lceil d/2 \rceil$-faces. We denote by $K_j^{(d)}(m, n)$ the maximum of $K_j^{(d)}(P, H)$ over all sets P, H as above.

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We now derive a recurrence for $K_j^{(d)}(m,n)$. Pick $h \in H$, remove it and add it back. Consider the j-faces that are not contained in h and bound cells of the arrangement that contain points of P. This number can increase when h is added to $\mathcal{A}(H \setminus \{h\})$, only when h splits a cell c containing points of P into two subcells, each containing points of P. In this case, the local increase in the number of j-faces under consideration is equal to the number of $(j-1)$-faces of the $(d-1)$-face $c \cap h$ of $\mathcal{A}(H)$. Denote by H/h the set $\{h \cap h' \mid h' \in H \setminus \{h\}\}$ of $(d-2)$-hyperplanes within h. Then the total increase in the number of j-faces under consideration that is caused by the re-insertion of h is equal to the number of $(j-1)$-faces in the 'splitting cells' of the $(d-1)$-dimensional arrangement $\mathcal{A}(H/h)$. If the number of cell splittings caused by the re-insertion of h is m_h, then the number of j-faces counted in $K_j^{(d)}(P,H)$ and not contained in h is at most $K_j^{(d)}(P_h, H \setminus \{h\}) + K_{j-1}^{(d-1)}(m_h, n-1)$, where P_h is a subset of P obtained by removing m_h points from the cells that got merged when h was removed. Repeating this analysis for all $h \in H$, summing the respective bounds, and taking the maximum over P, H, we obtain

$$(n-d+j)K_j^{(d)}(m,n) \le \sum_{h \in H} \left(K_j^{(d)}(m-m_h, n-1) + K_{j-1}^{(d-1)}(m_h, n-1) \right), \tag{1}$$

where the factor $n-d+j$ comes from the observation that a j-face is appears in the count for every $h \in H$, except for the $d-j$ hyperplanes containing it.

The case $d = 4$. We start with the base case $d = 4$ (and $j = 2$). The equation (1) becomes

$$(n-2)K_2^{(4)}(m,n) \le \sum_{h \in H} \left(K_2^{(4)}(m-m_h, n-1) + K_1^{(3)}(m_h, n-1) \right). \tag{2}$$

By the result of [1], we have

$$K_1^{(3)}(m,n) = \begin{cases} \Theta(m^{2/3}n) & \text{for } m \ge n^{3/2} \\ \Theta(n^2) & \text{for } n \le m \le n^{3/2} \\ \Theta(mn) & \text{for } m \le n. \end{cases} \tag{3}$$

Divide (2) by $n(n-1)(n-2)$, and put $F_2^{(4)}(m,n) = K_2^{(4)}(m,n)/(n(n-1))$, to obtain

$$F_2^{(4)}(m,n) \le \frac{1}{n} \sum_{h \in H} F_2^{(4)}(m-m_h, n-1) + O\left(\frac{1}{n} \sum_{h \in H} \frac{K_1^{(3)}(m_h, n-1)}{n^2} \right). \tag{4}$$

We now unwind the recurrence in (4) all the way down to $n_0 = m^{1/4}$ remaining hyperplanes. We obtain a recurrence tree T. The j-th level of T is the collection of all nodes whose corresponding substructure involves j hyperplanes of H; thus the root of T is at level n (it represents the whole set H) and the leaves are at level n_0. Let π be a path in T, let $v_j(\pi)$ denote the node of π at level j, and let $h_j(\pi)$ denote the hyperplane removed and reinserted at $v_j(\pi)$, for $j = n, n-1, \ldots, n_0+1$; in other words, $h_j(\pi)$ is the hyperplane that represents the edge of π between $v_j(\pi)$ (parent node) and $v_{j-1}(\pi)$ (child node). It is easily verified that the unwound recurrence can be rewritten as

$$F_2^{(4)}(m,n) \le \frac{n_0!}{n!} \sum_{\pi} \left[F_2^{(4)}(m^*(\pi), n_0) + O\left(\sum_{j=n_0+1}^{n} \frac{K_1^{(3)}(m_j(\pi), j-1)}{j^2} \right) \right], \tag{5}$$

where π ranges over all paths in T, and where $m_j(\pi)$ is the number of points removed from the current subset of P when $h_j(\pi)$ is removed from the subset of H associated with $v_j(\pi)$; the number of points remaining in P after all these removals is denoted by $m^*(\pi)$, and we have $m^*(\pi)+\sum_{j=n_0+1}^{n} m_j(\pi) = m$. In other words, $F_2^{(4)}(m,n)$ is the average, over all paths of T, of the path-dependent expression in the brackets in (5). Denote this expression by $E(\pi) = F_2^{(4)}(m^*(\pi), n_0) + O(\sum_{j=n_0+1}^{n} E_j(\pi))$, where $E_j(\pi) = K_1^{(3)}(m_j(\pi), j-1)/j^2$.

We fix a path π in T, and estimate $E(\pi)$. First we have

$$F_2^{(4)}(m^*(\pi), n_0) = F_2^{(4)}(m, m^{1/4}) = \frac{K_2^{(4)}(m, m^{1/4})}{m^{1/4}(m^{1/4}-1)} = O\left(\frac{O(m)}{m^{1/2}}\right) = O(m^{1/2}),$$

where we have used the fact that an arrangement of $m^{1/4}$ hyperplanes has $O(m)$ cells and total complexity $O(m)$. Partition the nodes of π into three subsets:

$$\begin{aligned} J_1 &= \{j \mid m_j(\pi) > (j-1)^{3/2}\} \\ J_2 &= \{j \mid j-1 < m_j(\pi) \le (j-1)^{3/2}\} \\ J_3 &= \{j \mid m_j(\pi) \le j-1\}. \end{aligned}$$

Using (3) and Hölder's inequality, we obtain

$$\begin{aligned} \sum_{j\in J_1} E_j(\pi) =& O\left(\sum_{j\in J_1} \frac{m_j(\pi)^{2/3}}{j}\right) \\ =& O\left[\left(\sum_{j\in J_1} m_j(\pi)\right)^{2/3} \left(\sum_{j>n_0} \frac{1}{j^3}\right)^{1/3}\right] \\ =& O\left(\frac{m^{2/3}}{n_0^{2/3}}\right) = O(m^{1/2}). \end{aligned}$$

Next we have

$$\sum_{j\in J_3} E_j(\pi) = O\left(\sum_{j\in J_3} \frac{m_j(\pi)}{j}\right) = O\left(\sum_{\substack{j\in J_3 \\ j\le m^{1/2}}} \frac{m_j(\pi)}{j} + \sum_{\substack{j\in J_3 \\ j>m^{1/2}}} \frac{m_j(\pi)}{j}\right).$$

In the first sum, we use the fact that $m_j(\pi) < j$ to conclude that the sum is $O(m^{1/2})$. As for the second sum, we have

$$\sum_{\substack{j\in J_3 \\ j>m^{1/2}}} \frac{m_j(\pi)}{j} < \frac{1}{m^{1/2}} \sum_{\substack{j\in J_3 \\ j>m^{1/2}}} m_j(\pi) \le \frac{1}{m^{1/2}} \cdot m = m^{1/2}.$$

Finally, we have

$$\sum_{j\in J_2} E_j(\pi) = O\left(\sum_{j\in J_2} 1\right) = O\left(\sum_{\substack{j\in J_2 \\ j\le m^{1/2}}} 1 + \sum_{\substack{j\in J_2 \\ j>m^{1/2}}} 1\right).$$

The first subsum is at most $m^{1/2}$, while the second is at most

$$\sum_{m_j \geq m^{1/2}} 1 = \frac{m}{m^{1/2}} = m^{1/2}.$$

To summarize, we have shown that $E(\pi) = O(m^{1/2})$ for each path π in T. Since $F_2^{(4)}(m,n)$ is the average of these expressions, we conclude that $F_2^{(4)}(m,n) = O(m^{1/2})$, and hence $K_2^{(4)}(m,n) = O(m^{1/2}n^2)$. This establishes the base case $d = 4$, since the Dehn-Sommerville relations imply that $K_j^{(4)}(m,n) = O(K_2^{(4)}(m,n))$, for $j = 0, 1, 3$, as already mentioned.

The case of odd d. Next assume that $d > 4$ is odd, say $d = 2q + 1$. In this case, we focus on $j = \lceil d/2 \rceil = q + 1$ and (1) becomes

$$(n-q)K_{q+1}^{(2q+1)}(m,n) \leq \sum_{h \in H} \left(K_{q+1}^{(2q+1)}(m - m_h, n-1) + K_q^{(2q)}(m_h, n-1) \right). \qquad (6)$$

By the induction hypothesis, we have

$$K_q^{(2q)}(m,n) = O(m^{1/2} n^q \log^{(q-2)/2} n).$$

We substitute this bound in (6), divide it by $n(n-1)\cdots(n-q)$, and put $F_{q+1}^{(2q+1)}(m,n) = K_{q+1}^{(2q+1)}(m,n)/(n(n-1)\cdots(n-q+1))$, to obtain

$$F_{q+1}^{(2q+1)}(m,n) \leq \frac{1}{n} \sum_{h \in H} F_{q+1}^{(2q+1)}(m - m_h, n-1) + O\left(\frac{1}{n} \sum_{h \in H} m_h^{1/2} \log^{(q-2)/2} n \right). \qquad (7)$$

We now unwind the recurrence in (7) until only one hyperplane remains. We obtain a recurrence tree T, and continue to use the same notations as in the case $d = 4$. It is easily verified that the unwound recurrence can be rewritten as

$$F_{q+1}^{(2q+1)}(m,n) \leq \frac{1}{n!} \sum_{\pi} \left[\sum_{j=1}^{n} O(m_j(\pi)^{1/2} \log^{(q-2)/2} j) \right], \qquad (8)$$

where π ranges over all paths in T. In other words, as above, $F_{q+1}^{(2q+1)}(m,n)$ is the average, over all paths of T, of the path-dependent expression in the brackets in (8). By the Cauchy-Schwarz inequality, we have

$$\sum_{j=1}^{n} m_j(\pi)^{1/2} \leq \left(\sum_{j=1}^{n} m_j(\pi) \right)^{1/2} n^{1/2} \leq m^{1/2} n^{1/2}.$$

Hence $F_{q+1}^{(2q+1)}(m,n) = O(m^{1/2} n^{1/2} \log^{(q-2)/2} n)$, and thus

$$K_{q+1}^{(2q+1)}(m,n) = O(m^{1/2} n^{q+1/2} \log^{(q-2)/2} n),$$

which is the asserted bound for $d = 2q + 1$.

The case of even d. Finally consider the case where d is even, say $d = 2q > 4$. Here we take $j = \lceil d/2 \rceil = q$. In this case, (1) becomes

$$(n-q)K_q^{(2q)}(m,n) \le \sum_{h\in H} \left(K_q^{(2q)}(m-m_h, n-1) + K_{q-1}^{(2q-1)}(m_h, n-1) \right). \tag{9}$$

As noted above, it follows from the Dehn-Sommerville relations that

$$K_{q-1}^{(2q-1)}(m_h, n-1) = O(K_q^{(2q-1)}(m_h, n-1)),$$

which allows us to rewrite (9) as

$$(n-q)K_q^{(2q)}(m,n) \le \sum_{h\in H} \left(K_q^{(2q)}(m-m_h, n-1) + O(K_q^{(2q-1)}(m_h, n-1)) \right). \tag{10}$$

By the induction hypothesis, we have

$$K_q^{(2q-1)}(m,n) = O(m^{1/2} n^{q-1/2} \log^{(q-3)/2} n).$$

We substitute this bound in (10), divide it by $n(n-1)\cdots(n-q)$, and put $F_q^{(2q)}(m,n) = K_q^{(2q)}(m,n)/(n(n-1)\cdots(n-q+1))$, to obtain

$$F_q^{(2q)}(m,n) \le \frac{1}{n}\sum_{h\in H} F_q^{(2q)}(m-m_h, n-1) + O\left(\frac{1}{n}\sum_{h\in H} \frac{K_q^{(2q-1)}(m_h, n-1)}{n^q} \right) = \tag{11}$$

$$\frac{1}{n}\sum_{h\in H} F_q^{(2q)}(m-m_h, n-1) + O\left(\frac{1}{n}\sum_{h\in H} \frac{m_h^{1/2}}{n^{1/2}} \log^{(q-3)/2} n \right).$$

We now unwind the recurrence in (11) until only one hyperplane remains. We obtain a recurrence tree T, and, as above, rewrite the unwound recurrence as

$$F_q^{(2q)}(m,n) \le \frac{1}{n!}\sum_{\pi} \left[\sum_{j=1}^{n} O\left(\frac{m_j(\pi)^{1/2}}{j^{1/2}} \log^{(q-3)/2} j \right) \right], \tag{12}$$

where π ranges over all paths in T. In other words, as above, $F_q^{(2q)}(m,n)$ is the average, over all paths of T, of the path-dependent expression in the brackets in (12). By the Cauchy-Schwarz inequality, we have

$$\sum_{j=1}^{n} \frac{m_j(\pi)^{1/2}}{j^{1/2}} \le \left(\sum_{j=1}^{n} m_j(\pi) \right)^{1/2} \left(\sum_{j=1}^{n} \frac{1}{j} \right)^{1/2} = O(m^{1/2} \log^{1/2} n).$$

Hence $F_q^{(2q)}(m,n) = O(m^{1/2} \log^{(q-2)/2} n)$, and thus $K_q^{(2q)}(m,n) = O(m^{1/2} n^q \log^{(q-2)/2} n)$, which is the asserted bound for $d = 2q$.

This completes the proof of the theorem. □

Remarks: (1) The bounds in the theorem are new, and improve, by a polylogarithmic factor, previous upper bounds given in [2].

(2) In 4 dimensions the bound is $O(m^{1/2}n^2)$. We do not know whether this bound is tight for the whole range of m. It is clearly tight for $m = \Theta(1)$ and for $m = \Theta(n^4)$. It is also tight for $m = \Theta(n^2)$. This has been noted in [2, Theorem 3.3(b)]. For the sake of completeness, here is a sketch of the construction. Take two orthogonal planes p, p' in 4-space. Construct in p an arbitrary arrangement of $n/2$ lines in general position, and construct in p' an arrangement of $n/2$ lines that has a cell c so that all lines appear on its boundary. Now extend each of these n lines to a hyperplane in 4-space by taking its Cartesian product with the complementary plane. The cells under consideration in the resulting 4-dimensional arrangement are the Cartesian products of each cell of the arrangement in p with c. We obtain a collection of $m = \Theta(n^2)$ cells whose overall complexity is $\Theta(n^2 \cdot n) = \Theta(n^3) = \Theta(m^{1/2}n^2)$.

(3) The method of proof employed above can also be used to derive the known bound of $O(m^{2/3}n^{d/3} + n^{d-1})$, $d \geq 4$, on $K_{d-1}^{(d)}(m, n)$, from the corresponding bound in three dimensions. We omit the details.

2 Sum of Squares of Cell Complexities in Hyperplane Arrangements

We next apply Theorem 1.1 to obtain a simple proof of the following result, originally established in [2].

Theorem 2.1 *The sum of squares of the cell complexities in an arrangement of n hyperplanes in d dimensions, for $d \geq 4$, is $O(n^d \log^{\lfloor d/2 \rfloor - 1} n)$.*

Proof: Let H be a set of n hyperplanes in d-space, and let $|C|$ denote the combinatorial complexity (number of faces of all dimensions) of a cell C in $\mathcal{A}(H)$. We wish to bound the quantity $\Sigma(H) = \sum_C |C|^2$, where the sum ranges over all cells C of $\mathcal{A}(H)$.

Let $\mathcal{C}_k$ denote the subset of cells whose complexity is exactly k, for $k \leq \Theta(n^{\lfloor d/2 \rfloor})$. Let $\mathcal{C}_{\geq k}$ denote the subset of cells whose complexity is at least k, and let m_k denote the cardinality of $\mathcal{C}_{\geq k}$. Apply the bound of Theorem 1.1 to $\mathcal{C}_{\geq k}$, to obtain

$$km_k = O(m_k^{1/2} n^{d/2} \log^{(\lfloor d/2 \rfloor - 2)/2} n),$$

which implies that

$$m_k = O\left(\frac{n^d \log^{\lfloor d/2 \rfloor - 2} n}{k^2}\right).$$

We thus have

$$\Sigma(H) = \sum_k k^2 |\mathcal{C}_k| = \sum_k k^2 (|\mathcal{C}_{\geq k}| - |\mathcal{C}_{\geq k+1}|) \leq O(n^d + \sum_k k m_k) =$$

$$O(n^d) + O\left(\sum_{k \leq \Theta(n^{\lfloor d/2 \rfloor})} \frac{n^d \log^{\lfloor d/2 \rfloor - 2} n}{k}\right) = O(n^d \log^{\lfloor d/2 \rfloor - 1} n),$$

as asserted. □

Remarks: (1) Lemma 3.4 of [2] provides an alternative derivation of this bound from the many-cell bound of Theorem 1.1.

(2) This proof shows that for any $\beta < 2$ we have

$$\sum_{C} |C|^{\beta} = O(n^{d} \log^{\lfloor d/2 \rfloor - 2} n).$$

This improves the bound of Theorem 2.1, and, for the cases $d = 4$ and $d = 5$, settles in the affirmative a conjecture in [2].

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