Combinatorial Geometry with Algorithmic Applications

The Alcala Lectures

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Contents

CHAPTER 1

Crossing Numbers of Graphs: Graph Drawing and its Applications

Our ancestors drew their pictures (pictographs or, simply, "*graphs*") on walls of caves, nowadays we use mostly computer screens for this purpose. From the mathematical point of view, there is not much difference: both surfaces are "flat," they are topologically equivalent.

1. Crossings – the Brick Factory Problem

Let G be a finite graph with vertex set $V(G)$ and edge set $E(G)$. By a *drawing* of G, we mean a representation of G in the plane such that each vertex is represented by a distinct point and each edge by a simple (non-selfintersecting) continuous arc connecting the corresponding two points. If it is clear whether we talk about an "abstract" graph G or its planar representation, these points and arcs will also be called vertices and edges, respectively. For simplicity, we assume that in a drawing (a) no edge passes through any vertex other than its endpoints, (b) no two edges are tangent to each other (i.e., if two edges have a common interior point, then at this point they properly cross each other), and (c) no three edges cross at the same point.

Every graph has many different drawings. If G can be drawn in such a way that no two edges cross each other, then G is *planar.* According to an observation of K. Wagner [**?**] and I. Fáry [**?**] that also follows from a theorem of Steinitz [**?**], if G is planar then it has a drawing, in which every edge is represented by a straight-line segment.

It is well known that K_5, the *complete graph* with 5 vertices, and $K_{3,3}$, the *complete bipartite graph* with 3 vertices in each of its classes are not planar. According to Kuratowski's theorem, a graph is planar if and only if it has no subgraph that can be obtained from K_5 or from $K_{3,3}$ by subdividing some (or all) of its edges with distinct new vertices. In the next section, we give a completely different representation of planar graphs (see Theorem 2.4).

If G is not planar then it cannot be drawn in the plane without crossing. Paul Turán [**?**] raised the following problem: find a drawing of G, for which the number of crossings is minimum. This number is called the *crossing number* of G and is denoted by $\text{CR}(G)$. More precisely, Turán's (still unsolved) original problem was to determine $\text{CR}(K_{n,m})$, for every $n, m \geq 3$. According to an assertion of Zarankiewicz, which was down-graded from theorem to conjecture [**?**], we have

$$\text{CR}(K_{n,m}) = \left\lfloor \frac{m}{2} \right\rfloor \cdot \left\lfloor \frac{m-1}{2} \right\rfloor \cdot \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n-1}{2} \right\rfloor,$$

but we do not even know the limits

$$\lim_{n\to\infty} \frac{\mathrm{CR}(K_{n,n})}{n^4}, \quad \lim_{n\to\infty} \frac{\mathrm{CR}(K_n)}{n^4}$$

(cf. [**?**], [**?**]). It is not hard to show, however, that these limits exist and are positive.

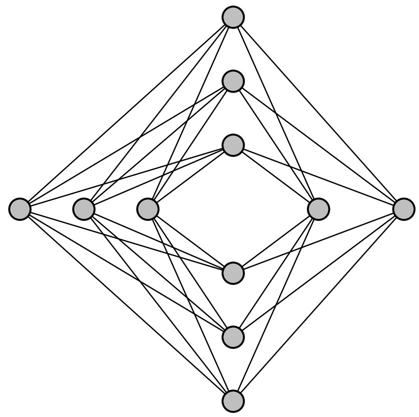

FIGURE 1. $K_{5,6}$ drawn with minimum number of crossings

Turán used to refer to the above question as the "brick factory problem," because it occurred to him at a factory yard, where, as forced labor during World War II, he moved wagons filled with bricks from kilns to storage places. According to his recollections, it was not a very tough job, except that they had to push much harder at the crossings. Had this been the only "practical application" of crossing numbers, much fewer people would have tried to estimate $\mathrm{CR}(G)$ during the past quarter of a century. In the early eighties, it turned out that the chip area required for the realization (VLSI layout) of an electrical circuit is closely related to the crossing number of the underlying graph [**?**]. This discovery gave an impetus to research in the subject.

2. Thrackles – Conway's Conjecture

A drawing of a graph is called a *thrackle*, if any two edges that do not share an endpoint cross precisely once, and if two edges share an endpoint then they have no other point in common.

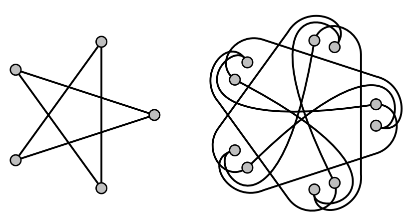

FIGURE 2. Cycles C_5 and C_{10} drawn as thrackles

It is easy to verify that C_4, a cycle of length 4, cannot be drawn as a thrackle, but any other cycle can [**?**]. If a graph cannot be drawn as a thrackle, then the same is true for all graphs that contain it as a subgraph. Thus, a thrackle does not contain a cycle of length 4, and, according to an old theorem of Erdős in extremal graph theory, the number of its edges cannot exceed $n^{3/2}$, where n denotes the number of its vertices (see [**?**]).

The following old conjecture states much more.

Conjecture 2.1 (J. Conway). *Every thrackle has at most as many edges as vertices.*

The first upper bound on the number of edges of a thrackle, which is linear in n, was found by Lovász *et al.* [**?**]: Every thrackle has at most twice as many edges as vertices. The constant two has been improved to one and a half.

Theorem 2.2 (Cairns-Nikolayevsky [**?**]). *Every thrackle has at most one and a half times as many edges as vertices.*

Thrackle and planar graph are, in a certain sense, opposite notions: in the former any two edges intersect, in the latter there is no crossing pair of edges. Yet the next theorem shows how similar these concepts are.

A drawing of a graph is said to be a *generalized thrackle* if every pair of its edges intersect an odd number of times. Here the common endpoint of two edges also counts as a point of intersection. Clearly, every thrackle is a generalized thrackle, but not the other way around. For example, a cycle of length 4 can be drawn as a generalized thrackle, but not as a thrackle.

Theorem 2.3 (Lovász *et al.* [**?**]). *A bipartite graph can be drawn as a thrackle if and only if it is planar.*

According to an old observation of Erdős, every graph has a bipartite subgraph which contains at least half of its edges. Clearly, every planar graph of $n \geq 3$ vertices has at most $2n - 4$ edges. Hence, Theorem 2.3 immediately implies that every thrackle with $n \geq 3$ vertices has at most $2(2n - 4) = 4n - 8$ edges. This bound is weaker than Theorem 2.2, but it is already linear in n.

In a drawing of a graph, a triple of internally disjoint paths ($P_1(u, v)$, $P_2(u, v)$, $P_3(u, v)$) between the same pair of vertices (u, v) is called a *trifurcation*. (The three paths cannot have any vertices in common, other than u and v, but they can cross at points different from their vertices.) A trifurcation ($P_1(u, v)$, $P_2(u, v)$, $P_3(u, v)$) is said to be a *converter* if the cyclic order of the initial pieces of P_1, P_2, and P_3 around u is opposite to the cyclic order of their final pieces around v.

Theorem 2.4 (Lovász *et al.* [**?**]). *A graph is planar if and only if it has a drawing, in which every trifurcation is a converter.*

Proof. The second half of the theorem is trivial: if a graph is planar, then it can be drawn without crossing, and, clearly, every trifurcation in this drawing is a converter. To establish the first half, by Kuratowski's theorem, it is sufficient to show that if every trifurcation in a graph G is a converter, then G does not contain a subdivision of $K_{3,3}$ or of K_5.

Suppose that G contains a subdivision of $K_{3,3}$ with vertex classes $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$. Denote this subdivision by K. Deleting from K the point u_3 together with the three paths connecting it to the v_j's, we obtain a converter between u_1 and u_2. Similarly, deleting u_2 (u_1) we obtain a converter between u_1 and u_3 (u_2 and u_3, respectively). We say that the type of u_i is *clockwise* or *counterclockwise* according to the circular order of the initial segments of the paths u_iv_1, u_iv_2, u_iv_3 around u_i. It follows from the definition of the converter that any two u_i's must have opposite types, which is impossible.

The case when G contains a subdivision of K_5 is left to the reader. □

3. Different Crossing Numbers?

As is illustrated by Theorem 2.4, the investigation of crossings in graphs often requires parity arguments. This phenomenon can be partially explained by the "banal" fact that if we start out from the interior of a simple (non-selfintersecting) closed curve in the plane, then we find ourselves inside or outside of the curve depending on whether we crossed it an even or an odd number of times.

Next we define three variants of the notion of crossing number.

(1) The *rectilinear crossing number*, LIU-CR(G), of a graph G is the minimum number of crossings in a drawing of G, in which every edge is represented by a straight-line segment.

(2) The *pairwise crossing number* of G, PAIR-CR(G), is the minimum number of crossing pairs of edges over all drawings of G. (Here the edges can be represented by arbitrary continuous curves, so that two edges may cross more than once, but every pair of edges can contribute to PAIR-CR(G) at most one.)

(3) The *odd-crossing number* of G, ODD-CR(G), is the minimum number of those pairs of edges which cross an odd number of times, over all drawings of G.

It readily follows from the definitions that

$$\text{ODD-CR}(G) \leq \text{PAIR-CR}(G) \leq \text{CR}(G) \leq \text{LIN-CR}(G).$$

Bienstock and Dean [?] exhibited a series of graphs with crossing number 4, whose rectilinear crossing numbers are arbitrary large. Pelsmajer, Schaefer, and Štefankovič [?] have shown that for any $\varepsilon > 0$ there exists a graph G with

$$\text{ODD-CR}(G) \leq \left(\frac{\sqrt{3}}{2} + \varepsilon\right) \text{PAIR-CR}(G).$$

A better construction was found by Tóth [?], with the constant $\frac{\sqrt{3}}{2}$ replaced by $\frac{3\sqrt{5}-5}{2}$. However, we cannot rule out the possibility that

CONJECTURE 3.1 ([?]). *There is a constant $\gamma > 0$ such that* ODD-CR$(G) \geq \gamma \cdot \text{CR}(G)$, *for every graph G.*

CONJECTURE 3.2. *For every graph G, we have* PAIR-CR$(G) = $ CR(G).

The determination of the odd-crossing number can be rephrased as a purely combinatorial problem, thus the above two conjectures may offer a spark of hope that there exists an efficient approximation algorithm for estimating their values.

According to a remarkable theorem of H. Hanani (alias Chojnacki) [**?**] and W. Tutte [**?**], if a graph G can be drawn in the plane so that any pair of its edges cross an even number of times, then it can also be drawn without any crossing. In other words, $\text{ODD-CR}(G) = 0$ implies that $\text{CR}(G) = 0$. Note that in this case, by the observation of Fáry mentioned in Section 1, we also have that $\text{LIU-CR}(G) = 0$.

The main difficulty in this problem is that a graph has so many essentially different drawings that the computation of any of the above crossing numbers, for a graph of only 15 vertices, appears to be a hopelessly difficult task even for very fast computers [**?**].

THEOREM 3.3 ([**?**], [**?**], [**?**]). *The computation of the crossing number, the pairwise crossing number, and the odd-crossing number are NP-complete problems.*

The growth rates of the three parameters in Theorem 3.3, $\text{CR}(G)$, $\text{PAIR-CR}(G)$, and $\text{ODD-CR}(G)$, are not completely unrelated. (See also [**?**] and [**?**].)

THEOREM 3.4 (Pach–Tóth [**?**]). *For any graph G, we have*

$$\text{CR}(G) \leq 2(\text{ODD-CR}(G))^2.$$

The proof of the last statement is based on the following sharpening of the Hanani–Tutte Theorem.

THEOREM 3.5 ([**?**]). *Any drawing of any graph in the plane can be redrawn in such a way that no edge, which originally crossed every other edge an even number of times, would participate in any crossing.*

PROOF. (Proof of Theorem 3.4 using Theorem 3.5) Let $G = (V, E)$ be a simple graph drawn in the plane with $\lambda = \text{ODD-CR}(G)$ pairs of edges that cross an odd number of times. Let $E_0 \subset E$ denote the set of edges in this drawing which cross every other edge an even number of times. Since every edge not in E_0 crosses at least one other edge an odd number of times, we obtain that

$$|E \setminus E_0| \leq 2\lambda.$$

By Theorem 3.5, there exists a drawing of G, in which no edge of E_0 is involved in any crossing. Pick a drawing with this property such that the total number of crossing points between all pairs of edges not in E_0 is minimal. Notice that in this drawing, any two edges cross at most once. Therefore, the number of crossings is at most

$$\binom{|E \setminus E_0|}{2} \leq \binom{2\lambda}{2} \leq 2\lambda^2,$$

and Theorem 3.4 follows. □

In [**?**], the original form of the Hanani–Tutte Theorem was applied to answer a question about the "complexity" of the boundary of the union of geometric figures [**?**]. A very elegant proof of a slight generalization of Theorem 3.5 was found by Pelsmajer *et al.* [**?**]. It is conjectured that Theorem 3.5 can be strengthened so that the conclusion remains true for every edge that in the original drawing meets all other edges *not incident to its endpoints* an even number of times.

In the original definition of the crossing number we assume that no three edges pass through the same point. Of course, this can be always achieved by slightly perturbing the drawing. Equivalently, we can say that *k-fold crossings* are permitted, but they are counted $\binom{k}{2}$ times.

G. Rote, M. Sharir, and others asked what happens if multiple crossings are counted only *once*? To what extent does this modification effect the notion of crossing number? It is important to assume here that *no tangencies are allowed* between the edges. Indeed, otherwise, given a complete graph with n vertices, one can easily draw it with only *one* crossing point p so that every pair of vertices is connected by an edge passing through p.

Let $\textsc{cr}'(G)$ denote the *degenerate crossing number* of G, that is, the minimum number of crossing *points* over all drawings of G satisfying this condition, where k-fold crossings are also allowed. Of course, we have

$$\textsc{cr}'(G) \leq \textsc{cr}(G),$$

and the two crossing numbers are not necessarily equal. For example, Kleitman [**?**] proved that the crossing number of the complete bipartite graph $K_{5,5}$ with five vertices in its classes is 16. On the other hand, the degenerate crossing number of $K_{5,5}$ in the plane is at most 15. Another example is depicted in Figure 3.

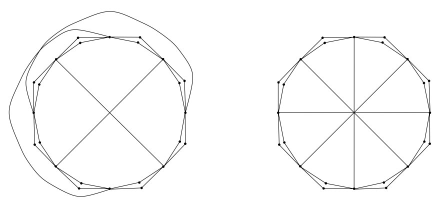

FIGURE 3. $\textsc{cr}(G) = 2$, $\textsc{cr}'(G) = 1$

Let $n = n(G)$ and $e = e(G)$ denote the number of vertices and the number of edges of a graph G. Ajtai, Chvátal, Newborn, Szemerédi [**?**] and, independently, Leighton [**?**] proved

THEOREM 3.6 (Leighton and Ajtai *et al.*). *For every graph G with $e(G) \geq 4n(G)$, we have*

$$\textsc{cr}(G) \geq \frac{1}{64}\frac{e^3(G)}{n^2(G)}.$$

As we have seen in Chapter 4, this statement has many interesting applications in combinatorial geometry. Does it remain asymptotically true for the *degenerate* crossing number? It is not hard to show [**?**] that the answer is "no" if we permit drawings in which two edges may cross an arbitrary number of times. More precisely, any graph G with $n(G)$ vertices and $e(G)$ edges has a proper drawing with fewer than $e(G)$ crossings, where each crossing point that belongs to the interior of several edges is counted only once. That is, we have $\textsc{cr}'(G) < e(G)$. The order of magnitude of this bound cannot be improved if $e \geq 4n$.

Therefore, we restrict our attention to so-called *simple* drawings, i.e., to proper drawings in which two edges are allowed to cross at most once. Let $\textsc{cr}^*(G)$ denote the minimum number of crossing points over all simple drawings, where several edges may cross at the same point. One can prove that in this sense the degenerate crossing number of very "dense" graphs is at least $\Omega(e^3/n^2)$. More precisely, we have

THEOREM 3.7 (Pach–Tóth [?]). *There exists a constant $c^* > 0$ such that*

$$\mathrm{CR}^*(G) \geq c^* \frac{e^4(G)}{n^4(G)}$$

holds for any graph G with $e(G) \geq 4n(G)$.

It is a challenging question to decide whether here the term $\frac{e^4(G)}{n^4(G)}$ can be replaced by $\frac{e^3(G)}{n^2(G)}$, just like in Theorem 3.6.

4. Straight-Line Drawings

For "straight-line thrackles," Conway's conjecture discussed in Section 2 had been settled by H. Hopf–E. Pannwitz [?] and (independently) by Paul Erdős much before the problem was raised!

If every edge of a graph is drawn by a straight-line segment, then we call the drawing a *geometric graph* [?], [?], [?]. We assume for simplicity that no three vertices of a geometric graph are collinear and that no segment representing an edge passes through any vertex other than its endpoints.

THEOREM 4.1 (Hopf–Pannwitz–Erdős theorem). *If any two edges of a geometric graph intersect (in an endpoint or an internal point), then it can have at most as many edges as vertices.*

PROOF. (Perles) We say that an edge uv of a geometric graph is a *leftmost* edge at its endpoint u if turning the ray uv around u through 180 degrees in the counterclockwise direction, it never contains any other edge uw. For each vertex u, delete the leftmost edge at u, if such an edge exists. We claim that at the end of the procedure, no edge is left. Indeed, if at least one edge uv remains, it follows that we did not delete it when we visited u and we did not delete it when we visited v. Thus, there exist two edges uw and vz such that the ray uw can be obtained from uv by a counterclockwise rotation about u through less than 180 degrees, and the ray vz can be obtained from vu by a counterclockwise rotation about v through less than 180 degrees. This implies that uw and vz cannot intersect, which contradicts our assumption. □

The systematic study of extremal problems for geometric graphs was initiated by Avital–Hanani [?], Erdős, Perles, and Kupitz [?]. In particular, they asked the following question: what is the maximum number of edges of a geometric graph of n vertices, which does not have k pairwise disjoint edges? (Here, by "disjoint" we mean that they cannot cross and cannot even share an endpoint.) Denote this maximum by $e_k(n)$.

Using this notation, the above theorem says that $e_2(n) = n$, for every $n > 2$. Noga Alon and Erdős [?] proved that $e_3(n) \leq 6n$. This bound was first reduced by a factor of two [?], and not long ago Černý [?] showed that $e_3(n) = 2.5n + O(1)$. It had been an open problem for a long time to decide whether $e_k(n)$ is linear in n for every fixed $k > 3$. Pach and Törőcsik [?] were the first to show that this is indeed the case. More precisely, they used Dilworth's theorem for partial orders to prove that $e_k(n) \leq (k-1)^4 n$. This bound was improved successively by G. Tóth–P. Valtr [?], and by Tóth [?].

THEOREM 4.2 (Tóth [?]). *For every k and every n, we have $e_k(n) \leq 2^{10}(k-1)^2 n$.*

It is very likely that the dependence of $e_k(n)$ on k is also (roughly) linear.

Analogously, one can try to determine the maximum number of edges of a geometric graph with n vertices, which does not have k pairwise crossing edges. Denote this maximum by $f_k(n)$. It follows from Euler's Polyhedral Formula that, for $n > 2$, every planar graph with n vertices has at most $3n-6$ edges. Equivalently, we have $f_2(n) = 3n - 6$.

THEOREM 4.3 (Agarwal *et al.* [?]). $f_3(n) = O(n)$.

Better proofs and generalizations were found in [?], [?], [?], [?].

Recently, Ackerman [?] proved that $f_4(n) = O(n)$. Plugging this bound into the result of [?], we obtain

THEOREM 4.4. *For a fixed $k > 4$, we have $f_k(n) = O(n \log^{2k-8} n)$.*

Valtr [?] has shown that $f_k(n) = O(n \log n)$, for any $k > 4$, but it can be conjectured that $f_k(n) = O(n)$. Moreover, it cannot be ruled out that there exists a constant c such that $f_k(n) \leq ckn$, for every k and n. However, we cannot even decide whether every complete geometric graph with n vertices contains at least (a positive) constant times n pairwise crossing edges. We are ashamed to admit that the strongest result in this direction is the following

THEOREM 4.5 (Aronov *et al.* [?]). *Every complete geometric graph with n vertices contains at least $\lfloor \sqrt{n/12} \rfloor$ pairwise crossing edges.*

Several *Ramsey-type* results for geometric graphs, closely related to the subject of this section, were established in [?], [?], [?]. In [?], some of these results have been generalized to *geometric hypergraphs* (systems of simplices).

5. Angular Resolution and Slopes

It is one of the major goals of graphic visualization to improve the readability of diagrams. If the angle between two adjacent edges is too small, it causes "blob effects" and it is hard to tell those edges apart. The minimum angle between two edges in a straight-line drawing of a graph G is called the *angular resolution* of the drawing. Of course, if the maximum degree of a vertex of G is d, then the angular resolution of any drawing of G is at most $\frac{2\pi}{d}$. It was shown by Formann *et al.* [?] that every graph G of maximum degree d can be drawn by straight-line edges with angular resolution at least constant times $\frac{1}{d^2}$ and that this bound is best possible up to a logarithmic factor. For planar graphs, one can achieve the asymptotically optimal resolution $\Omega\left(\frac{1}{d}\right)$, but then the optimal drawing is not necessarily crossing-free. In the case we insist on crossing-free (planar) straight-line drawings, Malitz and Papakostas [?] proved that there exists a constant $\alpha > 0$ such that any planar graph of maximum degree d permits a good drawing of angular resolution at least α^d.

Wade and Chu [?] defined the *slope number* $\mathrm{sl}(G)$ of G as the smallest number of distinct edge slopes used in a straight-line drawing of G. Dujmović *et al.* [?] asked whether the slope number of a graph of maximum degree d can be arbitrarily large. The following short argument of Pach and Pálvölgyi shows that the answer is yes for $d \geq 5$.

THEOREM 5.1 (Pach–Pálvölgyi [?], Barát *et al.* [?]). *For every $d \geq 5$, there exists a sequence of graphs of maximum degree d such that their slope numbers tend to infinity.*

PROOF. Define a "frame" graph F on the vertex set $\{1, \dots, n\}$ by connecting vertex 1 to 2 by an edge and connecting every $i > 2$ to $i-1$ and $i-2$. Adding a perfect matching M between these n points, we obtain a graph $G_M := F \cup M$. The number of different matchings is at least $(n/3)^{n/2}$. Let G denote the huge graph obtained by taking the union of disjoint copies of all G_M. Clearly, the maximum degree of the vertices of G is *five*. Suppose that G can be drawn using at most S slopes, and fix such a drawing.

For every edge $ij \in M$, label the points in G_M corresponding to i and j by the slope of ij in the drawing. Furthermore, label each frame edge ij ($|i-j| \leq 2$) by its slope. Notice that no two components of G receive the same labeling. Indeed, up to translation and scaling, the labeling of the edges uniquely determines the positions of the points representing the vertices of G_M. Then the labeling of the vertices uniquely determines the edges belonging to M. Therefore, the number of different possible labelings, which is $S^{|F|+n} < S^{3n}$, is an upper bound for the number of components of G. On the other hand, we have seen that the number of components (matchings) is at least $(n/3)^{n/2}$. Thus, for any S we obtain a contradiction, provided that n is sufficiently large. □

A more complicated proof has been found independently by Barát, Matoušek, and Wood [?].

With some extra care one can obtain

THEOREM 5.2 ([?], [?]). *For any $d \geq 5$ and $\varepsilon > 0$, there exist graphs G with n vertices of maximum degree d, whose slope numbers satisfy*

$$\mathrm{sl}(G) \geq \max\{n^{\frac{1}{2}-\frac{1}{d-2}-\varepsilon}, n^{1-\frac{8+\varepsilon}{d+4}}\}.$$

On the other hand, for *cubic* graphs we have

THEOREM 5.3 ([?], [?]). *Any connected graph of maximum degree three can be drawn with edges of at most four different slopes.*

This leaves open the annoying question whether graphs of maximum degree *four* can have arbitrarily large slope numbers.

6. An Application in Computer Graphics

It is a pleasure for the mathematician to see her research generate some interest outside her narrow field of studies. During the past twenty five years, combinatorial geometers have been fortunate enough to experience this feeling quite often. Automated production lines revolutionized *robotics*, and started an avalanche of questions whose solution required new combinatorial geometric tools [**?**]. *Computer graphics*, whose group of users encompasses virtually everybody from engineers to film-makers, has had a similar effect on our subject [**?**].

Most graphics packages available on the market contain some (so-called *warping* or *morphing*) program suitable for deforming figures or pictures. Originally, these programs were written for making commercials and animated movies, but today they are widely used.

An important step in programs of this type is to fix a few basic points of the original picture (say, the vertices of the straight-line drawing of a planar graph), and then to choose new locations for these points. We would like to redraw the graph without creating any crossing. In general, now we cannot insist that the edges be represented by segments, because such a drawing may not exist. Our goal is to produce a drawing with polygonal edges, in which the total number of segments is small. The complexity and the running time of the program is proportional to this number.

THEOREM 6.1 (Pach–Wenger [**?**]). *Every planar graph with n vertices can be redrawn in such a way that the new positions of the vertices are arbitrarily prescribed, and each edge is represented by a polygonal path consisting of at most $24n$ segments. There is an $O(n^5)$–time algorithm for constructing such a drawing.*

Badent *et al.* [**?**] have strengthened this theorem by constructing a drawing in which every edge consists of at most $3n+3$ segments, and the running time of their algorithm is only $O(n^2 \log n)$. The next result shows that Theorem 6.1 cannot be substantially improved.

THEOREM 6.2 (Pach–Wenger [**?**]). *For every n, there exist a planar graph G_n with n vertices and an assignment of new locations for the vertices such that in any polygonal drawing of G_n there are at least $n/100$ edges composed of at least $n/100$ segments.*

The proof of this theorem is based on a generalization of a result of Leighton [**?**], found independently by Pach *et al.* [**?**] and by Sýkora *et al.* [**?**]. It turned out to play a crucial role in the solution of several extremal and algorithmic problems related to graph embeddings.

The *bisection width* of a graph is the minimum number of edges whose removal splits the graph into two pieces such that there are no edges running between them and the larger piece has at most twice as many vertices as the smaller. The following result can be proved using a weighted version of the Lipton-Tarjan Separator Theorem for planar graphs [**?**].

THEOREM 6.3 ([?], [?]). *Let G be a graph of n vertices whose degrees are $d_1, d_2, \ldots, d_n$. Then the bisection width of G is at most*

$$1.58\left(16\text{CR}(G)+\sum_{i=1}^{n} d_i^2\right)^{1/2}.$$

In the next section, following Pach *et al.* [?], we use the last result to give an unusual proof of the Crossing Lemma of Leighton and Ajtai *et al.* (Theorem 3.6). For similar applications of Theorem 6.3, see [**?, ?, ?**]. Kolman and Matoušek [?] have established a similar relationship between the bisection and the pairwise crossing number of a graph.

7. An Unorthodox Proof of the Crossing Lemma

Let $b(G)$ denote the bisection width of G. By repeated application of Theorem 6.3, we obtain

COROLLARY 7.1. *Let G be a graph of n vertices with degrees $d_1, d_2, \ldots, d_n$. Then, for any edge disjoint subgraphs $G_1, G_2, \ldots, G_j \subseteq G$, we have*

$$\sum_{i=1}^{j} b(G_i) \le 1.58 j^{1/2}\left(16\text{CR}(G)+\sum_{k=1}^{n} d_k^2\right)^{1/2}.$$

PROOF. Let d_{ik} denote the degree of the k-th vertex in G_i. Applying Theorem 6.3 to each G_i separately and adding up the resulting inequalities, we obtain

$$\begin{aligned}\sum_{i=1}^{j} b^2(G_i) &\le (1.58)^2\left(16\sum_{i=1}^{j}\text{CR}(G_i)+\sum_{i=1}^{j}\sum_{k=1}^{n} d_{ik}^2\right)\\ &\le (1.58)^2\left(16\text{CR}(G)+\sum_{k=1}^{n} d_k^2\right).\end{aligned}$$

Therefore, we have

$$\left(\sum_{i=1}^{j} b(G_i)\right)^2 \le j\sum_{i=1}^{j} b^2(G_i) \le (1.58)^2 j\left(16\text{CR}(G)+\sum_{k=1}^{n} d_k^2\right),$$

as required. □

COROLLARY 7.2. *[Pach-Tardos [?]] Let G be a graph of n vertices with degrees $d_1, d_2, \ldots, d_n$. Then, for any $1 < s \le n$, one can remove at most*

$$8.6\left(\frac{n}{s}\right)^{1/2}\left(16\text{CR}(G)+\sum_{i=1}^{n} d_i^2\right)^{1/2}$$

edges from G so that every connected component of the resulting graph has fewer than s vertices.

PROOF. Partition G by subsequently subdividing each of its large components into two roughly equal halves as follows. Start the procedure by deleting $b(G)$ edges of G so that it falls into two parts, each having at most $\frac{2}{3}|V(G)| = \frac{2}{3}n$ vertices. As long as there exists a component $H \subset G$ whose size is at least s, by the removal of $b(H)$ edges, *cut* it into two smaller components, each of size at most $(2/3)|V(H)|$. When no such components are left, stop.

Let $\mathcal{H}$ denote the family of all components arising at *any* level of the above procedure (e.g., we have $G \in \mathcal{H}$ if G is connected). Define the *order* of any element $H \in \mathcal{H}$ as the largest integer k, for which there is a chain

$$H_0 \subsetneq H_1 \subsetneq \ldots \subsetneq H_k \tag{7.1}$$

in $\mathcal{H}$ such that $H_k = H$. For any k, let $\mathcal{H}_k$ denote the set of all elements of $\mathcal{H}$ of order k. Thus, $\mathcal{H}_0$ is the set of the components in the final decomposition.

For any fixed k, the elements of $\mathcal{H}_k$ are pairwise (vertex) disjoint. Recall that in a chain (7.1) we have $|V(H_1)| \geq s$ and the ratio of the sizes of any two consecutive members is at least $3/2$. Therefore, the number of vertices in any element of $\mathcal{H}_k$ is at least $(3/2)^{k-1}s$, which in turn implies that for $k \geq 1$

$$j_k := |\mathcal{H}_k| \leq \frac{n}{(3/2)^{k-1}s} = \frac{(2/3)^{k-1}n}{s}.$$

Applying Corollary 7.1 to the subgraphs in $\mathcal{H}_k$, we obtain that the total number of edges removed, when they are first subdivided during our procedure, is at most

$$1.58 \cdot (2/3)^{(k-1)/2} \left(\frac{n}{s}\right)^{1/2} \left(16\mathrm{CR}(G) + \sum_{i=1}^{n} d_i^2\right)^{1/2},$$

Summing up over all $k \geq 1$, we conclude that the total number of edges deleted during the whole procedure does not exceed the number claimed. □

Let G be a graph with $n(G) = n$ vertices and $e(G) = e > 4n$ edges. We want to prove that its crossing number $\mathrm{CR}(G)$ satisfies

$$\mathrm{CR}(G) \geq \gamma \frac{e^3(G)}{n^2(G)},$$

for an absolute constant $\gamma > 0$.

Let d denote the average degree of the vertices of G, that is, let $d = 2e/n$. Consider a drawing of G in which the number of crossings is minimum. We modify this drawing into a drawing of another graph, G', with maximum degree d, as follows. One by one we visit each vertex v of G, and if its degree $d(v)$ is larger than d, then we split v it into $\lceil d(v)/d \rceil$ vertices, each lying very close to the original location of v and each of them incident to at most d consecutive edges originally terminating at v. In the new drawing, no two edges cross in a small neighborhood of the old vertex v, including the new vertices it gave rise to, and outside of this neighborhood the new drawing is identical with the old one. Obviously, the resulting graph G' has the same number of edges as the original one and its number of vertices satisfies

$$n(G') = \sum_{v \in V(G)} \left\lceil \frac{d(v)}{d} \right\rceil < \sum_{v \in V(G)} \left(\frac{d(v)}{d} + 1\right) = 2n.$$

The number of crossings in the new drawing is precisely the same as in the original one, that is, $\mathrm{CR}(G)$.

Applying Corollary 7.2 to G', we obtain that for every $s > 1$ one can remove

$$e^* = 8.6\left(\frac{2n}{s}\right)^{1/2}\left(16\textsc{cr}(G') + \sum_{v'\in V(G')} d^2(v')\right)^{1/2} < 8.6\left(\frac{2n}{s}\right)^{1/2}\left(16\textsc{cr}(G) + 2n\left(\frac{2e}{n}\right)^2\right)^{1/2}$$

edges from G' so that every connected component of the resulting graph has fewer than s vertices.

Set $s := e/n$. After the removal of at most e^* edges, the remaining graph has at most $\frac{n}{s}\binom{s}{2} < \frac{e}{2}$ edges, so that we have $e^* > e/2$. This yields

$$(8.6)^2\frac{2n^2}{e}\left(16\textsc{cr}(G) + \frac{8e^2}{n}\right) > \frac{e^2}{4}, \quad \text{or}$$

$$2\textsc{cr}(G) + \frac{e^2}{n} > \frac{1}{5000}\frac{e^3}{n^2}.$$

Consequently, either $2\textsc{cr}(G) > 10^{-4}(e^3/n^2)$, in which case we are done, or $e^2/n > 10^{-4}(e^3/n^2)$. In the latter case, we have $e < 10^4 n$, and the relation $\textsc{cr}(G) = \Omega(e^3/n^2)$ follows from the easy observation that

$$\textsc{cr}(G) \geq e - 3n + 6,$$

for every graph G with $n > 2$ vertices and e edges (see, e.g., [**?**]).

The above argument can be easily modified to obtain better bounds for graphs without some forbidden subgraphs. Assume, for example, that G has no cycle of length *four*. According to an old theorem of Erdős, then G has at most $2n^{3/2}$ edges. Repeating essentially the same argument as above, we can argue that after the removal of e^* edges, each component C of the remaining graph has at most $2|C|^{3/2}$ edges. Setting $s = e^2/(4n)^2$, we conclude that $e^* > e/2$, and the proof can be completed analogously. We obtain

THEOREM 7.3 (Pach *et al.* [**?**]). *Let G be a graph with n vertices and $e \geq 4n$ edges, which does not contain a cycle of length* four. *Then the crossing number of G is at least $\gamma e^4/n^3$, where $\gamma > 0$ is a suitable constant. This result is tight, apart from the value of γ.*

The proof generalizes to other bipartite forbidden subgraphs in the place of the cycle of length *four*.