PREVENTION OF BLOW-UP BY INFINITE DIFFUSION

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The system of equations derived in [1] has the form

$$\partial_t p = D_2 \nabla_r \cdot [D(u) \nabla_r p] - \chi_0 \nabla_r [p \nabla_r c] \tag{1}$$

$$\partial_t v = D_v \nabla_r^2 v + ap - \gamma v \tag{2}$$

where in the case of one space dimension $r = \widetilde{x}$ with $\Gamma(u) = \dfrac{1+u^2}{(1-u)^2}$ and in the case of two space dimensions $r = (\widetilde{x}, \widetilde{y})$ with $\Gamma(u) = \dfrac{1+u}{1-u+ulog(u)}$. Here p denotes the cellular density, whereas u denotes the fraction of volume occupied by cells. After appropriate change of variables we obtain the system:

$$\partial_t u = \nabla \cdot [\Gamma(u) \nabla u] - \chi \nabla [u \nabla v] \tag{3}$$

$$\partial_t v = D \nabla^2 v + \alpha u - \gamma v \tag{4}$$

with Γ defined as above. This system is considered in a region $\Omega \in \mathbb{R}^n$, $n = 1, 2$, with a smooth boundary, subject to the initial condition $(u(0,r), v(0,r)) = (u_0(r), v_0(r))$ and the *no-flux* boundary conditions on $\partial\Omega$. We consider the following system of differential equations:

$$\partial_t u = \nabla \cdot [\Gamma(u) \nabla u] - \nabla [\chi(u,v) \nabla v] + g(u,v) \tag{5}$$

$$\partial_t v = \nabla \cdot [d(v) \nabla v] + f(u,v) \tag{6}$$

in $\Omega \times [0,T)$, where Ω is a bounded domain in $\mathbb{R}^n$, with the initial-boundary conditions

$$(u(0,x), v(0,x)) = (u_0(x), v_0(x)), \quad x \in \Omega, \qquad \frac{\partial u}{\partial \nu}(x) = \frac{\partial v}{\partial \nu}(x) = 0, \quad x \in \partial\Omega. \tag{7}$$

H0 $n \geq 2$. $\partial\Omega$ is of $C^{2+\eta}$ class, $\eta \in (0,1)$

H1 Let $\mathcal{D} := [0,1] \times [0,\infty)$. Let $\chi : \mathcal{D} \to [0,\infty)$ be of C^2 class, and $\chi(0,v) = 0$ for $v \geq 0$

H2 $g : \mathcal{D} \to \mathbb{R}$ is of C^2 class. $g(u,v) \leq M_g(1-u)$, $g(0,v) \geq 0$ for all $(u,v) \in \mathcal{D}$, $M_g \geq 0$; $f : \mathcal{D} \to \mathbb{R}$ is of C^2 class. $f(u,v) < 0$ for all $v \geq V > 0$, $f(u,0) \geq 0$ for all $u \in [0,1]$; $d : [0,\infty) \to [d_1, d_2)$ is of C^2 class, $0 < d_1 < d_2 < \infty$

H3 $u_0, v_0 \in C^{2+\eta}(\overline{\Omega})$, $0 \leq u_0(x) < 1$, $0 \leq v_0(x)$ for $x \in \overline{\Omega}$ and $\dfrac{\partial u_0}{\partial \nu}(x) = 0$, $\dfrac{\partial v_0}{\partial \nu}(x) = 0$ for $x \in \partial\Omega$

H4 $\Gamma : [0,1) \to (0,\infty)$ is of C^2 class. There exist positive numbers $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$ such that $\Gamma(u) > \varepsilon_1$ for all $u \in [0,1]$ and $\Gamma(u) \geq \varepsilon_2(1-u)^{-\alpha}$ for $u \in [1-\varepsilon_3, 1)$ and $\alpha \geq 2$

Theorem 1. *Let the conditions H0 to H4 hold. Then there exists a unique global solution (u,v) to system (5)-(6), (7) such that u and v are in $C^{1+\eta/2, 2+\eta}([0,\infty) \times \overline{\Omega})$. Moreover, there exists a constant $c_v \geq 0$ such that $0 \leq v(t,x) \leq c_v$ and $0 \leq u(t,x) < 1$ for all $x \in \overline{\Omega}$ and all $t > 0$.*

[1] Pavel M. Lushnikov, Nan Chen & Mark Alber, *Macroscopic dynamics of biological cells interacting via chemotaxis and direct contact*, Phys. Rev. E. **78** (2008)