The number of unit-area triangles in the plane: Theme and variations[*]

Orit E. Raz[†]   Micha Sharir[‡]

January 2, 2015

Abstract

We show that the number of unit-area triangles determined by a set S of n points in the plane is $O(n^{20/9})$, improving the earlier bound $O(n^{9/4})$ of Apfelbaum and Sharir [2]. We also consider two special cases of this problem: (i) We show, using a somewhat subtle construction, that if S consists of points on three lines, the number of unit-area triangles that S spans, with one vertex on each of the lines, can be $\Omega(n^2)$, for any triple of lines (this number is always $O(n^2)$ in this case). (ii) We show that if S is a *convex grid* of the form $A \times B$, where A, B are *convex* sets of $n^{1/2}$ real numbers each (i.e., the sequences of differences of consecutive elements of A and of B are both strictly increasing), then S determines $O(n^{31/14})$ unit-area triangles.

1   Introduction

In 1967, Oppenheim (see [10]) asked the following question: Given n points in the plane and $A > 0$, how many triangles spanned by the points can have area A? By applying a scaling transformation, one may assume $A = 1$ and count the triangles of *unit* area. Erdős and Purdy [9] showed that a $\sqrt{\log n} \times (n/\sqrt{\log n})$ section of the integer lattice determines $\Omega(n^2 \log \log n)$ triangles of the same area. They also showed that the maximum number of such triangles is at most $O(n^{5/2})$. In 1992, Pach and Sharir [11] improved the bound to $O(n^{7/3})$, using the Szemerédi-Trotter theorem [18] (see below) on the number of point-line incidences. More recently, Dumitrescu et al. [4] have further improved the upper bound to $O(n^{44/19}) = O(n^{2.3158})$, by estimating the number of incidences between the given points and a 4-parameter family of quadratic curves. In a subsequent improvement, Apfelbaum and Sharir [2] have obtained the upper bound $O(n^{9/4+\varepsilon})$, for any $\varepsilon > 0$, which has been slightly improved to $O(n^{9/4})$ in Apfelbaum [1]. This has been the best known upper bound so far.

[†]School of Computer Science, Tel Aviv University, Tel Aviv 69978, Israel. *oritraz@post.tau.ac.il*

[‡]School of Computer Science, Tel Aviv University, Tel Aviv 69978, Israel. *michas@post.tau.ac.il*

In this paper we further improve the bound to $O(n^{20/9})$. Our proof uses a different reduction of the problem to an incidence problem, this time to incidences between points and two-dimensional algebraic surfaces in $\mathbb{R}^4$. A very recent result of Solymosi and De Zeeuw [17] provides a sharp upper bound for the number of such incidences, similar to the Szemerédi–Trotter bound, provided that the points, surfaces, and incidences satisfy certain fairly restrictive assumptions. The main novel features of our analysis are thus (a) the reduction of the problem to this specific type of incidence counting, and (b) showing that the assumptions of [17] are satisfied in our setup.

After establishing this main result, we consider two variations, in which better bounds can be obtained.

We first consider the case where the input points lie on three arbitrary lines, and we are interested only in triangles that have one vertex on each of the lines. It is easily checked that in this case there are at most $O(n^2)$ unit-area triangles. We show, in Section 3, that this bound is tight, and can be attained for any triple of lines. Rather than just presenting the construction, we spend some time showing its connection to a more general problem studied by Elekes and Rónyai [6] (see also the recent developments in [7, 13, 14]), involving the zero set of a trivariate polynomial within a triple Cartesian product. Skipping over the details, which are spelled out in Section 3, it turns out that the case of unit-area triangles determined by points lying on three lines is an exceptional case in the theory of Elekes and Rónyai [6], which then leads to a construction with $\Theta(n^2)$ unit-area triangles. In contrast, if the input points lie on a general algebraic curve of constant degree then the number of unit-area triangles that they determine drops to $O(n^{11/6})$, as follows from the recent result of Raz et al. [14].

Another variation that we consider in Section 4, concerns unit-area triangles spanned by points in a *convex grid*. That is, the input set is of the form $A \times B$, where A and B are convex sets of $n^{1/2}$ real numbers each; a set of real numbers is called *convex* if the differences between consecutive elements form a strictly increasing sequence. We show that in this case $A \times B$ determine $O(n^{31/14})$ unit-area triangles. The main technical tool used in our analysis is a result of Schoen and Shkredov [15] on difference sets involving convex sets.

2 Unit-area triangles in the plane: The general case

In this section we establish our main result — an improved upper bound on the maximum number of unit-area triangles determined by an arbitrary set of n points in the plane. For the analysis, we first recall the Szemerédi–Trotter theorem [18] on point-line incidences in the plane.

Theorem 1 (Szemerédi and Trotter [18]). *(i) The number of incidences between M distinct points and N distinct lines in the plane is $O(M^{2/3}N^{2/3} + M + N)$. (ii) Given M distinct points in the plane and a parameter $k \leq M$, the number of lines incident to at least k of the points is $O(M^2/k^3 + M/k)$. Both bounds are tight in the worst case.*

The main result of this section is the following.

Theorem 2. *The number of unit-area triangles spanned by n points in the plane is $O(n^{20/9})$.*

Proof. Let S be a set of n points in the plane, and let U denote the set of unit-area triangles spanned by S. For any pair of distinct points, $p \neq q \in S$, let ℓ_{pq} denote the line through p and

q. We regard ℓ_{pq} as oriented from p to q. The points r for which the triangle pqr has unit-area lie on two lines ℓ_{pq}^-, ℓ_{pq}^+ parallel to ℓ_{pq} and at distance $2/|pq|$ from ℓ_{pq} on either side. We let $\ell_{pq}' \in \{\ell_{pq}^-, \ell_{pq}^+\}$ be the line that lies to the left of the line ℓ_{pq}. We then have[1]

$$|U| = \frac{1}{3} \sum_{(p,q)\in S\times S} |\ell_{pq}' \cap S|.$$

It suffices to consider only triangles pqr of U that have the property that at least one of the three lines ℓ_{pq}, ℓ_{pr}, ℓ_{qr} is incident to at most $n^{1/2}$ points of S, because the number of triangles in U that do not have this property is $O(n^{3/2})$. Indeed, by Theorem 1(ii), there exist at most $O(n^{1/2})$ lines in $\mathbb{R}^2$, such that each contains at least $n^{1/2}$ points of S. Since every triple of those lines supports (the edges of) at most one triangle (some of the lines might be mutually parallel, and some triples might intersect at points that do not belong to S), these lines support in total at most $O(n^{3/2})$ triangles, and, in particular, at most $O(n^{3/2})$ triangles of U. Since this number is subsumed in the asserted bound on $|U|$, we can therefore ignore such triangles in our analysis. In what follows, U denotes the set of the remaining unit-area triangles.

We charge each of the surviving unit-area triangles pqr to one of its sides, say pq, such that ℓ_{pq} contains at most $n^{1/2}$ points of S. That is, we have

$$|U| \leq \sum_{(p,q)\in(S\times S)^*} |\ell_{pq}' \cap S|,$$

where $(S \times S)^*$ denotes the subset of pairs $(p,q) \in S \times S$, such that $p \neq q$, and the line ℓ_{pq} is incident to at most $n^{1/2}$ points of S.

A major problem in estimating $|U|$, that also arises in the previous studies of the problem, is that the lines ℓ_{pq}', for $p, q \in S$, are not necessarily distinct, and the analysis has to take into account the (possibly large) multiplicity of these lines. (If the lines were distinct then $|U|$ would be bounded by the number of incidences between $n(n-1)$ lines and n points, which is $O(n^2)$ — see Theorem 1(i).) Let L denote the collection of lines $\{\ell_{pq}' \mid (p,q) \in (S\times S)^*\}$ (without multiplicity). For $\ell \in L$, we define $(S \times S)_\ell$ to be the set of all pairs $(p,q) \in (S\times S)^*$, for which $\ell_{pq}' = \ell$. We then have

$$|U| \leq \sum_{\ell\in L} |\ell \cap S||(S\times S)_\ell|.$$

Fix some integer parameter $k \leq n^{1/2}$, whose value will be set later, and partition L into the sets

$$\begin{aligned}
L^- &:= \{\ell \in L \mid |\ell \cap S| < k\}, \\
L^+ &:= \{\ell \in L \mid k \leq |\ell \cap S| \leq n/k\}, \\
L^{++} &:= \{\ell \in L \mid |\ell \cap S| > n/k\}.
\end{aligned}$$

We have

$$|U| \leq \sum_{\ell\in L^-} |\ell \cap S||(S\times S)_\ell| + \sum_{\ell\in L^+} |\ell \cap S||(S\times S)_\ell| + \sum_{\ell\in L^{++}} |\ell \cap S||(S\times S)_\ell|.$$

---

[1]In this sum, as well as in similar sums in the sequel, we only consider pairs of distinct points in $S \times S$.

The first sum is at most $k \sum_{\ell \in L^-} |(S \times S)_\ell| \le kn^2$, because $\sum_{\ell \in L^-} |(S \times S)_\ell|$ is at most $|(S \times S)^*| \le |S \times S| = n^2$. The same (asymptotic) bound also holds for the the third sum. Indeed, since $n/k \ge n^{1/2}$, the number of lines in L^{++} is at most $O(k)$, as follows from Theorem 1(ii), and, for each $\ell \in L^{++}$, we have $|\ell \cap S| \le n$ and $|(S \times S)_\ell| \le n$ (for any $p \in S$, $\ell \in L$, there exists at most one point $q \in S$, such that $\ell'_{pq} = \ell$). This yields a total of at most $O(n^2 k)$ unit-area triangles. It therefore remains to bound the second sum, over L^+.

Applying the Cauchy-Schwarz inequality to the second sum, it follows that

$$|U| \le O(n^2 k) + \left(\sum_{\ell \in L^+} |\ell \cap S|^2 \right)^{1/2} \left(\sum_{\ell \in L^+} |(S \times S)_\ell|^2 \right)^{1/2}.$$

Let N_j (resp., $N_{\ge j}$), for $k \le j \le n/k$, denote the number of lines $\ell \in L^+$ for which $|\ell \cap S| = j$ (resp., $|\ell \cap S| \ge j$). We then have

$$\sum_{\ell \in L^+} |\ell \cap S|^2 = \sum_{j=k}^{n/k} j^2 N_j \le k^2 N_{\ge k} + \sum_{j=k+1}^{n/k} (2j-1) N_{\ge j}.$$

By Theorem 1(ii), $N_{\ge j} = O\left(\dfrac{n^2}{j^3} + \dfrac{n}{j} \right)$, so we have

$$\sum_{\ell \in L^+} |\ell \cap S|^2 = O\left(\frac{n^2}{k} + nk + \sum_{j=k+1}^{n/k} \left(\frac{n^2}{j^2} + n \right) \right) = O\left(\frac{n^2}{k} \right)$$

(where we used the fact that $k \le n^{1/2}$). It follows that

$$|U| = O\left(n^2 k + \frac{n}{k^{1/2}} \left(\sum_{\ell \in L^+} |(S \times S)_\ell|^2 \right)^{1/2} \right).$$

To estimate the remaining sum, put

$$Q := \left\{ (p, u, q, v) \in S^4 \mid (p, u), (q, v) \in (S \times S)_\ell, \text{ for some } \ell \in L^+ \right\}.$$

That is, Q consists of all quadruples (p, u, q, v) such that $\ell'_{pu} = \ell'_{qv} \in L^+$, and each of ℓ_{pu}, ℓ_{qv} contains at most $n^{1/2}$ points of S. See Figure 1(a) for an illustration. The above bound on $|U|$ can then be written as

$$|U| = O\left(n^2 k + \frac{n |Q|^{1/2}}{k^{1/2}} \right). \tag{1}$$

The main step of the analysis is to establish the following upper bound on $|Q|$.

Proposition 3. *Let Q be as above. Then $|Q| = O\left(n^{8/3} \right)$.*

Assuming that the proposition is true, the overall bound on $|U|$ is then

$$|U| = O\left(n^2 k + \frac{n^{7/3}}{k^{1/2}} \right),$$

4
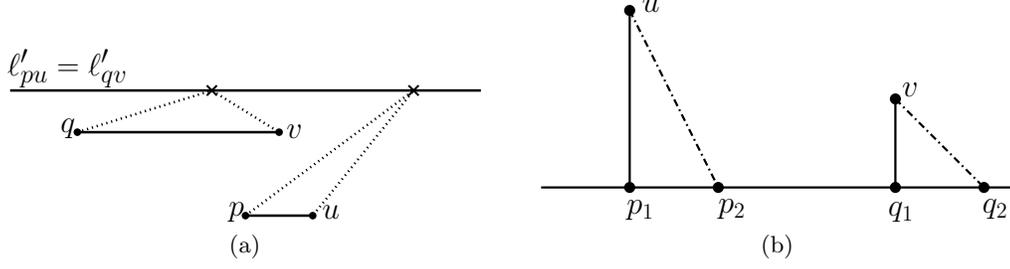
Figure 1: (a) A quadruple (p, u, q, v) in Q. (b) If p_1, q_1, p_2, q_2 are collinear and $|p_1 p_2| = |q_1 q_2|$ then $\ell_{p_2 u}, \ell_{q_2 v}$ are not parallel to one another, for every $(u, v) \in \sigma_{p_1 q_1} \setminus \ell_{p_1 q_1}$. Thus, in particular, $(u, v) \notin \sigma_{p_2 q_2}$.

which, if we choose $k = n^{2/9}$, becomes $|U| = O(n^{20/9})$. Since the number of triangles that we have discarded is only $O(n^{3/2})$, Theorem 2 follows. $\qquad\square$

Proof of Proposition 3. Consider first quadruples $(p, u, q, v) \in Q$, with all four points p, u, q, v collinear. As is easily checked, in this case (p, u, q, v) must also satisfy $|pu| = |qv|$. It follows that a line ℓ in the plane, which is incident to at most j points of S, can support at most j^3 such quadruples. By definition, $(S \times S)_\ell \subset (S \times S)^*$ for each $\ell \in L^+$, so the line $\ell_{pu} = \ell_{qv}$ is incident to at most $n^{1/2}$ points of S, and it suffices to consider only lines ℓ with this property. Using the preceding notations N_j, $N_{\geq j}$, the number of quadruples under consideration is

$$O\left(\sum_{j \leq n^{1/2}} j^3 N_j\right) = O\left(\sum_{j \leq n^{1/2}} j^2 N_{\geq j}\right) = O\left(\sum_{j \leq n^{1/2}} j^2 \cdot \frac{n^2}{j^3}\right) = O\left(n^2 \log n\right).$$

This is subsumed by the asserted bound on $|Q|$, so, in what follows we only consider quadruples $(p, u, q, v) \in Q$, such that p, u, q, v are not collinear.

For convenience, we assume that no pair of points of S share the same x- or y-coordinate; this can always be enforced by a suitable rotation of the coordinate frame. The property that two pairs of $S \times S$ are associated with a common line of L can then be expressed in the following algebraic manner.

Lemma 4. *Let $(p, u, q, v) \in S^4$, and represent $p = (a, b), u = (x, y), q = (c, d)$, and $v = (z, w)$, by their coordinates in $\mathbb{R}^2$. Then $\ell'_{pu} = \ell'_{qv}$ if and only if*

$$\frac{y - b}{x - a} = \frac{w - d}{z - c} \tag{2}$$
$$\frac{bx - ay + 2}{x - a} = \frac{dz - cw + 2}{z - c}.$$

Proof. Let $\alpha, \beta \in \mathbb{R}$ be such that $\ell'_{(a,b)(x,y)} = \{(t, \alpha t + \beta) \mid t \in \mathbb{R}\}$. Then, by the definition of $\ell'_{(a,b)(x,y)}$, we have

$$\frac{1}{2}\begin{vmatrix} a & x & t \\ b & y & \alpha t + \beta \\ 1 & 1 & 1 \end{vmatrix} = 1,$$

or

$$(b - y - \alpha(a - x))t - \beta(a - x) + ay - bx = 2,$$

5

for all $t \in \mathbb{R}$. Thus,

$$\alpha = \alpha(a, b, x, y) = \frac{y - b}{x - a},$$

$$\beta = \beta(a, b, x, y) = \frac{bx - ay + 2}{x - a}.$$

Then the constraint $\ell'_{(a,b)(x,y)} \equiv \ell'_{(c,d)(z,w)}$ can be written as

$$\alpha(a, b, x, y) = \alpha(c, d, z, w),$$
$$\beta(a, b, x, y) = \beta(c, d, z, w),$$

which is (2). $\qquad\square$

We next transform the problem of estimating $|Q|$ into an incidence problem. With each pair $(p = (a, b), q = (c, d)) \in S \times S$, we associate the two-dimensional surface $\sigma_{pq} \subset \mathbb{R}^4$ which is the locus of all points $(x, y, z, w) \in \mathbb{R}^4$ that satisfy the system (2). Technically, σ_{pq} is not well-defined when $x = a$ and $z = c$, but we remove this artificial constraint by clearing the denominators in (2), so the actual equations defining σ_{pq} are

$$(y - b)(z - c) = (x - a)(w - d) \tag{3}$$
$$(z - c)(bx - ay + 2) = (x - a)(dz - cw + 2).$$

The degree of σ_{pq} is at most 4, being the intersection of two quadratic hypersurfaces.

We let Σ denote the set of surfaces

$$\Sigma := \{\sigma_{pq} \mid (p, q) \in S \times S, \ p \neq q\}.$$

For $(p_1, q_1) \neq (p_2, q_2)$, the corresponding surfaces $\sigma_{p_1 q_1}, \sigma_{p_2 q_2}$ are distinct; this will follow from the proof of Proposition 7 below. We also consider the set $\Pi := S \times S$, regarded as a point set in $\mathbb{R}^4$ (identifying $\mathbb{R}^2 \times \mathbb{R}^2 \simeq \mathbb{R}^4$). We have $|\Pi| = |\Sigma| = O(n^2)$. The set $I(\Pi, \Sigma)$, the set of *incidences* between Π and Σ, is naturally defined as

$$I(\Pi, \Sigma) := \{(\pi, \sigma) \in \Pi \times \Sigma \mid \pi \in \sigma\}.$$

By Lemma 4, we have

$$(x, y, z, w) \in \sigma_{pq} \qquad \text{if and only if} \qquad \ell'_{pu} = \ell'_{qv}, \tag{4}$$

where $u := (x, y)$ and $v := (z, w)$. This implies that

$$|Q| \leq |I(\Pi, \Sigma)|.$$

Consider the subcollection $\mathcal{I}$ of incidences $((x, y, z, w), \sigma_{pq}) \in I(\Pi, \Sigma)$, such that $p, q, u := (x, y), v := (z, w)$ are non-collinear (as points in $\mathbb{R}^2$). As already argued, the number of collinear quadruples in Q is $O(n^2 \log n)$, and hence

$$|Q| \leq |\mathcal{I}| + O(n^2 \log n).$$

So to bound $|Q|$ it suffices to obtain an upper bound on $|\mathcal{I}|$.

For this we use the following (special case of a) recent result of Solymosi and De Zeeuw [17] (see also the related results in [16, 19]). To state it we need the following definition.

Definition 5. A two-dimensional constant-degree surface σ in $\mathbb{R}^4$ is said to be *slanted* (the original term used in [17] is *good*), if, for every $p \in \mathbb{R}^2$, $\rho_i^{-1}(p) \cap \sigma$ is finite, for $i = 1, 2$, where ρ_1 and ρ_2 are the projections of $\mathbb{R}^4$ onto its first and last two coordinates, respectively.

Theorem 6 (Solymosi and De Zeeuw [17]). *Let S be a subset of $\mathbb{R}^2$, and let Γ be a finite set of slanted surfaces. Set $\Pi := S \times S$, and let $\mathcal{I} \subset \Pi \times \Gamma$. Assume that for every pair of distinct points $\pi_1, \pi_2 \in \Pi$ there are at most four surfaces $\sigma \in \Sigma$ such that both pairs $(\pi_1, \sigma), (\pi_2, \sigma)$ are in $\mathcal{I}$. Then*

$$|\mathcal{I}| = O\left(|\Pi|^{2/3}|\Sigma|^{2/3} + |\Pi| + |\Sigma|\right).$$

To apply Theorem 6, we need the following key technical proposition, whose proof is given in the next subsection.

Proposition 7. *Let Π, Σ, and $\mathcal{I}$ be as above. Then, (a) the surfaces of σ are all slanted, and (b) for every pair of distinct points $\pi_1, \pi_2 \in \Pi$, there are at most four surfaces $\sigma \in \Sigma$ such that both pairs $(\pi_1, \sigma), (\pi_2, \sigma)$ are in $\mathcal{I}$.*

We have $|\Pi|, |\Sigma| = O(n^2)$. Therefore, Theorem 6 implies that $|\mathcal{I}| = O(n^{8/3})$, which completes the proof of Proposition 3 (and, consequently, of Theorem 2). $\qquad\square$

2.1   Proof of Proposition 7

By the symmetry of the setup, it is equivalent to prove the following dual statement of Proposition 7(b); part (a) is proved in Lemma 9 below.

Proposition 8. *For any $p_1 \neq q_1, p_2 \neq q_2 \in S$, such that $(p_1, q_1) \neq (p_2, q_2)$, either*

$$|\sigma_{p_1 q_1} \cap \sigma_{p_2 q_2}| \leq 4,$$

or all four points p_1, q_1, p_2, and q_2 are collinear.

For the proof, we first observe several useful properties of the surfaces. Consider a surface $\sigma_{pq} \in \Sigma$, where $p = (a, b)$ and $q = (c, d)$. Let us use the shorthand notation

$$X = x - a, \quad Y = y - b, \quad Z = z - c, \quad \text{and} \quad W = w - d$$

(so the coordinate shifts in this notation depend on the surface). Continuing to assume, for convenience, that $X, Z \neq 0$, the equations of σ_{pq} are thus

$$\frac{Y}{X} = \frac{W}{Z}, \tag{5}$$
$$\frac{bX - aY + 2}{X} = \frac{dZ - cW + 2}{Z}.$$

Eliminating Z and W, we get, by an easy calculation,

$$Z = \frac{2X}{(b-d)X + (c-a)Y + 2}, \tag{6}$$
$$W = \frac{2Y}{(b-d)X + (c-a)Y + 2}.$$

Note that the system (6) defines the same zero set as

$$Z\big((b - d)X + (c - a)Y + 2\big) = 2X, \tag{7}$$
$$W\big((b - d)X + (c - a)Y + 2\big) = 2Y,$$

so we can (and will) think of (6) as the algebraic variety defined by these two polynomials. To see that the zero set does not change, note that even after clearing the denominators, σ_{pq} is still undefined when $(b - d)X + (c - a)Y + 2 = 0$ (and only then), because then (7) would imply that $X = Y = 0$, which is impossible. Because of this observation, we can use (6) or (7) interchangeably to represent σ_{pq}. Note also that a pair $(x, y) \in \mathbb{R}^2$ satisfies the equation $(b - d)X + (c - a)Y + 2 = 0$ if and only if the line ℓ'_{pu}, where $u := (x, y)$, passes though the point q. In this case there is indeed no point v in the affine plane satisfying $\ell'_{pu} = \ell'_{qv}$.

The equations (6) imply several useful properties of the surfaces σ_{pq}.

Lemma 9. *For any $p \neq q \in \mathbb{R}^2$, the surface σ_{pq} is slanted.*

Proof. The representation (6) implies that every (x, y) defines at most one pair (z, w) such that $(x, y, z, w) \in \sigma_{pq}$. By the symmetry of the definition of σ_{pq}, every pair (z, w) also determines at most one pair (x, y) such that $(x, y, z, w) \in \sigma_{pq}$. $\qquad\square$

Lemma 10. *For any $p \neq q \in \mathbb{R}^2$, σ_{pq} is a ruled surface.*[2]

Proof. Consider the line

$$(b - d)X + (c - a)Y + 2 = s$$

in the xy-plane, where $s \in \mathbb{R} \setminus \{0\}$ is arbitrary. (These lines are parallel to, and distinct from, ℓ_{pq}.) Substituting this into (6'), we get

$$Z = (2/s)X \quad \text{and} \quad W = (2/s)Y.$$

That is, we have shown that σ_{pq} contains the line

$$\ell_s : \quad (b - d)X + (c - a)Y + 2 = s, \qquad Z = (2/s)X, \qquad W = (2/s)Y.$$

Since we can pass such a line through each point on σ_{pq}, it follows that σ_{pq} is a ruled surface. $\quad\square$

Lemma 11. *Let $p = (a, b), q = (c, d) \in S$. Each line ℓ in the xy-plane, other than the line $(b - d)X + (c - a)Y + 2 = 0$, is mapped by lifting it vertically to σ_{pq} and then by projecting the image onto the zw-plane, to a line ℓ^* in this latter plane.*

(Note that the "intermediate" curve $\gamma(\ell)$ obtained by lifting ℓ to σ_{pq} is in general not a line. Lemma 10 is a special case in which, for the special lines ℓ_s considered there, $\gamma(\ell_s)$ *is* a line. Also, for any line ℓ not of this form, the intersection point of ℓ with $(b - d)X + (c - a)Y + 2 = 0$, cannot be lifted to σ_{pq}, and the assertion in Lemma 11 applies to all the other points on ℓ.)

Proof. Let ℓ be any line in the xy-plane, other than $(b - d)X + (c - a)Y + 2 = 0$. If ℓ is parallel to ℓ_{pq} the assertion follows from Lemma 10, so we may assume that this is not the case. Assume next that ℓ passes through the point p. Let θ be the normal vector of ℓ in

---

[2]We use the definition that a surface σ is *ruled* if every point of σ is incident to a straight line that is fully contained in σ.

the original plane. Then $\langle (X, Y), \theta \rangle = 0$, for every $(x, y) \in \ell$. We claim that in this case ℓ^* is the line that passes through q and is parallel to ℓ. Indeed, the inner product $\langle (Z, W), \theta \rangle$ is again zero by the representation (6) (for every $(x, y) \in \ell$, except for its intersection with $(b - d)X + (c - a)Y + 2 = 0$).

We may therefore assume that ℓ is not parallel to ℓ_{pq} and does not pass through p. Assume further that ℓ is not parallel to the x-axis; this assumption can be made without loss of generality since the setup is symmetric in x, y. Let $x = uy + v$ be the equation of ℓ; rewrite it as $X = uY + v_0$, where $v_0 = ub + v - a \neq 0$, by assumption, and then as

$$1 = \frac{uY}{X} + \frac{v_0}{X},$$

or

$$\frac{2}{X} = \frac{2}{v_0} - \frac{(2u/v_0)Y}{X}. \tag{8}$$

We now rewrite (5) as

$$d - \frac{cW}{Z} + \frac{2}{Z} = b - \frac{aY}{X} + \frac{2}{X} = b - \frac{aW}{Z} + \frac{2}{X},$$

and, using (8), we get

$$d - \frac{cW}{Z} + \frac{2}{Z} = b - \frac{aW}{Z} + \frac{2}{v_0} - \frac{(2u/v_0)Y}{X} = b - \frac{aW}{Z} + \frac{2}{v_0} - \frac{(2u/v_0)W}{Z}.$$

Getting rid of the denominator Z, we get

$$(d - b - 2/v_0)Z - (c - a - 2u/v_0)W + 2 = 0. \tag{9}$$

In other words, the lifting and projection prescribed in the lemma map each line ℓ in the xy-plane that satisfies the assumptions in the last part of the proof to the line ℓ^* in the zw-plane, given by (9). (Note that the equation (9) indeed represents a line, using the assumption that ℓ is not parallel to ℓ_{pq}. That is, both coefficients of z, w in (9) can vanish only when $u = (c - a)/(d - b)$, i.e., when ℓ is parallel to ℓ_{pq}.) $\qquad \square$

Proof of Proposition 8. Consider the two distinct surfaces $\sigma_{p_1 q_1}$ and $\sigma_{p_2 q_2}$, and represent $p_1 = (a, b)$, $q_1 = (c, d)$, $p_2 = (a', b')$, and $q_2 = (c', d')$ by their coordinates in $\mathbb{R}^2$. (Here "distinct" means that the pairs of points defining the surfaces are distinct. That the surfaces themselves are distinct will follow from the forthcoming analysis.) We can find the locus of intersection $\sigma_{p_1 q_1} \cap \sigma_{p_2 q_2}$ by combining the equations in (6) for both surfaces, which yields

$$z - c = \frac{2(x - a)}{(b - d)(x - a) + (c - a)(y - b) + 2}$$

$$w - d = \frac{2(y - b)}{(b - d)(x - a) + (c - a)(y - b) + 2}$$

$$z - c' = \frac{2(x - a')}{(b' - d')(x - a') + (c' - a')(y - b') + 2}$$

$$w - d' = \frac{2(y - b')}{(b' - d')(x - a') + (c' - a')(y - b') + 2},$$

or

$$c' - c = \frac{2(x-a)}{D} - \frac{2(x-a')}{D'},$$

$$d' - d = \frac{2(y-b)}{D} - \frac{2(y-b')}{D'},$$

(10)

where

$$D := (b-d)(x-a) + (c-a)(y-b) + 2, \quad \text{and} \quad D' := (b'-d')(x-a') + (c'-a')(y-b') + 2.$$

(Note that $D = 0$ is just the equation defining the line $\ell'_{p_1 q_1}$ and $D' = 0$ is the equation defining the line $\ell'_{p_2 q_2}$.) Clearing the denominators, we get two quadratic equations, which represent two conic sections that we denote as δ_z and δ_w, respectively.

We first claim that the zero set of (10) is at most one-dimensional. We establish the stronger claim that it is impossible for neither of the equations in (10) to be an identity. Suppose to the contrary that this indeed happens, and that, say, the first equation in (10) is an identity. Fixing $x = x_0$ and letting $y \to \infty$, we see that the right hand-side of the equation goes to 0, and therefore we must have $c = c'$. (Recall that we assume that $p_1 \neq q_1$ and $p_2 \neq q_2$, so, by our coordinate uniqueness assumption, the coefficient of y in each of the expressions D and D' is nonzero.) The equation in (10) then becomes

$$\frac{x-a}{D} = \frac{x-a'}{D'}.$$

Substituting $x = a, y = b$, the left-hand side of this equation is zero. So the right-hand side must also be zero, which yields $a = a'$. Our assumption on the genericity of the coordinate frame then implies that $d = d'$ and $b = b'$, because no distinct pair of points in S share the same x-coordinate. That is, we have $p_1 = p_2$ and $q_1 = q_2$. But then $(p_1, q_1) = (p_2, q_2)$, a contradiction that establishes the claim.

Note that this argument implies that the surfaces $\sigma_{p_1 q_1}$ and $\sigma_{p_2 q_2}$ are indeed distinct, as claimed.

If δ_z and δ_w intersect in a finite number of points then this number is at most 4, and then, in this case, these points are the xy-projections of the at most four points of intersection of $\sigma_{p_1 q_1}$ and $\sigma_{p_2 q_2}$. It thus remains to consider the cases where δ_z and δ_w either coincide or have a common irreducible component, which must be a line.

Case 1: The curves δ_z and δ_w coincide. Denote this common curve as δ (which we may assume not to be a double line, as this is a special instance of Case 2, discussed below). Draw an arbitrary line ℓ in the xy-plane that is not parallel to the x-axis, does not pass through p_1 or through p_2, is different from the lines $\ell'_{p_1 q_1}$ and $\ell'_{p_2 q_2}$, and intersects δ at two distinct points $s_1 = (x_1, y_1)$, $s_2 = (x_2, y_2)$, neither of which lies on $\ell'_{p_1 q_1}$ or on $\ell'_{p_2 q_2}$. Since these points lie on δ, their respective liftings to $\sigma_{p_1 q_1}$ coincide, as do the liftings to $\sigma_{p_2 q_2}$, and we denote them as $\tilde{s}_1$ and $\tilde{s}_2$, respectively. Let s_1^*, s_2^* denote the respective ρ_2-projections of $\tilde{s}_1$ and $\tilde{s}_2$. Each of the lines ℓ^* obtained in Lemma 11, one for each of the surfaces $\sigma_{p_1 q_1}$, $\sigma_{p_2 q_2}$, must pass through both s_1^* and s_2^*, and thus these lines must coincide. (By Lemma 9, the projection of each of $\sigma_{p_1 q_1}$, $\sigma_{p_2 q_2}$ onto the zw-plane is injective, and so the points s_1^*, s_2^* are necessarily distinct.)

Denote the equation of ℓ as $x = uy + v$. Using (9), ℓ^* satisfies both equations

$$(d - b - 2/(ub + v - a))(z - c) - (c - a - 2u/(ub + v - a))(w - d) + 2 = 0$$
$$(d' - b' - 2/(ub' + v - a'))(z - c') - (c' - a' - 2u/(ub' + v - a'))(w - d') + 2 = 0,$$

where both expressions $ub + v - a$, $ub' + v - a'$ are nonzero, or

$$((d - b)(ub + v - a) - 2)(z - c) - ((c - a)(ub + v - a) - 2u)(w - d) \tag{11}$$
$$+ 2(ub + v - a) = 0$$
$$((d' - b')(ub' + v - a') - 2)(z - c') - ((c' - a')(ub' + v - a') - 2u)(w - d')$$
$$+ 2(ub' + v - a') = 0.$$

Moreover, the choice of ℓ is rather arbitrary, and there is a nonempty open region in the uv-plane consisting of pairs (u, v) for which the corresponding line ℓ satisfies the assumptions made above and meets δ at two points, which also satisfy the above requirements. It follows that the coincidence (up to a scalar multiple) of the two equations in (11) holds for *every* choice of u and v.

Focusing on the coefficients of z and w in (11), the coincidence implies, for every u, v,

$$\frac{(d - b)(ub + v - a) - 2}{(c - a)(ub + v - a) - 2u} = \frac{(d' - b')(ub' + v - a') - 2}{(c' - a')(ub' + v - a') - 2u}. \tag{12}$$

Fix $u \neq 0$, and choose v so that $ub + v - a = 0$. (Although the corresponding line ℓ does not satisfy the above assumptions, as it passes through p_1, (12) holds for ℓ too by a continuity argument.) Then, putting

$$\Delta := ub' + v - a' = u(b' - b) - (a' - a),$$

(12) becomes

$$\frac{1}{u} = \frac{(d' - b')\Delta - 2}{(c' - a')\Delta - 2u},$$

or

$$(c' - a')\Delta - 2u = u((d' - b')\Delta - 2),$$

or

$$(c' - a')\Delta = u(d' - b')\Delta.$$

Since $p_2 \neq q_2$, and since this equation holds for every choice of u, we must have

$$\Delta = u(b' - b) - (a' - a) = 0,$$

and again, since this holds for every u, we must have $p_1 = p_2$. A symmetric argument yields that $q_1 = q_2$ too, contradicting the assumption that (p_1, q_1) and (p_2, q_2) are distinct. That is, we have shown that Case 1 is impossible under our assumptions.

Case 2: The curves δ_z and δ_w have a common line λ. We first observe that

$$\lambda \cap \ell'_{p_1 q_1} = \lambda \cap \ell'_{p_2 q_2}.$$
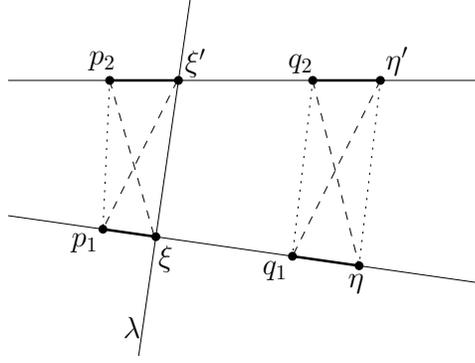
Figure 2: The case where δ_z and δ_w share a common line λ.

Indeed, recall that the lines $\ell'_{p_1q_1}$ and $\ell'_{p_2q_2}$ are given by the equations $D = 0$ and $d' = 0$, respectively. Then, substituting $D = 0$ in (10) for a point $u = (x, y)$ on λ, we get $(x - a)D' = (y - b)D' = 0$, so either $D' = 0$ or $u = p_1$. Since $p_1 \notin \ell'_{p_1q_1}$, we must have $D' = 0$, and the claim follows. In particular, we have that if one of the lines $\ell'_{p_1q_1}$, $\ell'_{p_2q_2}$ is parallel to λ, then both lines must be parallel to λ. Moreover, this latter case occurs only when $\ell_{p_1q_1} \parallel \ell_{p_2q_2}$, because these lines are parallel to $\ell'_{p_1q_1}$ and to $\ell'_{p_2q_2}$, respectively. Note that if $\lambda = \ell'_{p_1q_1} = \ell'_{p_2q_2}$ then each of $\sigma_{p_1q_1}$ and $\sigma_{p_2q_2}$ is undefined over λ, and we can safely ignore this possibility. It is therefore sufficient to consider only the following two subcases.

Case 2.1: The three lines λ, $\ell'_{p_1q_1}$, and $\ell'_{p_2q_2}$ intersect in exactly one common point.
Let ξ (resp., ξ') denote the unique intersection point of λ with $\ell_{p_1q_1}$ (resp., $\ell_{p_2q_2}$); it is easily checked, using the previous observations, that these points exist. Clearly, by construction, ξ does not lie on $\ell'_{p_1q_1}$, and, similarly, ξ' does not lie on the line $\ell'_{p_2q_2}$. Let η (resp., η') denote the point that satisfies $(\xi, \eta) \in \sigma_{p_1q_1} \cap \sigma_{p_2q_2}$ (resp., $(\xi', \eta') \in \sigma_{p_1q_1} \cap \sigma_{p_2q_2}$); these points exist since $\xi, \xi' \in \lambda$, and they do not lie on the respective lines $\ell'_{p_1q_1}$, $\ell'_{p_2q_2}$. See Figure 2 for an illustration.

Assume first that $\xi \neq p_1$, $\xi' \neq p_2$, $\eta \neq q_1$, and $\eta' \neq q_2$; see Figure 2. By construction,

$$\ell'_{p_1\xi} = \ell'_{q_1\eta}, \quad \ell'_{p_1\xi'} = \ell'_{q_1\eta'}, \quad \ell'_{p_2\xi} = \ell'_{q_2\eta}, \quad \ell'_{p_2\xi'} = \ell'_{q_2\eta'}.$$

In particular, we have

$$\ell_{p_1\xi} \parallel \ell_{q_1\eta}, \quad \ell_{p_1\xi'} \parallel \ell_{q_1\eta'}, \quad \ell_{p_2\xi} \parallel \ell_{q_2\eta}, \quad \ell_{p_2\xi'} \parallel \ell_{q_2\eta'}.$$

It follows that p_1, ξ, q_1, and η are collinear, and then we must also have $|p_1\xi| = |q_1\eta|$. Similarly, we have that p_2, ξ', q_2, and η' are also collinear, and $|p_2\xi'| = |q_2\eta'|$.

In addition, the lines $\ell_{p_1\xi'}$ and $\ell_{q_1\eta'}$ are parallel, and so are the lines $\ell_{p_2\xi}$ and $\ell_{q_2\eta}$. All this is easily seen to imply that the two quadrilaterals $p_1p_2\xi'\xi$ and $q_1q_2\eta'\eta$ are congruent, from which we conclude that p_1p_2 is parallel to q_1q_2 and they both have the same length. Moreover, we have $|p_1\xi'| = |q_1\eta'|$. But then, to have $\ell'_{p_1\xi'} = \ell'_{q_1\eta'}$, the points p_1, ξ', q_1, η' must be collinear, which is easily seen to imply that p_1, p_2, q_1, q_2 are collinear too.

It is easy to verify that similar arguments apply also for the cases where (exactly) one of $\xi = p_1$, $\xi' = p_2$ holds, and they yield the same conclusion, that p_1, p_2, q_1, q_2 are collinear.
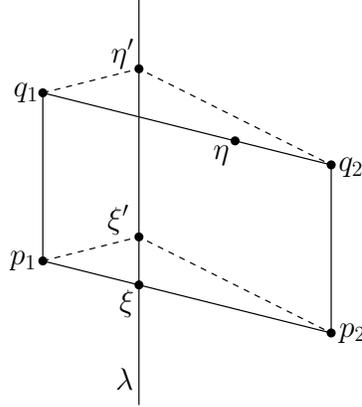
Figure 3: The case where λ is parallel to both lines $\ell_{p_1q_1}, \ell_{p_2q_2}$.

Assume then that $\xi = p_1$ and $\xi' = p_2$, and thus also $\eta = q_1$ and $\eta' = q_2$. By construction, we have $(p_1, q_1) \in \sigma_{p_2q_2}$ and $(p_2, q_2) \in \sigma_{p_1q_1}$. That is, $\ell'_{p_1p_2} = \ell'_{q_1q_2}$ and $\ell'_{p_2p_1} = \ell'_{q_2q_1}$. In particular, the lines $\ell_{p_1p_2}, \ell_{q_1q_2}$ are parallel. We claim that this property is possible only when p_1, q_1, p_2, q_2 are collinear. Indeed, assume that they are not, and suppose, without loss of generality, that $|p_1p_2| \le |q_1q_2|$. Let $\ell \in \{\ell^+_{p_1p_2}, \ell^-_{p_1p_2}\}$ be the line that is farther from $\ell_{q_1q_2}$; say it is $\ell'_{p_1p_2}$ (otherwise, it is $\ell'_{p_2p_1}$). Note that, unless p_1, q_1, p_2, q_2 are collinear, ℓ is well defined. Then, for any point $v \in \ell$, the triangle q_1q_2v is of area strictly larger than the area of p_1p_2v. Since the latter is of unit area, this contradicts $\ell'_{p_1p_2} = \ell'_{q_1q_2}$, a contradiction the implies the collinearity of p_1, p_2, q_1, q_2.

Case 2.2: λ is parallel to both lines $\ell'_{p_1q_1}$ and $\ell'_{p_2q_2}$, and does not coincide with either of them. In this case we have $\ell_{p_1q_1} \parallel \ell_{p_2q_2}$ (since they are parallel to $\ell'_{p_1q_1}$ and $\ell'_{p_2q_2}$, respectively, and thus to λ). If in addition λ is parallel to either of $\ell_{p_1p_2}, \ell_{q_1q_2}$, then, together with the assumption in this subcase, we get that p_1, p_2, q_1, q_2 are collinear, and we are done.

Otherwise, let ξ denote the unique point in the intersection $\ell_{p_1p_2} \cap \lambda$. Let η be the point such that (ξ, η) satisfies (6) with respect to both surfaces $\sigma_{p_1q_1}, \sigma_{p_2q_2}$ (the same point arises for both surfaces because $\xi \in \delta_z \cap \delta_w$). That is, $\ell'_{p_1\xi} = \ell'_{q_1\eta}$, and $\ell'_{p_2\xi} = \ell'_{q_2\eta}$. In particular, $\ell_{p_1\xi} \parallel \ell_{q_1\eta}$ and $\ell_{p_2\xi} \parallel \ell_{q_2\eta}$. Since, by construction, $\xi \in \ell_{p_1p_2}$, we have $\ell_{p_1\xi} = \ell_{p_2\xi}$, which yields that also $\ell_{q_1\eta} \parallel \ell_{q_2\eta}$. Thus necessarily q_1, q_2, η are collinear, and $\ell_{q_1q_2} \parallel \ell_{p_1p_2}$. Thus $p_1q_1q_2p_2$ is a parallelogram. See Figure 3 for an illustration.

Now pick any other point $\xi' \in \lambda$ and let η' be the corresponding such that (ξ', η') satisfies (6) with respect to both surfaces $\sigma_{p_1q_1}, \sigma_{p_2q_2}$. Since $\ell_{p_1\xi'} \parallel \ell_{q_1\eta'}$ and $\ell_{p_2\xi'} \parallel \ell_{q_2\eta'}$, it follows that the triangles $p_1p_2\xi'$ and $q_1q_2\eta'$ are congruent. In particular, $|p_1\xi'| = |q_1\eta'|$ and $|p_2\xi'| = |q_2\eta'|$. However, in this case, similar to the argument used in Case 2.1, we can have $\ell'_{p_1\xi'} = \ell'_{q_1\eta'}$ and $\ell_{p_2\xi'} \parallel \ell_{q_2\eta'}$ only when p_1, q_1, p_2, q_2 are collinear, as is easily checked. This concludes the proof of the proposition for this subcase.

As we have exhausted all possible cases, Proposition 8 is now established. $\square$

3 Unit-area triangles spanned by points on three lines

In this section we consider the special case where S is contained in the union of three pairwise distinct lines l_1, l_2, l_3. More precisely, we write $S = S_1 \cup S_2 \cup S_3$, with $S_i \subset l_i$, for $i = 1, 2, 3$, and we are only interested in the number of unit-area triangles spanned by triples of points in $S_1 \times S_2 \times S_3$. It is easy to see that in this case the number of unit-area triangles spanned by $S_1 \times S_2 \times S_3$ is $O(n^2)$. Indeed, for any pair of points $p, q \in S_1 \times S_2$, the line ℓ'_{pq} intersects l_3 in at most one point, unless ℓ'_{pq} coincides with l_3. The number of unit-area triangles for which the latter situation does not arise is thus $O(n^2)$. If no two lines among l_1, l_2, l_3 are parallel to one another, it can be checked that the number of pairs (p, q) such that $\ell'_{pq} = l_3$ is at most a constant, thus contributing a total of at most $O(n)$ unit-area triangles. For the case where two (or more) lines among l_1, l_2, l_3 are parallel, the number of unit-area triangles is easily seen to be $O(n^2)$.

In this section we present a rather subtle construction that shows that this bound is tight in the worst case, for any triple of distinct lines. Instead of just presenting the construction, we spend some time showing its connection to a more general setup considered by Elekes and Rónyai [6] (and also, in more generality, by Elekes and Szabó [7]).

Specifically, the main result of this section is the following.

Theorem 12. *For any triple of distinct lines l_1, l_2, l_3 in $\mathbb{R}^2$, and for any integer n, there exist subsets $S_1 \subset l_1$, $S_2 \subset l_2$, $S_3 \subset l_3$, each of cardinality $\Theta(n)$, such that $S_1 \times S_2 \times S_3$ spans $\Theta(n^2)$ unit-area triangles.*

Proof. The upper bound has already been established (for any choice of sets S_1, S_2, S_3 and of lines ℓ_1, ℓ_2, ℓ_3), so we focus on the lower bound.

We recall that by the area formula for triangles in the plane, if

$$\frac{1}{2} \begin{vmatrix} p_x & q_x & r_x \\ p_y & q_y & r_y \\ 1 & 1 & 1 \end{vmatrix} = 1, \tag{13}$$

then the points $p = (p_x, p_y)$, $q = (q_x, q_y)$ and $r = (r_x, r_y)$ form the vertices of a positively oriented unit-area triangle in $\mathbb{R}^2$. (Conversely, if Δpqr has area 1 then the left-hand side of (13) has value ± 1, depending on the orientation of (p, q, r).)

To establish the lower bound, we distinguish between three cases, depending on the number of pairs of parallel lines among l_1, l_2, l_3.

The three lines l_1, l_2, l_3 are mutually parallel. In this case we may assume without loss of generality that they are of the form

$$l_1 = \{(t, 0) \mid t \in \mathbb{R}\},$$
$$l_2 = \{(t, 1) \mid t \in \mathbb{R}\},$$
$$l_3 = \{(t, \alpha) \mid t \in \mathbb{R}\},$$

for some $1 < \alpha \in \mathbb{R}$. (We translate and rotate the coordinate frame so as to place ℓ_1 at the x-axis and then apply an area-preserving linear transformation that scales the x- and y-axes by reciprocal values.)

We set

$$S_1 := \{(x_i := \tfrac{i}{1-\alpha}, 0) \mid i = 1, \dots, n\} \subset l_1,$$
$$S_2 := \{(y_j := \tfrac{j}{\alpha}, 1) \mid j = 1, \dots, n\} \subset l_2,$$
$$S_3 := \{(z_{ij} := i + j - 2, \alpha) \mid i, j = 1, \dots, n\} \subset l_3.$$

Clearly each of the sets S_i, $i = 1, 2, 3$, is of cardinality $\Theta(n)$. Note that for every pair of indices $1 \leq i, j \leq n$, we have $(1-\alpha)x_i + \alpha y_j - z_{ij} = 2$. By (13), every such pair i, j corresponds to a unit-area triangle with vertices $(x_i, 0) \in S_1$, $(y_j, 1) \in S_2$ and $(z_{ij}, \alpha) \in S_3$. That is, $S_1 \times S_2 \times S_3$ spans $\Omega(n^2)$ unit-area triangles.

There is exactly one pair of parallel lines among l_1, l_2, l_3. Using an area-preserving affine transformation[3] of $\mathbb{R}^2$ (and possibly re-indexing the lines), we may assume that

$$l_1 = \{(t, 0) \mid t \in \mathbb{R}\},$$
$$l_2 = \{(t, 1) \mid t \in \mathbb{R}\},$$
$$l_3 = \{(0, t) \mid t \in \mathbb{R}\}.$$

We claim that in this case the sets

$$S_1 := \{(x_i := 2^i + 2, 0) \mid i = 1, \dots, n\} \subset l_1,$$
$$S_2 := \{(y_j := 2^j + 2, 1) \mid j = 1, \dots, n\} \subset l_2,$$
$$S_3 := \{(0, z_{ij} := \tfrac{1}{1 - 2^{j-i}}) \mid i, j = 1, \dots, n, \; i \neq j\} \subset l_3,$$

span $\Omega(n^2)$ unit-area triangles. As before, S_1, S_2 and S_3 are each of cardinality $\Theta(n)$.

Using (13), the triangle spanned by $(x_i, 0)$, $(y_j, 1)$, and $(0, z_{ij})$ has unit area if

$$\frac{1}{2} \begin{vmatrix} x_i & y_j & 0 \\ 0 & 1 & z_{ij} \\ 1 & 1 & 1 \end{vmatrix} = \frac{x_i - z_{ij}(x_i - y_j)}{2} = 1,$$

or

$$z_{ij} = \frac{x_i - 2}{x_i - y_j} = \frac{1}{1 - \frac{y_j - 2}{x_i - 2}} = \frac{1}{1 - 2^{j-i}}.$$

Since the latter equality holds for every pair $1 \leq i \neq j \leq n$, we get $\Omega(n^2)$ unit-area triangles, as claimed.

No pair of lines among l_1, l_2, l_3 are parallel. Using an area-preserving affine transformation of $\mathbb{R}^2$, we may assume that the lines are given by

$$l_1 = \{(t, 0) \mid t \in \mathbb{R}\},$$
$$l_2 = \{(0, t) \mid t \in \mathbb{R}\},$$
$$l_3 = \{(t, -t + \alpha) \mid t \in \mathbb{R}\},$$

---

[3]In more generality than the transformation used in the first case, these are linear transformations with determinant ± 1.

for some $\alpha \in \mathbb{R}$. By (13) once again, , the points $(x, 0) \in l_1$, $(0, y) \in l_2$, and $(z, -z + \alpha) \in l_3$ span a unit-area triangle if

$$\frac{1}{2} \begin{vmatrix} x & 0 & z \\ 0 & y & -z + \alpha \\ 1 & 1 & 1 \end{vmatrix} = 1,$$

or

$$z = f(x, y) := \frac{xy - \alpha x - 2}{y - x}.$$

Thus it suffices to find sets $X, Y, Z \subset \mathbb{R}$, each of cardinality $\Theta(n)$, such that

$$\left| \{(x, y, z) \in X \times Y \times Z \mid z = f(x, y)\} \right| = \Omega(n^2);$$

then the sets

$$\begin{aligned} S_1 &:= \{(x, 0) \mid x \in X\} \subset l_1, \\ S_2 &:= \{(0, y) \mid y \in Y\} \subset l_2, \\ S_3 &:= \{(z, -z + \alpha) \mid z \in Z\} \subset l_3, \end{aligned}$$

are such that $S_1 \times S_2 \times S_3$ spans $\Omega(n^2)$ unit-area triangles.

The construction of S_1, S_2, S_3: General context. As mentioned at the beginning of this section, rather than stating what S_1, S_2, S_3 are, we present the machinery that we have used for their construction, thereby demonstrating that this problem is a special case of the theory of Elekes and Rónyai [6]; we also refer the reader to the more recent related studies [7, 13, 14].

One of the main results of Elekes and Rónyai is the following. (Note that the bound in (i) has recently been improved to $O(n^{11/6})$ in [13, 14].)

Theorem 13 (Elekes and Rónyai [6]). *Let $f(x, y)$ be a bivariate real rational function. Then one of the following holds.*
(i) For any triple of sets $A, B, C \subset \mathbb{R}$, each of size n,

$$\left| \{(x, y, z) \in A \times B \times C \mid z = f(x, y)\} \right| = o(n^2).$$

(ii) There exist univariate real rational functions h, φ, ψ, such that f has one of the forms

$$\begin{aligned} f(x, y) &= h(\varphi(x) + \psi(y)), \\ f(x, y) &= h(\varphi(x)\psi(y)), \\ f(x, y) &= h\left(\frac{\varphi(x) + \psi(y)}{1 - \varphi(x)\psi(y)}\right). \end{aligned}$$

Note that if one does not insist on φ, ψ being rational functions, then in all three cases specified in (ii) f can be written as $f(x, y) = h(\varphi(x) + \psi(y))$, for suitable h, φ, ψ (not necessarily rational functions).

Our problem is thus a special instance of the context in Theorem 13. Specifically, we claim that

$$f(x, y) = \frac{xy - \alpha x - 2}{y - x}$$

satisfies condition (ii) of the theorem, which in turn will lead to the (natural) construction of the desired sets S_1, S_2, S_3 (see below for details). Again, we could just state this special

16

representation of f, but we use this opportunity to show the derivation of this representation, using the machinery sketched in Elekes and Rónyai [6].

So we set the task of describing a necessary and sufficient condition that a real bivariate (twice differentiable) function $F(x, y)$ is locally[4] of the form $F(x, y) = h(\varphi(x) + \psi(y))$, for suitable univariate twice differentiable functions h, φ, ψ (not necessarily rational functions). This condition is presented in [6] where its (rather straightforward) necessity is argued. It is mentioned in [6] that the sufficiency of this test was observed by A. Jarai Jr. (apparently in an unpublished communication). Since no proof is provided in [6], we present here a proof, for the sake of completeness.

Lemma 14. *Let $F(x, y)$ be a bivariate twice-differentiable real function, and assume that neither of F_x, F_y is identically zero. Let $D(F) \subset \mathbb{R}^2$ denote the domain of definition of F, and let U be a connected component of the relatively open set $D(F) \setminus (\{F_y = 0\} \cup \{F_x = 0\}) \subset \mathbb{R}^2$. We let $q(x, y) := F_x/F_y$, which is defined, with a constant sign, over U. Then*

$$\frac{\partial^2 (\log |q(x, y)|)}{\partial x \partial y} \equiv 0 \tag{14}$$

over U if and only if F, restricted to U, is of the form

$$F(x, y) = h(\varphi(x) + \psi(y)), \tag{15}$$

for some (twice-differentiable) univariate real functions φ, ψ, and h.

Proof. We show only sufficiency of the condition (14), as its necessity can easily be verified (as argued in [6]). Setting $g(x, y) := \log |q(x, y)|$, equation (14) becomes

$$\frac{\partial^2 g}{\partial x \partial y} \equiv 0,$$

and then clearly g must have the form

$$g(x, y) = g_1(x) - g_2(y),$$

for suitable differentiable univariate functions g_1, g_2. That is

$$\log |q(x, y)| = g_1(x) - g_2(y),$$

or

$$q(x, y) = \pm e^{g_1(x)}/e^{g_2(y)}.$$

That is,

$$F_x/F_y = \varphi'(x)/\psi'(y), \tag{16}$$

where $\varphi(x) := \pm \int e^{g_1(x)} dx$ and $\psi(y) := \int e^{g_2(y)} dy$ (the arbitrary constants in these indefinite integrals clearly do not matter). Note that φ and ψ are twice differentiable strictly monotone functions, and are thus injective.

---

[4]Note that such a local representation of F allows one to construct sets A, B, C showing that property (i) of Theorem 13 does not hold for F, i.e., sets such that there are $\Theta(n^2)$ solutions of $z = F(x, y)$ in $A \times B \times C$. This, using Theorem 13, implies the validity of property (ii) (globally, and with rational functions).

Next, we express the function F in terms of new coordinates (u, v), given by

$$u = \varphi(x) + \psi(y),$$
$$v = \varphi(x) - \psi(y),$$

where (u, v) range over the image of U under this transformation; since φ and ψ are injections, the above system is invertible. Then by the chain rule we have

$$F_x = F_u u_x + F_v v_x = \varphi'(x)(F_u + F_v)$$

$$F_y = F_u u_y + F_v v_y = \psi'(y)(F_u - F_v),$$

or

$$\frac{F_x}{\varphi'(x)} = F_u + F_v,$$

$$\frac{F_y}{\psi'(y)} = F_u - F_v,$$

and thus

$$\frac{F_x}{\varphi'(x)} - \frac{F_y}{\psi'(y)} \equiv 2F_v.$$

Using (16), the last equation is

$$F_v \equiv 0.$$

This means that F depends only on the variable u, so it has the form

$$F(x, y) = h(\varphi(x) + \psi(y)),$$

as claimed. $\qquad\square$

The construction of S_1, S_2, S_3: Specifics. We next apply Lemma 14 to our specific function $f(x, y) = \frac{xy - \alpha x - 2}{y - x}$. In what follows we fix a connected open set $U \subset D(f) \setminus (\{f_x = 0\} \cup \{f_y = 0\})$, and restrict the analysis only to points $(x, y) \in U$. We have

$$f_x = \frac{y^2 - \alpha y - 2}{(y - x)^2}, \quad \text{and} \quad f_y = \frac{-x^2 + \alpha x + 2}{(y - x)^2}.$$

By assumption, the numerators are nonzero and of constant signs, and the denominator is nonzero, over U. In particular, we have

$$\frac{f_x}{f_y} = \frac{(-x^2 + \alpha x + 2)^{-1}}{(y^2 - \alpha y - 2)^{-1}}. \tag{17}$$

That is, without explicitly testing that (14) holds, we see that f_x/f_y has the form in (16). Hence Lemma 14 implies that $f(x, y)$ can be written as

$$f(x, y) = h(\varphi(x) + \psi(y)),$$

for suitable twice-differentiable univariate functions φ, ψ, and h, where φ and ψ are given (up to additive constants) by

$$\varphi'(x) = -\frac{1}{x^2 - \alpha x - 2}, \qquad \psi'(y) = \frac{1}{y^2 - \alpha y - 2}.$$

18

As explained above, this already implies that f satisfies property (ii) of Theorem 13.

Straightforward integration of these expressions yields that, up to a common multiplicative factor, which can be dropped, we have[5]

$$\varphi(x) = \ln \left| \frac{x - s_2}{x - s_1} \right|, \qquad \psi(y) = \ln \left| \frac{y - s_1}{y - s_2} \right|,$$

where s_1, s_2 are the two real roots of $s^2 - \alpha s - 2 = 0$.

We conclude that $f(x, y) = \dfrac{xy - \alpha x - 2}{y - x}$ is a function of

$$\varphi(x) + \psi(y) = \ln \left| \frac{x - s_2}{x - s_1} \right| + \ln \left| \frac{y - s_1}{y - s_2} \right| = \ln \left| \frac{x - s_2}{x - s_1} \right| \cdot \left| \frac{y - s_1}{y - s_2} \right|,$$

or, rather, a function of

$$u = \frac{x - s_2}{x - s_1} \cdot \frac{y - s_1}{y - s_2}.$$

A tedious calculations, which we omit, shows that

$$f(x, y) = \frac{s_2 - s_1 u}{1 - u},$$

confirming that f does indeed have one of the special forms in Theorem 13 above. That is,

$$f(x, y) = h(\varphi(x)\psi(y)),$$

where h, φ, ψ are the rational functions

$$h(u) = \frac{s_2 - s_1 u}{1 - u}, \qquad \varphi(x) = \frac{x - s_2}{x - s_1}, \qquad \psi(y) = \frac{y - s_1}{y - s_2}$$

(these are not the φ, ψ in the derivation above).

We then choose points $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$ such that

$$\frac{x_i - s_2}{x_i - s_1} = \frac{y_i - s_2}{y_i - s_1} = 2^i,$$

or

$$x_i = y_i = \frac{2^i s_1 - s_2}{2^i - 1},$$

for $i = 1, \ldots, n$, and let $X := \{x_1, \ldots, x_n\}$ and $Y := \{y_1, \ldots, y_n\}$. For $x = x_i$, $y = y_j$, the corresponding value of u is 2^{i-j}. Hence, setting

$$Z := \left\{ f(x_i, y_j) \mid 1 \le i, j \le n \right\} = \left\{ \frac{s_2 - s_1 \cdot 2^{i-j}}{1 - 2^{i-j}} \mid 1 \le i, j \le n \right\},$$

which is clearly also of size $\Theta(n)$, completes the proof. $\square$

---

[5]Note also that f is defined over $y \ne x$, whereas in our derivation we also had to exclude $\{f_x = 0\} \cup \{f_y = 0\}$, i.e. $\{x = s_1\} \cup \{x = s_2\} \cup \{y = s_1\} \cup \{y = s_2\}$. Nevertheless, the final expression coincides with f also over these excluded lines.

4   Unit-area triangles in convex grids

A set $X = \{x_1, \ldots, x_n\}$, with $x_1 < x_2 < \cdots < x_n$, of real numbers is said to be *convex* if

$$x_{i+1} - x_i > x_i - x_{i-1},$$

for every $i = 2, \ldots, n - 1$. See [5, 15] for more details and properties of convex sets.

In this section we establish the following improvement of Theorem 2 for convex grids.

Theorem 15. *Let $S = A \times B$, where $A, B \subset \mathbb{R}$ are convex sets of size $n^{1/2}$ each. Then the number of unit-area triangles spanned by the points of S is $O(n^{31/14})$.*

Proof. With each point $p = (a, b, c) \in A^3$ we associate a plane $h(p)$ in $\mathbb{R}^3$, given by

$$\frac{1}{2} \begin{vmatrix} a & b & c \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 1, \tag{18}$$

or equivalently by

$$(c - b)x + (a - c)y + (b - a)z = 2.$$

We put

$$H := \{h(p) \mid p \in A^3\}.$$

A triangle with vertices $(a_1, x_1), (a_2, x_2), (a_3, x_3)$ has unit area if and only if the left-hand side of (18) has absolute value 1, so for half of the permutations (i_1, i_2, i_3) (i.e., three permutations) of $(1, 2, 3)$, we have $(x_{i_1}, x_{i_2}, x_{i_3}) \in h(a_{i_1}, a_{i_2}, a_{i_3})$. In other words, the number of unit-area triangles is at most one third of the number of incidences between the points of B^3 and the planes of H. In addition to the usual problematic issue that arise in point-plane incidence problems, where many planes can pass through a line that contains many points (see, e.g., [3]), we need to face here the issue that the planes of H are in general not distinct, and may arise with large multiplicity. Denote by $w(h)$ the *multiplicity* of a plane $h \in H$, that is, $w(h)$ is the number of points $p \in A^3$ for which $h(p) = h$. Observe that, for $p, p' \in A^3$,

$$h(p) \equiv h(p') \quad \text{if and only if} \quad p' \in p + (1, 1, 1)\mathbb{R}. \tag{19}$$

We can transport this notion to points of A^3, by defining the *multiplicity* $w(p)$ of a point $p \in A^3$ by

$$w(p) := (p + (1, 1, 1)\mathbb{R}) \cap A^3.$$

Then we clearly have $w(h(p)) = w(p)$ for each $p \in A^3$. Similarly, for $q \in B^3$, we put, by a slight abuse of notation,

$$w(q) := (q + (1, 1, 1)\mathbb{R}) \cap B^3,$$

and refer to it as the *multiplicity* of q. (Clearly, the points of B^3 are all distinct, but the notion of their "multiplicity" will become handy in one of the steps of the analysis — see below.)

Fix a parameter $k \in \mathbb{N}$, whose specific value will be chosen later. We say that $h \in H$ (resp., $p \in A^3$, $q \in B^3$) is *k-rich*, if its multiplicity is at least k; otherwise we say that it is *k-poor*. For a unit-area triangle T, with vertices $(a, x), (b, y), (c, z)$, we say that T is *rich-rich* (resp., *rich-poor*, *poor-rich*, *poor-poor*) if $(a, b, c) \in A^3$ is k-rich (resp., rich, poor, poor), and

20

$(x, y, z) \in B^3$ is k-rich (resp., poor, rich, poor). (These notions depend on the parameter k, which is fixed throughout the rest of the analysis.)

Next, we show that our assumption that A and B are convex allows us to have some control on the multiplicity of the points and the planes, which we need for the proof.

For two given subsets $X, Y \subset \mathbb{R}$, and for any $s \in \mathbb{R}$, denote by $\delta_{X,Y}(s)$ the number of representations of s in the form $x - y$, with $x \in X$, $y \in Y$. We order the elements of $X - Y = \{s_1, \ldots, s_{|X-Y|}\}$, so that

$$\delta_{X,Y}(s_1) \geq \delta_{X,Y}(s_2) \geq \cdots$$

The following lemma is taken from Schoen and Shkredov.

Lemma 16 (Schoen and Shkredov [15]). *Let $X, Y \subset \mathbb{R}$, with X convex. Then, for any $\tau \geq 1$, we have*

$$\left| \{ s \in X - Y \mid \delta_{X,Y}(s) \geq \tau \} \right| = O\left(\frac{|X||Y|^2}{\tau^3} \right).$$

Lemma 16 implies that the number of points $(a, b) \in A^2$, for which the line $(a, b) + (1, 1)\mathbb{R}$ contains at least k points of A^2, is $O(n^{3/2}/k^2)$. Indeed, the number of differences $s \in A - A$ with $\delta_{A,A}(s) \geq \tau$ is $O(n^{3/2}/\tau^3)$. Each difference s determines, in a 1-1 manner, a line in $\mathbb{R}^2$ with orientation $(1, 1)$ that contains the $\delta_{A,A}(s)$ pairs $(a, b) \in A^2$ with $b - a = s$. Let M_τ (resp., $M_{\geq \tau}$) denote the number of differences $s \in A - A$ with $\delta_{A-A}(s) = \tau$ (resp., $\delta_{A-A}(s) \geq \tau$). Then the desired number of points is

$$\sum_{\tau \geq k} \tau M_\tau = k M_{\geq k} + \sum_{\tau > k} M_{\geq \tau} = O(n^{3/2}/k^2) + \sum_{\tau > k} O(n^{3/2}/\tau^3) = O(n^{3/2}/k^2).$$

We next establish the following simple claim.

Lemma 17. *The number of k-rich points in A^3 and in B^3 is $O(n^2/k^2)$.*

Proof. Let $(a, b, c) \in A^3$ be k-rich. Then, by definition, the line $l := (a, b, c) + (1, 1, 1)\mathbb{R}$ contains at least k points of A^3. We consider the line $l' := (a, b) + (1, 1)\mathbb{R}$, which is the (orthogonal) projection of l onto the xy-plane, which we identify with $\mathbb{R}^2$. Note that the projection of the points of $l \cap A^3$ onto $\mathbb{R}^2$ is injective and its image is equal to $l' \cap A^2$. In particular, l' contains at least k points of A^2. As just argued, the total number of such points in A^2 (lying on some line of the form l', that contains at least k points of A^2) is $O(n^{3/2}/k^2)$. Each such point is the projection of at most $n^{1/2}$ k-rich points of A^3 (this is the maximum number of lines of the form $(a, b, c) + (1, 1, 1)\mathbb{R}$ that project onto the same line l'). Thus, the number of k-rich points in A^3 is $O(\frac{n^{3/2}}{k^2} \cdot n^{1/2}) = O(n^2/k^2)$. The same bound applies to the number of k-rich points in B^3, by a symmetric argument. $\qquad\square$

Remark. The proof of Lemma 17 shows, in particular, that the images of the sets of k-rich points of A^3 and of B^3, under the projection map onto the xy-plane, are of cardinality $O(n^{3/2}/k^2)$.

In what follows, we bound separately the number of unit-area triangles that are rich-rich, poor-rich (and, symmetrically, rich-poor), and poor-poor.

Rich-rich triangles. Note that for $((a, b, c), (\xi, \eta)) \in A^3 \times B^2$, with $a \neq b$, there exists at most one point $\zeta \in B$ such that $T((a, \xi), (b, \eta), (c, \zeta))$ has unit area. Indeed, the point (c, ζ) must lie on a certain line $l((a, \xi), (b, \eta))$ parallel to $(a, \xi) - (b, \eta)$. This line intersects $x = c$ in exactly one point (because $a \neq b$), which determines the potential value of ζ. Thus, since we are now concerned with the number of rich-rich triangles (and focusing at the moment on the case where $a \neq b$), it suffices to bound the number of such pairs $((a, b, c), (\xi, \eta))$, with $(a, b, c) \in A^3$ being rich, and $(\xi, \eta) \in B^2$ being the projection of a rich point of B^3, which is $O((n^2/k^2) \cdot (n^{3/2}/k^2)) = O(n^{7/2}/k^4)$, using Lemma 17 and the Remark following it.

It is easy to check that the number of unit-area triangles $T(p, q, r)$, where $p, q, r \in P$ and p, q share the same abscissa (i.e., A-component), is $O(n^2)$. Indeed, there are $\Theta(n^{3/2})$ such pairs (p, q), and for each of them there exist at most $n^{1/2}$ points $r \in P$, such that $T(p, q, r)$ has unit area (because the third vertex r must lie on a certain line $l(p, q)$, which passes through at most this number of points of P); here we do not use the fact that we are interested only in rich-rich triangles.

We thus obtain the following lemma.

Lemma 18. *The number of rich-rich triangles spanned by P is $O\left(\dfrac{n^{7/2}}{k^4} + n^2\right)$.*

Poor-rich and rich-poor triangles. Without loss of generality, it suffices to consider only poor-rich triangles.

We put
$$H_i := \{h \in H \mid 2^{i-1} \leq w(h) < 2^i\},$$
for $i = 1, \ldots, \log k$, and
$$S_{\geq k} := \{q \in B^3 \mid w(q) \geq k\}.$$
That is, by definition, $\bigcup_i H_i$ is the collection of k-poor planes of H, and $S_{\geq k}$ is the set of k-rich points of B^3. Since each element of H_i has multiplicity at least 2^{i-1}, we have the trivial bound $|H_i| \leq n^{3/2}/2^{i-1}$.

Consider the family of horizontal planes $\mathcal{F} := \{\xi_z\}_{z \in B}$, where $\xi_{z_0} := \{z = z_0\}$. Our strategy is to restrict $S_{\geq k}$ and H_i, for $i = 1, \ldots, \log k$, to the planes $\xi \in \mathcal{F}$, and apply the Szemerédi–Trotter incidence bound (see Theorem 1) to the resulting collections of points and intersection lines, on each such ξ. Note that two distinct planes $h_1, h_2 \in H$ restricted to ξ, become two distinct lines in ξ. Indeed, each plane of H contains a line parallel to $(1, 1, 1)$, and two such planes, that additionally share a horizontal line within ξ, must be identical. Similarly to the argument in the Remark following Lemma 17, we have that the number of rich points $(x, y, z_0) \in S_{\geq k}$, with z_0 fixed, is $O\left(n^{3/2}/k^2\right)$; that is, $|S_{\geq k} \cap \xi_{z_0}| = O\left(n^{3/2}/k^2\right)$ for every fixed z_0.

The number of incidences between the points of $S_{\geq k}$ and the poor planes of H, counted with multiplicity (of the planes) is at most
$$\sum_{z \in B} \sum_{i=1}^{\log k} 2^i \cdot \mathcal{I}(S_{\geq k} \cap \xi_z, H_{iz}),$$

where $H_{iz} := \{h \cap \xi_z \mid h \in H_i\}$. By Theorem 1, this is at most

$$\sum_{z \in B} \sum_{i=1}^{\log k} 2^i \cdot O\left(\left(\frac{n^{3/2}}{k^2}\right)^{2/3} \left(\frac{n^{3/2}}{2^{i-1}}\right)^{2/3} + \frac{n^{3/2}}{k^2} + \frac{n^{3/2}}{2^{i-1}}\right)$$

$$= \sum_{z \in B} O\left(\frac{n^2}{k^{4/3}} \sum_{i=1}^{\log k} 2^{i/3} + \frac{n^{3/2}}{k^2} \sum_{i=1}^{\log k} 2^i + n^{3/2} \log k\right)$$

$$= \sum_{z \in B} O\left(\frac{n^2}{k} + \frac{n^{3/2}}{k} + n^{3/2} \log k\right)$$

$$= O\left(\frac{n^{5/2}}{k} + n^2 \log k\right).$$

This bounds the number of poor-rich triangles spanned by P. Clearly, using a symmetric argument, this bound also applies to the number of rich-poor triangles spanned by P. We thus obtain the following lemma.

Lemma 19. *The number of poor-rich triangles and of rich-poor triangles spanned by P is*
$O\left(\dfrac{n^{5/2}}{k} + n^2 \log k\right)$.

Poor-poor triangles. Again we are going to use Theorem 1. For $i = 1, \ldots, \log k$, put

$$S_i := \{q \in B^3 \mid 2^{i-1} \le w(q) < 2^i\},$$

and let S_i', H_i' be the respective (orthogonal) projections of S_i, H_i to the plane $\eta := \{x+y+z = 1\}$. Note that H_i' is a collection of lines in η. Moreover, arguing as above, two distinct planes of H_i project to two distinct lines of H_i', and thus the multiplicity of the lines is the same as the multiplicity of the original planes of H_i. Similarly, a point $q \in S_i$ with multiplicity t projects to a point $q' \in S_i'$ with multiplicity t (by construction, there are exactly t points of S_i that project to q'). These observations allow us to use here, as before, the trivial bounds $|S_i'| \le n^{3/2}/2^{i-1}$, $|H_i'| \le n^{3/2}/2^{i-1}$, for $i = 1, \ldots, \log k$.

Applying Theorem 1 to the collections S_i', H_j' in η, for $i, j = 1, \ldots, \log k$, taking into account the multiplicity of the points and of the lines in these collections, we obtain that the number of incidences between the poor points and the poor planes, counted with the appropriate multiplicity, is at most

$$\sum_{i,j=1}^{\log k} 2^{i+j} \cdot I(S_i', H_j') = \sum_{i,j=1}^{\log k} 2^{i+j} \cdot O\left(\left(\frac{n^{3/2}}{2^{i-1}}\right)^{2/3} \left(\frac{n^{3/2}}{2^{j-1}}\right)^{2/3} + \frac{n^{3/2}}{2^{i-1}} + \frac{n^{3/2}}{2^{j-1}}\right)$$

$$= O\left(n^2 \sum_{i,j=1}^{\log k} 2^{(i+j)/3} + n^{3/2} \sum_{i,j=1}^{\log k} \left(2^i + 2^j\right)\right)$$

$$= O\left(n^2 k^{2/3} + n^{3/2} k \log k\right).$$

Thus, we obtain the following lemma.

Lemma 20. *The number of poor-poor triangles spanned by P is $O\left(n^2 k^{2/3} + n^{3/2} k \log k\right)$.*

In summary, the number of unit-area triangles spanned by P is

$$O\left(\frac{n^{7/2}}{k^4} + \frac{n^{5/2}}{k} + n^2 k^{2/3} + n^{3/2} k \log k\right). \tag{20}$$

Setting $k = n^{9/28}$ makes this bound $O(n^{31/14})$, and Theorem 15 follows. $\square$

5 Discussion

The results obtained in this paper raise several interesting open problems.

Theorem 2 constitutes a significant improvement over the previous bounds, but it still leaves a substantial gap from the near-quadratic lower bound. One key feature in our analysis is the incidence bound between points and 2-surfaces in $\mathbb{R}^4$. Although this bound is known to be tight for the general case, one could hope that it is potentially much smaller for the special instance under consideration.

In Section 3 we have considered unit-area triangles spanned by points on three (arbitrary) lines, and have shown that, perhaps somewhat surprisingly, this becomes an instance of the special case in the Elekes–Rónyai theory, leading to the quadratic lower bound that we have derived. If instead the points lie on some general constant-degree algebraic curve, one would expect that this special case does not arise, and consequently the points of S would span only $O(n^{11/6})$ unit-area triangles. It would be interesting to characterize those curves for which the special case does arise.

Finally, it would be interesting to extend the analysis to other properties of triangles, such as bounding the number of unit-perimeter triangles spanned by n arbitrary points in the plane, or by n points lying on an algebraic curve, and so on.

References

[1] R. Apfelbaum, *Geometric Incidences and Repeated Configurations*, Ph.D. Dissertation, School of Computer Science, Tel Aviv University, 2013.

[2] R. Apfelbaum and M. Sharir, An improved bound on the number of unit-area triangles, *Discrete Comput. Geom.* 44 (2010), 753–761.

[3] R. Apfelbaum and M. Sharir, Large bipartite graphs in incidence graphs of points and hyperplanes, *SIAM J. Discrete Math.* 21 (2007), 707–725.

[4] A. Dumitrescu, M. Sharir and Cs. D. Tóth, Extremal problems on triangle areas in two and three dimensions, *J. Combinat. Theory, Ser. A* 116 (2009), 1177–1198.

[5] G. Elekes, M. Nathanson, and I. Ruzsa, Convexity and sumsets, *J. Number Theory* 83(2) (2000), 194–201.

[6] G. Elekes and L. Rónyai, A combinatorial problem on polynomials and rational functions, *J. Combinat. Theory Ser. A* 89 (2000), 1–20.

[7] G. Elekes and E. Szabó, How to find groups? (And how to use them in Erdős geometry?), *Combinatorica* 32 (2012), 537–571.

[8] G. Elekes and E. Szabó, On triple lines and cubic curves: The Orchard Problem revisited, in `arXiv:1302.5777`.

[9] P. Erdős and G. Purdy, Some extremal problems in geometry, *J. Combinat. Theory* 10 (1971), 246–252.

[10] P. Erdős and G. Purdy, Extremal problems in combinatorial geometry. in *Handbook of Combinatorics* (R. Graham, M. Grötschel and L. Lovász, editors), Vol. 1, 809–874, Elsevier, Amsterdam, 1995.

[11] J. Pach and M. Sharir, Repeated angles in the plane and related problems, *J. Combinat. Theory Ser. A* 59 (1992), 12–22.

[12] Pach, J. and Sharir, S., On the nunber of incidences between points and curves, *Combinat. Probab. Comput.* 7 (1998), 121–127.

[13] O. E. Raz, M. Sharir, and J. Solymosi, Polynomials vanishing on grids: The Elekes-Rónyai problem revisited, *Amer. J. Math.*, to appear. Also in *Proc. 30th Annu. ACM Sympos. Comput. Geom.*, 2014, 251–260, and in `arXiv:1401.7419`.

[14] O. E. Raz, M. Sharir, and F. de Zeeuw, Polynomials vanishing on Cartesian products: The Elekes-Szabó Theorem revisited, manuscript, 2014.

[15] T. Schoen and I. D. Shkredov, On sumsets of convex sets, *Combinat. Probab. Comput.* 20 (2011), 793–798.

[16] J. Solymosi and T. Tao, An incidence theorem in higher dimensions, *Discrete Comput. Geom.* 48 (2012), 255–280.

[17] J. Solymosi and F. de Zeeuw, Incidence bounds on Cartesian products, manuscript, 2014.

[18] E. Szemerédi and W. T. Trotter, Extremal problems in discrete geometry, *Combinatorica* 3 (1983), 381–392.

[19] J. Zahl, A Szemerédi-Trotter type theorem in $\mathbb{R}^4$, in arXiv:1203.4600.