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On Geometric Graphs with No *k* Pairwise Parallel Edges*

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Abstract. A *geometric graph* is a graph $G = (V, E)$ drawn in the plane so that the vertex set V consists of points in general position and the edge set E consists of straight-line segments between points of V. Two edges of a geometric graph are said to be *parallel* if they are opposite sides of a convex quadrilateral.

In this paper we show that, for any fixed $k \geq 3$, any geometric graph on n vertices with no k pairwise parallel edges contains at most $O(n)$ edges, and any geometric graph on n vertices with no k pairwise crossing edges contains at most $O(n \log n)$ edges. We also prove a conjecture by Kupitz that any geometric graph on n vertices with no pair of parallel edges contains at most $2n - 2$ edges.

1. Introduction

A *geometric graph* is a graph $G = (V, E)$ drawn in the plane so that the vertex set V consists of points in general position and the edge set E consists of straight-line segments between points of V. See [9] for a survey of results about geometric graphs. Two edges of a geometric graph are said to be *parallel*, if they are opposite sides of a convex quadrilateral. Two edges *cross*, if their relative interiors intersect.

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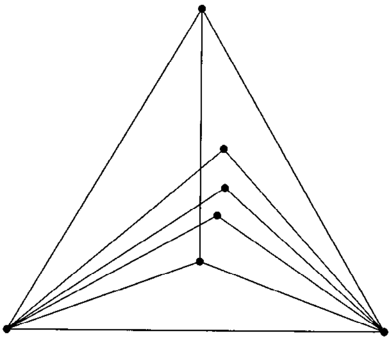

Fig. 1. Kupitz's construction for $n = 7$.

Pach and Törőcsik [11] proved that any geometric graph on n vertices with no k pairwise disjoint edges contains at most $(k-1)^4 n$ edges. In particular, their result implies that for any fixed $k \geq 2$ any geometric graph on n vertices with no k pairwise disjoint edges contains at most $O(n)$ edges. In this paper we show that the number of edges is at most linear in n also in a more general case.

Theorem 1. *Let $k \geq 2$ be a constant. Then any geometric graph on n vertices with no k pairwise parallel edges has at most $O(n)$ edges.*

The case $k = 2$ of Theorem 1 was considered first by Kupitz [7] who constructed, for any $n \geq 4$, a geometric graph with n vertices and $2n-2$ edges containing no pair of parallel edges (see Fig. 1). Kupitz [7] also conjectured that the lower bound $2n-2$ given by his construction is tight. A nearly tight upper bound $2n-1$ was shown by Katchalski and Last [5] and [8]. In this paper we show a refinement of the proof by Katchalski and Last [5] and [8] giving Kupitz's conjecture.

Theorem 2. *For $n \geq 4$, any geometric graph on n vertices with no pair of parallel edges contains at most $2n-2$ edges.*

A related question is: How many edges can be contained in a geometric graph with no k pairwise crossing edges? For $k = 2$, Euler's formula gives the upper bound $3n-6$ ($n \geq 3$). Pach et al. [10] proved that any geometric graph on n vertices with no k pairwise crossing edges contains at most $O(n \log^{2k-4} n)$ edges ($k \geq 3$). This bound was improved to $O(n \log^{2k-6} n)$ in [2] ($k \geq 3$) (thus, in particular, to $O(n)$ for $k = 3$). For fixed $k \geq 4$ we further improve this bound to $O(n \log n)$.

Theorem 3. *Any geometric graph on n vertices with no k pairwise crossing edges has at most $c_k n \log n$ edges.*

In a very recent paper [12], we give a different proof of Theorem 3 and generalize

Theorem 3 to graphs whose vertices are represented by distinct points in the plane and edges by x-monotone curves (Jordan arcs).

Theorem 3 is derived from Theorem 1. In the proof of Theorem 1 we apply a projection method of Katchalski and Last [5] and [8], Dilworth's theorem, and results on generalized Davenport–Schinzel sequences. Theorem 1 is proved in Section 2, Theorem 2 in Section 3, and Theorem 3 in Section 4.

2. Geometric Graphs with No *k* Pairwise Parallel Edges

2.1. *Generalized Davenport–Schinzel Sequences*

For $l \geq 1$, a sequence is called *l-regular*, if any l (or fewer) consecutive terms are pairwise different. For $l \geq 2$, a sequence

$$S = s_1, s_2, \ldots, s_{3l-2}$$

of length $3l - 2$ is said to be *of type* up-down-up(l), if the first l terms are pairwise different and, for $i = 1, 2, \ldots, l$,

$$s_i = s_{2l-i} = s_{(2l-2)+i}.$$

Thus, a sequence is of type up-down-up(2) if and only if it is an alternating sequence of length 4. It is well known [3] (and not difficult to prove) that any 2-regular sequence over an n-element alphabet containing no alternating subsequence of length 4 has length at most $2n - 1$. In the proof of Theorem 1 we apply the following related result:

Theorem 4 [6]. *Let $l \geq 2$ be a constant. Then the length of any l-regular sequence over an n-element alphabet containing no subsequence of type* up-down-up(l) *is at most $O(n)$.*

2.2. *Proof of Theorem* 1

Let $G = (V, E)$ be a geometric graph on n vertices, with no k pairwise parallel edges. Let $V = \{v_1, v_2, \ldots, v_n\}$. Without loss of generality, we assume that no two points lie on a horizontal line.

Let $e \in E$. An oriented edge $\overrightarrow{e}$ is defined as the edge e oriented upward. The *direction of e*, $\mathrm{dir}(e)$, is defined as the direction of the vector $\overrightarrow{v_i v_j}$, where $\overrightarrow{e} = (v_i, v_j)$. Let $E = \{e_1, e_2, \ldots, e_m\}$, where

$$0 < \mathrm{dir}(e_1) < \mathrm{dir}(e_2) < \cdots < \mathrm{dir}(e_m) < \pi$$

(if necessary, we perturb the vertices of G to make the directions of edges of G pairwise different).

Let P_1 and P_2 be the sequences of m integers obtained from the sequence

$$\overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_m}$$

by replacing each edge $\overrightarrow{e_k} = (v_i, v_j)$ by integer i and by integer j, respectively. We call the sequences P_1, P_2 the *pattern sequences of* G. Inspired by [5] and [8], our proof of Theorem 1 is based on a careful analysis of the pattern sequences.

Lemma 5. *For each $l \geq 1$, at least one of the pattern sequences P_1, P_2 contains an l-regular subsequence of length at least $|E|/(4l) = m/(4l)$.*

Lemma 6. *Neither of the pattern sequences P_1, P_2 contains a subsequence of type* up-down-up(k^3).

Before proving Lemmas 5 and 6 we complete the proof of Theorem 1.

Proof of Theorem 1. According to Lemma 5, at least one of the sequences P_1, P_2 contains a k^3-regular subsequence S of length at least $|E|/(4k^3)$. According to Lemma 6, the sequence S contains no subsequence of type up-down-up(k^3). Theorem 4 implies that the length of S is at most $O(n)$. Consequently,

$$|E| \leq 4k^3 \cdot O(n) = O(n). \qquad \square$$

It remains to prove Lemmas 5 and 6.

2.3. *Proof of Lemma* 5

We apply a simple greedy algorithm [1] which, for given integer $l \geq 1$ and finite sequence A, returns an l-regular subsequence $B(A, l)$ of A. In the first step, an auxiliary sequence B is taken empty. Then the terms of A are considered one by one from left to right, and in each step the considered term is placed at the end of B iff this does not violate the l-regularity of B. Finally, the obtained l-regular subsequence B of A is taken for $B(A, l)$.

For example, if $A = 1, 3, 1, 3, 5, 2, 2, 5, 1, 5, 1, 2$ and $l = 3$, then the algorithm returns the sequence $B(A, 3) = 1, 3, 5, 2, 1, 5, 2$. We prove Lemma 5 by showing that, given $l \geq 1$, at least one of the sequences $B(P_1, l)$, $B(P_2, l)$ obtained by the algorithm has length at least $|E|/(4l) = m/(4l)$.

Let $l \geq 1$ be given. For $i = 1, 2$ and for $1 \leq j_1 \leq j_2 \leq m$, let $P_{i,[j_1,j_2]}$ denote the part of P_i starting with the j_1th term and ending with the j_2th term. Thus, $P_{i,[j_1,j_2]}$ consists of $j_2 - j_1 + 1$ terms.

Let $|T|$ denote the length of a sequence T, and $I(T)$ the set of integers appearing in T.

Claim 7. *For each $j = 1, 2, \ldots, m$,*

$$|B(P_{1,[1,j]}, l)| + |B(P_{2,[1,j]}, l)| \geq j/(2l).$$

Proof. First, consider two integers j_1, j_2 such that $1 \leq j_1 \leq j_2 \leq m$. Obviously,

$$\{\overrightarrow{e_{j_1}}, \overrightarrow{e_{j_1+1}}, \ldots, \overrightarrow{e_{j_2}}\} \subseteq \{(v_a, v_b) | a \in I(P_{1,[j_1,j_2]}), b \in I(P_{2,[j_1,j_2]})\}.$$

Thus,

$$|\{\overrightarrow{e_{j_1}}, \overrightarrow{e_{j_1+1}}, \ldots, \overrightarrow{e_{j_2}}\}| \leq |\{(v_a, v_b) | a \in I(P_{1,[j_1,j_2]}), b \in I(P_{2,[j_1,j_2]})\}|.$$

Consequently,

$$j_2 - j_1 + 1 \leq |I(P_{1,[j_1,j_2]})| \cdot |I(P_{2,[j_1,j_2]})|.$$

By the inequality between algebraic and geometric means,

$$\frac{|I(P_{1,[j_1,j_2]})| + |I(P_{2,[j_1,j_2]})|}{2} \geq \sqrt{j_2 - j_1 + 1}. \quad (1)$$

We can now prove the claim by induction on j. If $j \leq \min\{16l^2, m\}$, then by (1) and by $j \leq 16l^2$

$$\begin{aligned} |B(P_{1,[1,j]}, l)| + |B(P_{2,[1,j]}, l)| &\geq |I(P_{1,[1,j]})| + |I(P_{2,[1,j]})| \\ &\geq 2\sqrt{j} \\ &\geq j/(2l). \end{aligned}$$

Suppose now that $16l^2 < j_0 \leq m$ and that Claim 7 holds for $j = 1, 2, \ldots, j_0 - 1$. Since for $i = 1, 2$ each integer of $I(P_{i,[j_0-4l^2+1,j_0]})$ not appearing among the last $l - 1$ terms in $B(P_{i,[1,j_0-4l^2]}, l)$ appears more times in $B(P_{i,[1,j_0]}, l)$ than in $B(P_{i,[1,j_0-4l^2]}, l)$, we have

$$|B(P_{i,[1,j_0]}, l)| \geq |B(P_{i,[1,j_0-4l^2]}, l)| + |I(P_{i,[j_0-4l^2+1,j_0]})| - (l-1).$$

Consequently, by the inductive hypothesis and by (1),

$$\begin{aligned} |B(P_{1,[1,j_0]}, l)| + |B(P_{2,[1,j_0]}, l)| &\geq (j_0 - 4l^2)/(2l) + 2\sqrt{4l^2} - 2(l-1) \\ &> j_0/(2l). \end{aligned}$$ □

Proof of Lemma 5. Lemma 5 follows easily from Claim 7 (with $j = m$) and from the pigeon-hole principle. □

2.4. *Proof of Lemma* 6

In the proof of Lemma 6 we apply the following easy consequence of Dilworth's theorem [4]:

Theorem 8. *If the union of three partial orderings on a set I of size at least $(k-1)^3+1$ is a linear ordering on I, then at least one of the partial orderings contains a chain of length k.*

Proof. Let $\preceq_1, \preceq_2, \preceq_3$ be the three partial orderings on I. If $(I, \preceq_1)$ does not contain a chain of length k then, by Dilworth's theorem, it can be covered by at most $k - 1$ antichains. Consequently, there is an antichain A of size $(k-1)^2 + 1$ in $(I, \preceq_1)$. If we restrict our attention to A and to orderings $\preceq_2, \preceq_3$, another application of Dilworth's theorem gives k elements in A which form a chain with respect to $\preceq_2$ or to $\preceq_3$. □

Proof of Lemma 6. Because of symmetry, it suffices to prove Lemma 6 for the pattern sequence P_1. Suppose to the contrary that P_1 contains a subsequence of type up-down-up(k^3). Thus, there is a subsequence

$$S = s_1, s_2, \ldots, s_{3k^3-2}$$

of P_1 such that the integers $s_1, s_2, \ldots, s_{k^3}$ are pairwise different and that, for $i = 1, 2, \ldots, k^3$, $s_i = s_{2k^3-i} = s_{(2k^3-2)+i}$. For simplicity of notation, suppose that $s_i = i$ ($i = 1, \ldots, k^3$) and that $S = P_{1,[1,3k^2-2]}$. We obtain a contradiction by showing that k of the edges $e_1, e_2, \ldots, e_{3k^3-2}$ are pairwise parallel.

Define three partial orderings $\preceq_1, \preceq_2, \preceq_3$ on the set $I = \{1, 2, \ldots, k^3\}$ as follows:

Definition 9. Let $i, j \in I$, and let $\mathrm{dir}(\overrightarrow{v_i v_j})$ denote the direction of the vector $\overrightarrow{v_i v_j}$. Then:

(i) $i \prec_1 j$, if $i < j$ and $\mathrm{dir}(\overrightarrow{v_i v_j}) \in [\mathrm{dir}(\overrightarrow{e_{k^3}}), \pi)$;
(ii) $i \prec_2 j$, if $i < j$ and $\mathrm{dir}(\overrightarrow{v_i v_j}) \in (\pi, \pi + \mathrm{dir}(\overrightarrow{e_{2k^3-1}}))$; and
(iii) $i \prec_3 j$, if $i < j$ and $\mathrm{dir}(\overrightarrow{v_i v_j}) \in [\pi + \mathrm{dir}(\overrightarrow{e_{2k^3-1}}), 2\pi) \cup (0, \mathrm{dir}(\overrightarrow{e_{k^3}}))$.

Since the union of $\preceq_1, \preceq_2, \preceq_3$ is a linear ordering on I, Theorem 8 implies that one of the orderings $\preceq_1, \preceq_2, \preceq_3$ contains a chain $i_1, i_2, \ldots, i_k$ of length k. We distinguish the corresponding three possible cases.

If $i_1 \prec_1 i_2 \prec_1 \cdots \prec_1 i_k$, then the edges $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$ are pairwise parallel. Indeed, if $1 \le j < j' \le k$, then the inequalities

$$0 \le \mathrm{dir}(\overrightarrow{e_{i_j}}) < \mathrm{dir}(\overrightarrow{e_{i_{j'}}}) \le \mathrm{dir}(\overrightarrow{e_{k^3}}) \le \mathrm{dir}(\overrightarrow{v_{i_j} v_{i_{j'}}}) < \pi$$

show that the edges $e_{i_j}, e_{i_{j'}}$ are parallel (see Fig. 2).

Similarly, if $i_1 \prec_2 i_2 \prec_2 \cdots \prec_2 i_k$, then the edges $e_{(2k^3-2)+i_1}, e_{(2k^3-2)+i_2}, \ldots, e_{(2k^3-2)+i_k}$ are pairwise parallel, and if $i_1 \prec_3 i_2 \prec_3 \cdots \prec_3 i_k$, then the edges $e_{2k^3-i_1}, e_{2k^3-i_2}, \ldots, e_{2k^3-i_k}$ are pairwise parallel. □

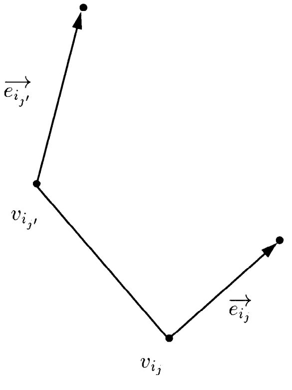

Fig. 2. The edges $e_{i_j}, e_{i_{j'}}$ are parallel.

3. Proof of Kupitz's Conjecture

In this section we prove Theorem 2. Our proof is based on the proof [5] of a weaker form of Theorem 2 (with $2n-2$ replaced by $2n-1$).

In the proof we apply the following lemma from [5]:

Lemma 10. *Let A_i, A_j, A_k, A_ℓ be four points appearing in this order on a closed convex curve γ. Let P, Q be two points inside γ. Consider the four (closed) segments*

$$PA_i,\ QA_j,\ PA_k,\ QA_\ell$$

and assume that among them there is no segment s such that s contains only one of the points P, Q and the line supporting s contains both of them. Then two of the four segments are in convex position.

Proof of Theorem 2. Let G be a geometric graph on n vertices with no pair of edges in convex position. We use the definitions and notation from [5]. The following bound on the number of edges in G was shown in [5]:

$$e \leq 2n-1.$$

We need to show that

$$e < 2n-1.$$

Suppose to the contrary that $e = 2n-1$. It then follows from the proof in [5] that each vertex v_i of G has a leftmost edge $v_i v_{l(i)}$ and a rightmost edge $v_i v_{r(i)}$ ($l(i) \neq r(i)$), and that the length of the pattern sequence $PS(G)$ is exactly $2n-2$ (see [5] for definitions).

For each vertex v_i of G, we define an interval $I(v_i)$ on C by

$$I(v_i) = C \cap \operatorname{conv}\big(\overrightarrow{v_i v_{l(i)}} \cup \overrightarrow{v_i v_{r(i)}}\big).$$

Certainly, $I(v_i)$ contains all points of $D(G)$ colored by color i.

Observation 11. *The intervals $I(v_i)$, $i = 1, \ldots, n$, form a nested set, i.e., if two of them intersect, then one of them is contained in the other one.*

Proof. If two intervals $I(v_i)$, $I(v_j)$, $i \neq j$, intersect and none of them is contained in the other one, then the points $\alpha_{il(i)}, \alpha_{jl(j)}, \alpha_{ir(i)}, \alpha_{jr(j)}$ appear in this order on C, and Lemma 10 implies that two of the edges $v_i v_{l(i)}, v_i v_{r(i)}, v_j v_{l(j)}, v_j v_{r(j)}$ are in convex position, a contradiction. □

Let v_k be a vertex of G such that the angle between $\overrightarrow{v_k v_{r(k)}}$ and $\overrightarrow{v_k v_{l(k)}}$ is maximal. Consider the two points $\alpha_{r(k)k}$ and $\alpha_{l(k)k}$. They both lie outside $I(v_k)$, since the angle between $\overrightarrow{v_k v_{r(k)}}$ and $\overrightarrow{v_k v_{l(k)}}$ is smaller than π.

Observation 12. *No interval $I(v_i)$ contains both $\alpha_{r(k)k}$ and $\alpha_{l(k)k}$.*

Proof. If an interval $I(v_i)$ contains both $\alpha_{r(k)k}$ and $\alpha_{l(k)k}$, then, by the maximality of the angle between $\overrightarrow{v_k v_{r(k)}}$ and $\overrightarrow{v_k v_{l(k)}}$, one of the rays $\overrightarrow{v_i v_{r(i)}}$, $\overrightarrow{v_i v_{l(i)}}$ intersects one of the rays $\overrightarrow{v_{r(k)} v_k}$, $\overrightarrow{v_{l(k)} v_k}$ outside C, and the corresponding two edges of G are in convex position. □

It follows from Observation 12 that there are at least three maximal intervals $I(v_i)$ (the interval $I(v_k)$ and the two maximal intervals containing $\alpha_{r(k)k}$ and $\alpha_{l(k)k}$, respectively).

By Observation 11, maximal intervals $I(v_i)$ are pairwise disjoint.

Observation 13. *Each maximal interval $I(v_i)$ containing m_i intervals $I(v_{i'})$ (including the interval $I(v_i)$ itself) contains at most $2m_i - 1$ points of $D(G)$.*

Proof. The part of $PS(G)$ corresponding to $I(v_i) \cap D(G)$ is a Davenport–Schinzel sequence of order 2 on m_i integers (i.e., it contains no alternating subsequence of length 4). As we have already mentioned in Section 2.1, it is well known [3] that the length of such a sequence is at most $2m_i - 1$. □

Thus, the length of $D(G)$ (and also of $PS(G)$) is at most

$$2n - (\sharp \text{ of maximal intervals } I(v_i)) \leq 2n - 3,$$

a contradiction. This completes the proof of Theorem 2. □

4. Graphs with No *k* Pairwise Crossing Edges

Here we show Theorem 3. Let $k \geq 2$ be a constant, and let $f_k(n)$ be the maximum number of edges in a geometric graph on n vertices with no k pairwise crossing edges. Let $G = (V, E)$ be a geometric graph on n vertices with no k pairwise crossing edges. Introduce a Cartesian coordinate system so that the y-axis partitions V into two parts which are as equal as possible, thus the sets

$$V^- = \{v \in V | \text{the } x\text{-coordinate of } v \text{ is negative}\},$$

$$V^+ = \{v \in V | \text{the } x\text{-coordinate of } v \text{ is positive}\},$$

have sizes

$$|V^-| = \lfloor n/2 \rfloor, \qquad |V^+| = \lceil n/2 \rceil.$$

Partition E into three subsets E^+, E^-, E' such that E^+ contains the edges with both endpoints in V^+, E^- contains the edges with both endpoints in V^-, and E' contains the edges crossing the y-axis.

To obtain a bound on the size of E', consider the mapping T given by $(x, y) \mapsto (1/x, y/x)$ $(x \neq 0)$. Further, consider the graph $\tilde{G}$ on the vertex set $\tilde{V} = \{T(v) | v \in V\}$ with two vertices $T(v), T(w)$ connected by an edge if and only if $\{v, w\} \in E'$. The graph $\tilde{G}$ contains no k pairwise parallel edges, since otherwise the corresponding k edges in

E' would be pairwise crossing. By Theorem 1, $\tilde{G}$ contains $O(n)$ edges. Consequently,

$$|E'| = O(n).$$

Obviously,

$$|E^-| \leq f_k(\lfloor n/2 \rfloor), \qquad |E^+| \leq f_k(\lceil n/2 \rceil).$$

Thus,

$$f_k(n) \leq f_k(\lfloor n/2 \rfloor) + f_k(\lceil n/2 \rceil) + O(n).$$

Consequently,

$$f_k(n) = O(n \log n). \qquad \square$$

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