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Simplicity v/s Complexity in the Framework of Geometric Asymptotic Analysis and Some New Applications of the Concentration Phenomenon

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Abstract. We introduce a concept of Simplicity which in a sense corresponds to the reverse direction to the concept of Complexity. Many problems of Asymptotic Geometric Analysis rotate around this notion. We also describe the concept of Concentration and suggest a new direction of possible applications of the Concentration Phenomenon.

Keywords. Complexity, Concentration Phenomenon, explicit v/s random.

1. Introduction

The study of many different high dimensional (or high parametric) systems turns out to have a similar flavor despite the very differently sounding questions and a huge diversity of techniques and approaches. Computer Science type problems as well as other high parametric combinatorial type problems (we often call it now Asymptotic Combinatorics) often rotate around the notion of Complexity. In the main part of this paper, section 2, we describe the concept corresponding to the reverse direction, which we call Simplicity, and show that many (if not most) problems of Asymptotic Geometric Analysis, i.e. asymptotic properties of high dimensional objects, rotate around this concept. The first time I introduced and discussed this concept was in June, 2004, in the Learning Theory Conference in Barcelona [18]. In section 3 we discuss a possibility of "mixing" explicit steps with random ones to derive most interesting features of the Theory. Section 4 is very speculative. I want to open a discussion on some applications of the Concentration Phenomenon to some problems and directions which are very different from the standard and well known problems of this Theory.

Actually, the question I would like to understand here on some conceptual (and almost philosophical) level is a well known rule of Casinos: the closure of a table where significant wins have happened. What relation does it have to this Theory? We will discuss it in Section 4.

Let me start by providing a few references on articles which were specially written to introduce the subject of Asymptotic Geometric Analysis and some phenomena which govern behavior of High Dimensional Spaces. They are [15] and [16]. Also, I recommend two recent detailed surveys on this subject [4] and [5]. (All these papers are available through the "survey article" part of my home page).

2. The concept of Simplicity

Let a family of procedures (we call them "steps") be described which we will call "simple steps" and a family of bodies are specified which we also call "simple" objects. Starting with some, supposedly complicated, object we would like to estimate the minimal number N of simple steps (i.e. the steps from the family of simple procedures we introduced earlier) that may be applied to our object in order to bring it to some other object which belongs to the pre-defined class of simple objects. Then we say that N is the simplicity of our object, of course, with respect to the defined families of simple steps and simple objects.

Note, that this philosophy is exactly opposite to the standard understanding of Complexity: in the process of constructing an algorithm which estimates Complexity we start with a simple object (system) and recover the original structure; but in estimating Simplicity, we "destroy" all specific information of our system to derive a simple one with very little remaining information. However, of course, the procedures are not reversible and "small simplicity" may co-exist with huge complexity (at least theoretically).

A lot of recent results of Asymptotic Geometric Analysis are directed to this goal: how quickly we may destroy all specification of a given (arbitrary, and a priori very complicated) object (which is in our case a normed high dimensional space, or a convex body in high dimension) and to derive, say, some isomorphic copy of a Euclidean space (or an ellipsoid).

There are a number of breakthroughs in this direction which we describe next using this language.

2.1. Minkowski and Steiner Symmetrizations

For some classical symmetrizations, let us check how many elementary steps are needed to approximate an ellipsoid, starting with an arbitrary convex body $K \subset \mathbb{R}^n$. We will analyse two symmetrization procedures: Steiner symmetrization (see, e.g. [12] for a definition) and Minkowski symmetrization, which is also called Blaschke symmetrization. In both cases, each elementary step consists of selecting a hyperplane h in $\mathbb{R}^n$. We will now define the lesser known Minkowski symmetrization. Consider the reflection $r_h K$ of the body K with respect to h, then the elementary step of this symmetrization is $S_h K := \frac{K + r_h K}{2}$, where by "+" we mean the Minkowski sum of sets. The elementary step of the Steiner symmetrization, associated with a hyperplane h, is denoted by $St_h K$.

Fix some $c > 1$. What is the smallest N such that for any $K \subset \mathbb{R}^n$ we may find $\{h_i\}_1^N$ such that there is an ellipsoid $\mathcal{E}$ and

$$\mathcal{E} \subset \prod_{h=1}^{N} St_{h_i} K := K_N \subset c \cdot \mathcal{E} \ ? \tag{2.1}$$

So, in our scheme, we are estimating the Simplicity of an arbitrary convex set K with respect to the family of Steiner symmetrizations as simple steps and the family of convex bodies on the (Banach-Mazur) distance from the Euclidean ball at most c as the family of simple bodies.

Here is a brief history of this question:

- Hadwiger (∼1955) estimated N from above by $\lesssim (c_1 \cdot n)^{n/2}$ (where c_1 is a universal constant, here and below).
- Bourgain–Lindenstrauss–Milman (∼1988) made a dramatic improvement to $N \lesssim c_1 n \log n$ (for some $c \sim 3$). However, one does not even need the logarithmic factor and the answer is $N \leq \frac{3}{2} n$ (Klartag–Milman [12]).

Actually, for every $\varepsilon > 0$ there is a constant $c(\varepsilon)$, depending only on ε, instead of c in (2.1) s.t.

$$N \leq (1+\varepsilon)n$$

(and, for some K, at least $n - c_1 \log n$ is necessary). If we denote by $\mathcal{D}$ the standard Euclidean ball then additional $n-1$ symmetrizations turn $\mathcal{E}$ into $r \cdot \mathcal{D}$ for some r, and altogether $N \leq (2+\varepsilon)n$ steps are enough to approximate the Euclidean ball up to $c(\varepsilon)$. (Note that the answer has an isomorphic flavor. We did not approximate the Euclidean ball "almost isometrically", but up to some constant, independent of dimension and the body K. Such isomorphic results in geometry actually have meaning as *asymptotic* results which are interesting for large dimensions.)

In the case of Minkowski symmetrizations the best known result on the same question, until 1987, was by [3]: for any $\varepsilon > 0$ there is a $c(\varepsilon)$ such that one may find $\{h_i\}_1^N$ for $N \leq \frac{1}{2} n \log n + c(\varepsilon)n$ and (for some $r > 0$)

$$r\mathcal{D} \subset \prod_{i=1}^{N} S_{h_i} K \subset (1+\varepsilon) r\mathcal{D} \tag{2.2}$$

(and a random selection of h_i leads to (2.2) with high probability). So, the estimate for Simplicity in this case was $\lesssim cn \log n$. Klartag later showed [9] that, for some K and random selection of hyperplanes, this answer is precise (up to a constant factor).

However, the best selection of hyperplanes happens to be different and Klartag showed [10] that the smallest Simplicity N is always $N \leq 5n$; this is true for $\varepsilon \sim c \sqrt{\frac{\log \log n}{\log n}}$. So, in this case, we observe an isomorphic answer which turns out to be "asymptotically isometric" for large dimension n.

The above estimates on Simplicity of any convex body are so good (i.e., so small) that for a long period of time we thought that they are the consequence of isomorphic answers, i.e., of these universal constants which accompany every result above, and that almost isometric results should have large Simplicity, perhaps even exponential. However, recent results of B.Klartag [11] show that also in the almost isometric case (ε-isometric, for any $\varepsilon > 0$) the simplicity of the above problems is very low:

There is a universal constant c such that for any dimension n and $\varepsilon > 0$ the number $N(n;\varepsilon)$ of Minkowski symmetrizations needed to bring any convex body K to an ε-neighborhood of a Euclidean ball is at most $cn \log 1/\varepsilon$.

A similar estimate was proved by Klartag also for the family of Steiner symmetrizations.

2.2.

In a joint paper with A.Pajor [20, Theorem 7] we selected a different family of simple steps (actually much simpler than the symmetrizations we described above) and a different family of simple final objects (not bodies with distance from ellipsoids which is bounded by a fixed constant, but convex bodies with slightly more involved description) and again we demonstrated a very low Simplicity of any convex body.

2.3.

The approximation of the Euclidean norm by averaging rotations of the given norm $\|\cdot\|$ has a similar flavor. What is the minimal number N of (orthogonal) rotations $\{u_i\}$ from $O(n)$ such that the Euclidean norm $\|\cdot\|$ is well approximated, say, up to 2, by the averaging of N rotations:

$$|x| \sim \frac{1}{N} \sum_{i=1}^{N} \|u_i x\|_K, \quad u_i \in O(n) \ ?$$

In [3] and [21] the precise formula for N is given, i.e., the Simplicity of this problem was computed and it is never above Cn (see also [15]). However, if we will define Simple steps as linear maps either in the space or in its dual, and the goal is the same, i.e., the approximation of a Euclidean norm up to some universal constant, then the picture is changed. Now the simplicity is just the number 2. More precisely, the following statement is correct:

For any n and any norm $\|\cdot\|$ in $\mathbb{R}^n$ there are two linear maps u_1, u_2 which are enough for approximation: for every $x \in \mathbb{R}^n$ let

$$|||x||| := \|x\| + \|u_1 x\|$$

and, dualizing, consider the new norm $|||x|||^* + |||u_2 x|||^*$: it is already C-equivalent to a Euclidean norm (and C is a universal constant).

This is a global version of the so-called "quotient of subspace theorem" [19].

3. Explicit versus random

Let us move to another complexity related subject. Standardly, we are describing some very interesting features of normed spaces and convex bodies (like, say, existence of Euclidean subspaces of very high dimension, or Euclidean quotients of subspaces of dimension proportional to the dimension of the whole space) through random selection of corresponding subspaces in a specific Euclidean structure. To estimate the complexity of such features it seems appropriate to demonstrate explicit constructions which lead to these properties. However, such explicit constructions are unknown. In the same time, very recently, we discovered [1] that one may start with very few randomly selected bits which are complemented by a number of explicit simple and short constructions and lead to the features we are searching for. Then this number of remaining random steps will estimate the remaining ("randomized") complexity of the feature we are studying.

As an example, let us consider the very famous example of ℓ_1^n. It is well known [8] that this space contains isomorphic copies of Euclidean subspaces of any dimension proportional to n (with isomorphic constant depending only on this proportion). In a recent joint paper with S.Artstein-Avidan [1] we demonstrated how starting with $n \log n$ random bits (i.e. $\log n$ random sign-vectors) one may construct (using already explicit steps) a subspace isomorphic to Euclidean of dimension, say, $n/2$ and with an absolute constant of isomorphism to the Euclidean space. We are using in such constructions expanders and some typical derandomization schemes borrowed from Computer Sciences. However, some more delicate derandomization results are needed for applications in Asymptotic Geometry (see [1]).

Many more typical features of an arbitrary high dimensional convex bodies are studied in [1], and in all examples the derandomized complexity is very low; it is on the logarithmic level with respect to dimension.

4. New applications of the Concentration Phenomenon; instability of equilibrium and stability of losses/gains

4.1. The Concept of Concentration

Let (X, d, μ) be a compact metric space with metric d and diameter $\mathrm{diam}(X) \geq 1$, which is also equipped with a Borel probability measure μ. We then define the *concentration function* (or "isoperimetric constant") of X by

$$\alpha(X; \varepsilon) = 1 - \inf\{\mu(A_\varepsilon) : A \text{ Borel subset of } X, \mu(A) \geq \tfrac{1}{2}\},$$

where $A_\varepsilon = \{x \in X : d(x, A) \leq \varepsilon\}$ is the ε-extension of A. As a consequence of the isoperimetric inequality on the Euclidean sphere S^{n+1} it is known that

$$\alpha(S^{n+1}; \varepsilon) \leq \sqrt{\pi/8}\, \exp(-\varepsilon^2 n/2),$$

an estimate which is crucial in many results of Asymptotic Geometric Analysis, in particularly for the proof of Dvoretzky's theorem.

P. Lévy (1919) was the first to observe the role of the dimension in this particular example. For this reason, a family (X_n, d_n, μ_n) of metric probability spaces is called a *normal Lévy family with constants* (c_1, c_2) if

$$\alpha(X_n, \varepsilon) \leq c_1 \exp(-c_2\varepsilon^2 n), \tag{4.1}$$

or, more generally, a *Lévy family* if for every $\varepsilon > 0$

$$\alpha(X_n; \varepsilon) \to 0$$

as $n \to \infty$. It is a non-trivial and important observation that many "natural" families of such spaces are Lévy families. This observation is called the Concentration Phenomenon. There are many examples of Lévy families which have been discovered and used for Asymptotic Geometric Analysis purposes. In most cases, new and very interesting techniques were invented in order to estimate the concentration function $\alpha(X; \varepsilon)$. We list some of them (and refer the reader to [17] and [13] for more information):

(1) The family of the orthogonal groups $(SO(n), \rho_n, \mu_n)$ equipped with the Hilbert-Schmidt metric and the Haar probability measure is a Lévy family with constants $c_1 = \sqrt{\pi/8}$ and $c_2 = 1/8$.

(2) The family $X_n = \prod_{i=1}^{m_n} S^n$ (where S^n denotes the n dimensional Euclidean sphere) with the natural Riemannian metric and the product probability measure is a Lévy family with constants $c_1 = \sqrt{\pi/8}$ and $c_2 = 1/2$.

(3) All homogeneous spaces of $SO(n)$ inherit the property of forming Lévy families. In particular, any family of Stiefel manifolds W_{n,k_n} or any family of Grassman manifolds G_{n,k_n} is a Lévy family with the same constants as in (1).

[All these examples of normal Lévy families come from [6].]

(4) The space $F^n = \{-1, 1\}^n$ with the normalized Hamming distance $d(\eta, \eta') = \#\{i \leq n : \eta_i \neq \eta'_i\}/n$ and the normalized counting measure is a Lévy family with constants $c_1 = 1/2$ and $c_2 = 2$. This follows from an isoperimetric inequality of Harper [7], and was first put in such form and used in [2].

(5) The group Π_n of permutations of $\{1, \dots, n\}$ with the normalized Hamming distance $d(\sigma, \tau) = \#\{i \leq n : \sigma(i) \neq \tau(i)\}/n$ and the normalized counting measure satisfies $\alpha(\Pi_n; \varepsilon) \leq 2\exp(-\varepsilon^2 n/64)$. This was proved by Maurey [14] with a martingale method, which was further developed in [22].

The rate of decay of the concentration function for different families (of course, computed in a natural normalization of the metric and a natural enumeration of the elements of the family) plays a crucial role in most proofs of Asymptotic Geometric Analysis. It is the main technical tool of the Theory. There are many techniques to estimate this rate.

In fact, the concept of a Lévy family (and especially a normal Lévy family) generalizes the concept behind the law of large numbers in two directions: a) the measures are not necessarily product measures (that is, there is no condition of "independence") and b) Lipschitz functions on the space are considered instead of linear functionals only.

To explain the reason for the terminology of "concentration" and also outline why a bound of the form (4.1) is so crucial, let us consider a 1-Lip function $f(x)$ defined on (X, ρ, μ), i.e.

$$\big|f(x) - f(y)\big| \le \rho(x,y)\,.$$

Denote by L_f the median of $f(x)$, i.e.

$$\mu\big\{x \in X \bigm| f(x) \ge L_f\big\} \ge \tfrac{1}{2} \quad \text{and} \quad \mu\big\{x \in Z \bigm| f(x) \le L_f\big\} \ge \tfrac{1}{2}.$$

Then

$$\mu\big\{x \in X \bigm| |f(x) - L_f| < \varepsilon\big\} \ge 1 - 2\alpha(X, \varepsilon). \tag{4.2}$$

So, if the value of $\alpha(X, \varepsilon)$ is very small, then the values of a Lipschitz function "concentrate" in measure around one value, meaning that the function is almost constant with high probability. This is the case when $X = S^n$ and the dimension n is large, as well as for large n for Π_n or SO_n or other examples we mentioned above. It is, in fact, a general property of high dimensional metric probability spaces which is called "*concentration phenomenon*".

Such a "concentration" of measure (these types of estimates) balances the exponentially high entropy of n-dimensional spaces (or other n-parametric families) and leads to a "regularity" in high dimension, keeping "diversity" under control.

4.2. Naive discussion

We would like to discuss a few new possible applications of the concept of concentration to problems similar in spirit to the problems of Game Theory. Consider a function $f(x)$ defined on the permutation group Π_n for some fixed and large n, say, $n = 52$ (as the number of cards in a deck). Of course, then every x may be identified with a given ordering of the cards in one game, and the result of the game is described by the function $f(x)$: the level of loss or gain in this game (in this scheme the art of the player is encoded into the function f).

Our function in a "fair situation" should be balanced, which is usually expressed by $\mathbb{E}f = 0$. But let us discuss how "fair" indeed such a situation is. To demonstrate in a more clear way our thoughts let us change the setting and consider a function $f(x)$ defined on the sphere S^{n-1}. Our assumption is $\mathbb{E}f := \int_{S^{n-1}} f(x)d\mu(x) = 0$. However, it does not necessarily mean that also the median L_f should be 0. Assume that $L_f < 0$. Of course, for large n and a function f which is locally well behaved, say, Lipschitz constant $Lipf$ of f being not too large, concentration reasons imply that the value L_f must be very close to $\mathbb{E}f = 0$. However, who stated that a very complicated function f which describes the loss and the gain of the game should have a good Lip constant? [Returning for a moment to our previous example of a function on Π_n we should actually realize that, quite oppositely, we should not expect a good Lip. constant.] But then "fairness" of the game is easy to beat. Indeed, as follows from (4.1), the $C/\sqrt{n}$ - neighborhood of the level L_f of the function f contains, for a suitable universal constant C, over half of the whole measure of the sphere. Assume that $f(x) < 0$ for these vectors x. This is natural because it is very close to the vectors where our function is equal to $L_f < 0$. But then it implies that

$$Prob\left\{x \in S^{n-1} : f(x) < 0\right\} > \tfrac{3}{4}. \tag{4.3}$$

So, by Chernoff estimates, in N events of computing $f(x_i)$ (assuming that x_i are selected randomly and independently in every event) with exponentially close to 1 probability around $3N/4$ times we have $f(x_i) < 0$ (i.e., we lost). Of course, because $\mathbb{E}f = 0$, it means that, although we seldom win, some positive values of $f(x)$ may be very large to compensate many losses. However, if your opponent ("the house") introduces (what sounds very fair) an "insurance policy": any loss which is too big as well as too big a win are cancelled (i.e., truncation of function f), then the balance is changed and you surely lose (depending, of course, on the level of truncation).

In reality the situation is not so visible, and the gain in our game (of random selection of x and computing the function $f(x)$), if exists, is small; I mean, that the probability

$$Prob\left\{x \in S^{n-1} : f(x) < 0\right\} > \tfrac{1}{2} + \varepsilon$$

for a small but fixed $\varepsilon > 0$. However, the "compensation region" for $f(x)$, i.e. the set A inside S^{n-1} on which our gain $f(x)$ is very large, $f(x) > C > 0$, C is large, may be very small, and actually exponentially small in dimension n if f takes exponentially large values and C is exponentially large. Then, instead of introducing an "insurance policy", your opponent may just limit the duration of the game.

The above discussion was very preliminary and naive. I wanted to explain that even a small "dis-balance" in the function f may lead to dramatic unfairness because of high dimension and strong concentration. However, if, say, our function $f(x)$ has only two values, $+1$ and -1 , then automatically $\mathbb{E}f = L_f$ and such dis-balance will not happen.

4.3. Instability of equilibrium

Let now $f_0(x)$, $x \in S^{n-1}$, be a fair game, i.e.

$$\mathbb{E}f_0 = L_{f_0} = 0.$$

Assume that our function f_0 may be very slightly perturbed and in reality we are playing with another function $f(x) = f_0(x) + p(x)$. So, we fix this new function for a series of games.

I would like to explain now that even a very minuscule perturbation $p(x)$ may dramatically change the fairness and outcome of the game. Again, a high dimension n and the strong (exponential) concentration are responsible for this. Actually, I already provided the needed explanation before. Indeed, let $|p(x)| < \varepsilon$ for some positive but very small ε, which may be of the order $1/n^a$, for some $a > 1/2$. In a case where $Lip(f_o)$ is large (as generally expected in many problems) also a much smaller order of ε may dramatically change the outcome. Will it go in your favor or not?

This depends on the new L_f; note, I ignore the new $\mathbb{E}f$; it is not very relevant. If $L_f < 0$, the series of games is not going in your favor (and oppositely). The non-trivial remark is that even an extremely small perturbation $p(x)$ (say, the atmosphere around) may have a very visible influence on the outcome.

Of course, even if the function $f_0(x)$ would be known (which is not the case usually) the perturbed function $f(x)$ is definitely not known. So, can we "beat" the fate and, with small losses, predict if we have a favorable game or not?

4.4. Stability of losses and gains

The concentration estimates are doing it, and with very few events - games (and even one may be good enough) we may state with a high probability if this game is in our favor or not. Indeed, if $L_f > 0$ (a favorable game), i.e. the new median L_f is inside the positive region for the function $f(x)$, and it is not too small (say, of the order $1/\sqrt{n}$), then already a probability of the first gain is much above 1/2, and after very few events (games) we may be sure with a very high probability that this series is favorable for us.

I would like to emphasize that the conclusion of this discussion is opposite to what we, mathematicians, would consider to be the right strategy. We usually tend to accept an absolute, ideal independence of outcome of the next game from the previous one. So, say, three losses in a row will not discourage us from the next attempt (and, perhaps, without admitting it even to ourselves, we would believe in the next luck even more). However, the ideas we discussed above suggest to stop to play this series (after three losses) immediately. I would like to end this section by explaining what pushed me to create this mixture of the firm mathematical concept with some philosophy, instead of computing some precise examples (which I actually did for myself). I was always surprised by some well known rules of Casinos (as, say, the closure of a table where significant wins happened; but why?), or by some expressions, like, say, "a period of luck" (or , oppositely, "a period of bad luck"). It sounded as complete non-sense to my mathematical culture and taste. I am not so sure in this now, taking into account the enormously high number of parameters which imply the sensitivity of functions- outcomes to unknown very small perturbations which may lead to dramatic influences on these functions.

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