Ordinal Arithmetic with List Structures

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Abstract

We provide a set of "natural" requirements for well-orderings of (binary) list structures. We show that the resultant order-type is the successor of the first critical epsilon number.

The checker has to verify that the process comes to an end. Here again he should be assisted by the programmer giving a further definite assertion to be verified. This may take the form of a quantity which is asserted to decrease continually and vanish when the machine stops. To the pure mathematician it is natural to give an ordinal number. In this problem the ordinal might be $(n-r)\omega^2 + (r-s)\omega + k$*. A less highbrow form of the same thing would be to give the integer* $2^{80}(n-r) + 2^{40}(r-s) + k$*.*

—Alan M. Turing (1949)

1 Introduction

A riddle—consider the Lisp-like function f,

$$\begin{aligned} f(a) &= a \\ f(b) &= b \end{aligned}$$

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$$
f(cons(x,y)) = \begin{cases} a & \text{if } x \equiv y \equiv a, \\ cons(cons(\ldots cons(b, f(y)), y) \ldots, y) & \text{if } x \equiv b \text{ and } y \not\equiv b, \\ cons(cons(\ldots cons(a, y), y) \ldots, y) & \text{if } x \equiv b \text{ and } y \equiv a, \\ cons(x, cons(x, \ldots cons(x, cons(f(x), b) \ldots))) & \text{if } y \equiv b \text{ and } x \not\equiv a, \\ cons(x, cons(x, \ldots cons(x, cons(x, a) \ldots))) & \text{if } y \equiv b \text{ and } x \equiv a, \\ cons(f(x), cons(f(x), \ldots cons(f(x), a) \ldots)) & \text{if } x \not\equiv a, b \text{ and } y \equiv a, \\ cons(x, f(y)) & \text{otherwise.} \end{cases}
$$

that maps binary trees with leaves labeled a or b to themselves. Ellipses represent repetitions of *arbitrary* length, so f is actually a multivalued function. Question: Is there any expression z over a, b, and *cons*, such that $z, f(z), f(f(z)), f(f(f(z))), \ldots$ is an infinite sequence, or must every such sequence $\{f^{(n)}(z)\}_n$ end in all as or bs? This function is depicted in Figure 1, where we use bullets ($\bullet$) for internal nodes ("cons cells") and squares for leaves (atoms).

The surprising answer is that no other infinite sequences are possible.

In general, such questions can be answered by using the notion of well-ordering, stemming from the fundamental work of Cantor [1915]. Floyd, in his landmark paper [1967], envisioned proving termination of programs by showing that some ordinal-valued function decreases strictly with each repetition of a loop, as did Turing before him (see the quotation above). The well-ordering most commonly used is ω, the natural ordering of the natural numbers [Dijkstra, 1976; Gries, 1981], but lexicographic orderings (ω^n) also play an important part [Manna, 1974]. Occasionally, "larger" orderings have been used (for example, [Dershowitz and Manna, 1979; Dershowitz, 1987]); see [Dershowitz, 1987; Dershowitz and Okada, 1988; Cichon, 1990].

The riddle above is a termination question on binary trees, one of the most pervasive data structures used in computer science. Like numbers, binary trees can be well-ordered in many ways. In this paper, we give "natural" principles that such orderings ought to satisfy. We consider infinite binary trees, and show how a "regular" subclass—the trees representable as list structures in Lisp—more than suffice for all ordinals up to and including $\epsilon_{\epsilon_{\epsilon_{\cdots}}}$, the first critical epsilon number. (Different notions of "naturalness" of ordinal notations are surveyed in [Crossley and Kister, 1986/1987].) Conversely, ordinals up to and *including* $\epsilon_{\epsilon_{\epsilon_{\cdots}}}$ can be neatly represented by this subclass of infinite binary trees.

In the next section, we consider natural orderings on binary trees, and some (known) consequences of those principles for finite trees. By imposing a lexicographic rule, we get—not surprisingly—an ϵ_0 ordering. Then, in

Section 3, we present our main results, the extension of the natural ordering to arbitrary list structures, which correspond to the "rational" subset [Courcelle, 1983] of infinite binary trees. We show that $\epsilon_{\epsilon_{\epsilon_{\cdots}}} + 1$ can be proved well-ordered by the Homeomorphic Embedding Theorem on infinite binary trees. Section 4 mentions related work on orderings of (finite) ordered trees, leading to orderings of type , $_0$, the first impredicative ordinal; the last section includes a few remarks on implications for program verification.

Nonempty lists are built from "cons" cells $cons(x, y)$ containing two pointers, x and y; pointers may point either to the empty list nil or to a cons cell. We use $|l|$ for the size of a list structure l, that is, the number of cons cells and nil pointers in l. Thus, for example, $|\,nil\,| = 1$, $|\,cons(nil, nil)| = 3$, and $|z| = 2$, when $z \equiv cons(nil, z)$.

The orderings we deal with are really quasi-orderings; that is, they are not anti-symmetric. For a quasi-ordering $\geq$, we use $\simeq$ for the intersection of $\geq$ and its inverse $\leq$; the strict ordering $>$ is $\geq \cap \not\leq$. We use $\equiv$ for structural equality, and $\not\equiv$ for its complement.

2 Small Ordinals

The ordering principles we propose apply equally well to cyclic and acyclic list structures. We begin, therefore, with the more mundane, acyclic variety—that is, with finite binary trees.

2.1 Axioms of Ordering

Principle 1 (Growth). *A tree is greater than or equivalent to its subtrees; that is,*

$$cons(x, y) \geq x, y,$$

for all trees x, y.

Principle 2 (Monotonicity). *Replacing a subtree by a greater or equivalent one results in a greater or equivalent tree; that is,*

$$x \geq y \Rightarrow \begin{cases} cons(x, z) \geq cons(y, z) \\ cons(z, x) \geq cons(z, y), \end{cases}$$

for all trees x, y, z.

Okada and Steele [1988] relate any ordering on finite trees satisfying such principles to Ackermann's ordinal notation.

By "deleting" in a tree, we mean replacing a subtree by one of its subtrees; "inserting" is the inverse operation.

Lemma 1. *Deleting (inserting) results in a smaller (greater) or equivalent tree.*

Proof. Follows from Growth and Monotonicity. □

So, if t_1 is homeomorphically embedded in t_2, then $t_1 \leq t_2$, where $\leq$ is any ordering satisfying Principles 1 and 2. (A tree t is *homeomorphically embedded* in a tree t' if there's a mapping of nodes of t_1 into nodes of t_2 such that each edge of t_1 corresponds to a disjoint path in t_2.)

Monotonicity implies that if $x' \geq x$ and $y' \geq y$, then $cons(x', y') \geq cons(x, y)$. What, however should the ordering of $cons(x, y)$ and $cons(x', y')$ be when $x' > x$ and $y > y'$? We choose a lexicographic rule in which "left" is more significant than "right". Note, however, that Lemma 1 implies that $cons(x', y') < cons(x, y)$ whenever $y > cons(x', y')$. So, we can't just say that $x' > x$ implies $cons(x', y') \geq cons(x, y)$. Hence, the following lexicographic principle is the strongest that can be formulated without violating our prior principles.

Principle 3 (Lexicography). *If $x' > x$ and $cons(x', y') \geq y$, then $cons(x', y') \geq cons(x, y)$.*

Let $\geq$ be a minimal ordering satisfying Principles 1, 2, and 3. (A "minimal" ordering is one that violates one of the principles if any pair $s \geq t$ is removed from the ordering.)

Theorem 1. *The ordering $\geq$ is total; that is $t_1 \geq t_2$, or $t_2 \geq t_1$, or both. Specifically,*

$$cons(x', y') \geq cons(x, y) \text{ if and only if } \begin{cases} y' \geq y & \text{if } x' \simeq x, \quad (a) \\ cons(x', y') \geq y & \text{if } x' > x, \quad (b) \\ y' \geq cons(x, y) & \text{if } x' < x \quad (c). \end{cases}$$

Proof. By induction on size of the trees, this definition—combined with the fact that the empty tree, *nil*, is comparable with all trees (it is the smallest by virtue of the Growth Principle)—gives a total ordering. (Transitivity of this definition can be shown by induction and case analysis.) This ordering clearly satisfies the principles. Furthermore, any ordering satisfying the principles must satisfy the "if" direction, the first case of which follows from Monotonicity; the second, from Lexicography; and the third, from the Growth Principle and transitivity. □

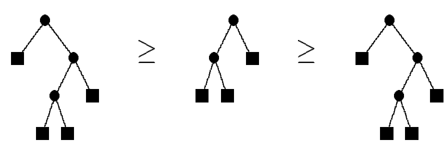
Lemma 2. *For any trees x and y, $cons(x, y) > nil$.*

Proof. Making $cons(x, y) \geq nil \not\geq cons(x, y)$ still gives an ordering satisfying the principles. □

Theorem 2. *Tree comparison of finite trees t_1 and t_2 can be done in time $O(|t_1| \times |t_2|)$.*

Proof. Follows from Theorem 1, Lemma 2, and induction on $|t_1|$ and $|t_2|$. □

The ordering $\geq$ is actually a quasi-ordering, for



because, in general,

Lemma 3. *If $x < y$, then $cons(x, cons(y, z)) \simeq cons(y, z)$.*

Proof. The inequality $cons(x, cons(y, z)) \geq cons(y, z)$ follows from the Growth Principle; the other direction follows from Lexicography, using Lemma 2. □

2.2 Order-Preserving Mapping

One can map finite binary trees, under the given ordering, to ordinals below ϵ_0 in the following straightforward way:

Proposition 1. *There is an order-preserving mapping from trees under $\geq$ to the ordinals up to ϵ_0:*

$$\begin{aligned} [\![nil]\!] &= 0 \\ [\![cons(x, y)]\!] &= \omega^{[\![x]\!]} + [\![y]\!] \end{aligned}$$

In other words, lists $(l_1, \ldots, l_n)$ are interpreted as the noncommutative sum $\omega^{[\![l_1]\!]} + \cdots + \omega^{[\![l_n]\!]}$.

This mapping is not one-to-one; as we just saw, there are equivalent, non-isomorphic trees. It is order-preserving. This means that for two finite binary trees t and t', $t \geq t'$ if and only if $[\![t]\!] \geq [\![t']\!]$. Furthermore, there is a one-to-one correspondence between binary trees and expressions involving (non-commutative) addition and exponentiation. Since such expressions give all ordinals below ϵ_0, our ordering is of order-type ϵ_0, too. Thus, expressions in Cantor Normal Form are in one-to-one correspondence with the equivalence classes on binary trees imposed by $\simeq$.

2.3 Embedding Theorem

As a special case of Higman's Lemma [Higman, 1952], we know that, in any infinite sequence $\{t_i\}_{i<\omega}$ of finite binary trees, there must be two trees t_j and t_k $(j < k)$ such that t_j is homeomorphically embedded in t_k. In other words, t_k can be obtained from t_j by deletion only. By Lemma 1, it follows that $t_j \leq t_k$; hence, an infinite descending sequence of trees is impossible. In other words, our ordering is well-founded. We have already seen that $\leq$ is order-isomorphic to ϵ_0. Since ϵ_0 induction is equivalent to the consistency of Peano Arithmetic, this means that the Embedding Lemma of Higman cannot be proved in Peano Arithmetic [Friedman, 19??].

2.4 Arithmetic

The mapping from ordinals to binary trees gives a convenient data structure for representing ordinals below ϵ_0. Arithmetic operations (commutative addition $\oplus$, commutative multiplication $\otimes$, and exponentiation), and a predecessor operation to get fundamental sequences, are now easy to define;

the following correspondences are suggestive:

$$\begin{array}{rcll}
0 & \mapsto & nil & \\
1 & \mapsto & cons(nil, nil) & \\
x \oplus nil & \mapsto & x & \\
cons(x, y) \oplus cons(x', y') & \mapsto & cons(x, y \oplus cons(x', y')) & \text{if } x \leq x' \\
x \otimes nil & \mapsto & nil & \\
cons(x, nil) \otimes cons(x', y') & \mapsto & cons(x \oplus x', cons(x, nil) \otimes y') & \\
cons(x, y) \otimes z & \mapsto & (cons(x, nil) \otimes z) \oplus (y \otimes z) & \\
\omega^x & \mapsto & cons(x, nil) & \\
pred_n(cons(nil, nil)) & \mapsto & nil & \\
pred_n(cons(x, nil)) & \mapsto & cons(pred_n(x), nil) \otimes n & \text{if } x \text{ is a successor ordinal} \\
pred_n(cons(x, nil)) & \mapsto & cons(pred_n(x), nil) & \text{if } x \text{ is a limit ordinal} \\
pred_n(cons(x, y)) & \mapsto & cons(x, pred_n(y)) & \text{if } y \not\equiv nil
\end{array}$$

For example, this binary-tree data structure could be used in implementing the computation of the various extensions of Ackermann's function (see, for example, [Ketonen and Solovay, 1981]). An ordinal-indexed function $A_\alpha(n)$ can be defined for ordinals α and natural numbers n by

$$A_\alpha(n) = \begin{cases} 2n & \text{if } \alpha = 0, n \geq 1, \\ A_\beta^{(n)}(1) & \text{if } \alpha \text{ is a successor ordinal } \beta + 1, \\ A_{pred_n(\alpha)}(n) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

The computation of this function plays an important role in the unbounded search procedures of Reingold and Shen [1991]. Moreover, these search procedures themselves use ordinals to index the recursive calls.

These operations also make it easy to encode problems like the "Battle of Hydra and Hercules" of Kirby and Paris [1982] as hard-to-prove-well-defined functions on binary trees.

3 Medium Sized Ordinals

List structures, in general, correspond to "rational" binary trees, which are like ordinary binary trees, but paths may be of length ω, as long as there are only a finite number of distinct subtrees.

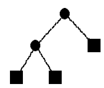
3.1 Axioms of Ordering

All the principles of Section 2.1 apply to this case as well, but an infinite number of deletions could increase a tree without violating Principles 1–3.

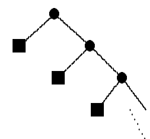
So, we take the following extension of Principle 2 as axiomatic:

Principle 4 (Continuity). *Replacing infinitely many subtrees by greater or equivalent ones results in a greater or equivalent tree.*

Principles 1–4 do not, however, give a total ordering. We do not, for example, know how to order



and



An additional principle is called for:

Principle 5 (Dominance). *If* $x > y_i$*, for all* $i = 1, 2, \ldots$*, then* $cons(x, nil) \geq cons(y_1,\ cons(y_2, \ldots))$.

For finite trees, this is a direct consequence of Theorem 1.

3.2 Order-Preserving Mapping

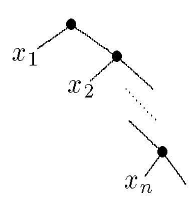
It turns out that we can restrict ourselves to the class of list structures in which there are no cycles except self-loops. Call such a list *normalized.*

Theorem 3. *For every rational binary tree* t *there is a normalized list* ℓ *such that* $t \leq \ell \leq t$.

When comparing structures, like ℓ, under $\leq$, we mean to compared its (possibly) infinite tree expansion.

Proof. All cycles in the graph representation of a rational tree can be reduced to self loops as follows: If a full binary tree is homeomorphically embedded in t, then t is equivalent to the structure z such that $z \equiv cons(z, z)$, which is just a double self-loop:

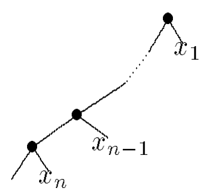
Consider a cyclic graph $z \equiv cons(x_1, cons(x_2, \ldots, cons(x_n, z)))$:



If any of the x_k contains all of z as a subterm, then z both contains the full binary tree (obtained by deleting all other x_i and pruning x_k to what is left of z) and is contained by it (as are all binary trees). Hence, z is equivalent to the full binary tree.

If none of the x_i have z as a subterm, then, by induction on $|l|$, we can suppose that there is a normalized list among the x_i that has a maximal ordinal assignment. We have z less than or equal to the structure $z' \equiv cons(\max\{x_i\}, z')$ by Monotonicity, and z greater than or equal to z' by Continuity. Hence, we can replace the loop in z with the self-loop of z'.

Similarly, $z \equiv cons(\ldots(cons(cons(z, x_n), x_{n-1}), \ldots), x_1)$, that is,



can be replaced by the double-self-loop corresponding to the full tree or by a self-loop $z' \equiv cons(\max\{x_i\}, z')$. $\square$

An attempt to prove a result like Theorem 3 appears in [Brown, 1979].

Proposition 2. *There is an order-preserving mapping from normalized lists, under the above ordering, onto the ordinals up to and including $\epsilon_{\epsilon_{\epsilon_{\cdots}}}$.*

Proof. The mapping from lists to ordinals is:

$$
\begin{array}{lcl}
[\![nil]\!] & = & 0, \\
[\![t \text{ such that } t \equiv cons(t, x)]\!] & = & \epsilon_{[\![x]\!]}, \\
[\![t \text{ such that } t \equiv cons(x, t)]\!] & = & \omega^{[\![x]\!]+1} \\
[\![t \text{ such that } t \equiv cons(t, t)]\!] & = & \epsilon_{\epsilon_{\epsilon_{\cdots}}}, \\
[\![cons(x, y)]\!] & = & \omega^{[\![x]\!]} + \left\{ \begin{array}{ll} 1 & [\![x]\!] \text{ not a limit ordinal} \\ 1 & [\![x]\!] \text{ not an epsilon number and } y \not\equiv nil \\ 0 & \text{otherwise} \end{array} \right\} + \left\{ \begin{array}{ll} \beta & \text{if } [\![\\ [\![y]\!] & \text{oth} \end{array} \right.
\end{array}
$$

(Addition, here, is not commutative.) Its inverse is:

1. $\langle 0 \rangle = nil$
2. $\langle \alpha + \beta \rangle = \begin{cases} succ(\langle \alpha \rangle) & \text{if } \beta = 1, \\ append(succ(\langle \alpha \rangle), \langle \beta \rangle) & \text{otherwise.} \end{cases}$

3. $succ(t) = \begin{cases} cons(nil, nil) & \text{if } t \equiv nil, \\ cons(succ(car(t), nil)) & \text{if } cdr(t) \equiv t, \\ cons(car(t), succ(cdr(t))) & \text{otherwise.} \end{cases}$

4. $\langle \omega^\alpha \rangle = \begin{cases} z \text{ such that } z \equiv cons(\langle \beta \rangle, z) & \text{if } \alpha = \beta + 1, \\ cons(\langle \alpha \rangle, nil) & \text{otherwise.} \end{cases}$

5. $\langle \epsilon_\alpha \rangle = z$ such that $z \equiv cons(z, \langle \alpha \rangle)$.

□

Arithmetic and predecessors can be defined via these mappings, or independently, as operations on lists, in a manner parallel to that of the previous section.

Theorem 4. *For normalized lists ℓ and ℓ', $\ell' \geq \ell$ if and only if $[\![\ell']\!] \geq [\![\ell]\!]$.*

Proof. There are three cases derived from the above mapping:

1. $\epsilon_\alpha \geq \epsilon_\beta$ if and only if $\alpha \geq \beta$.
2. $\omega^\alpha + \beta \geq \epsilon_\gamma$ if and only if $\alpha \geq \epsilon_\gamma$ $\beta \geq \epsilon_\gamma$.
3. $\omega^\alpha + \beta \geq \omega^{\alpha'} + \beta'$ if and only if $\alpha > \alpha'$ or ($\alpha = \alpha'$ and $\beta \geq \beta'$).

□

Corollary. *For rational trees t and t', $t' \geq t$ if and only if $[\![t']\!] \geq [\![t]\!]$, where $[\![t]\!]$ is the ordinal assigned to the normalized list equivalent to t.*

Theorem 5. *Normalized lists ℓ_1 and ℓ_2 can be compared in time $O(|\ell_1| \times |\ell_2|)$.*

Proof. Use the mapping in the above proposition and induction over $|\ell_1|$ and $|\ell_2|$. □

Theorem 6. *An arbitrary list ℓ can be normalized in time $O(|\ell|^2)$.*

Returning to the riddle, we interpret a as 0, and b as a self-loop. Then, we have $\llbracket cons(b,b) \rrbracket = \epsilon_{\epsilon_{\epsilon_{\dots}}}$ and, in all cases (except a and b), f gives a smaller ordinal:

$$\llbracket f(z) \rrbracket = \begin{cases} 0 & \text{if } \llbracket z \rrbracket = 1, \\ \omega^{\omega^{\omega^{\cdots^{\omega}}}} & \text{if } \llbracket z \rrbracket = \epsilon_0, \\ n & \text{if } \llbracket z \rrbracket = \omega, \\ \epsilon_{\epsilon_{\dots \epsilon_0}} & \text{if } \llbracket z \rrbracket = \epsilon_{\epsilon_{\epsilon_{\dots}}}, \\ \omega^{\omega^{\cdots^{\epsilon_{\llbracket f(y) \rrbracket} + \llbracket y \rrbracket} + \llbracket y \rrbracket} + \llbracket y \rrbracket} + \llbracket y \rrbracket & \text{if } \llbracket z \rrbracket = \epsilon_{\llbracket y \rrbracket}, \\ \omega^{\llbracket x \rrbracket} n + \omega^{\llbracket f(x) \rrbracket + 1} & \text{if } \llbracket z \rrbracket = \omega^{\llbracket x \rrbracket + 1},\ \llbracket x \rrbracket > 0, \\ \omega^{pred_n(\llbracket x \rrbracket)} \times n & \text{if } \llbracket z \rrbracket = \omega^{\llbracket x \rrbracket} \text{ for limit ordinal } \llbracket x \rrbracket, \\ \omega^{\llbracket x \rrbracket} + \llbracket f(y) \rrbracket & \text{if } \llbracket z \rrbracket = \omega^{\llbracket x \rrbracket} + \llbracket y \rrbracket \text{ and } \llbracket y \rrbracket > 0. \end{cases}$$

3.3 Embedding Theorem

Nash-Williams' version of the Embedding Theorem [Nash-Williams, 1965] also holds for infinite ordered trees: In any infinite sequence $\{t_i\}_{i<\omega}$ of (finite or infinite) binary trees, there must be two trees t_j and t_k ($j < k$) such that t_j is homeomorphically embedded in t_k. Since our ordering contains the embedding relation, we have:

Theorem 7. *The Embedding Theorem for infinite (rational) binary trees suffices to prove the well-ordering of* $\epsilon_{\epsilon_{\epsilon_{\dots}}} + 1$.

A similar analysis of infinite, not necessarily rational, binary trees may also be possible.

4 Bigger Ordinals

The epsilon number $\epsilon_{\epsilon_{\epsilon_{\dots}}}$ is $\phi_2(0)$ in the Veblen-Feferman-Schütte hierarchy [Veblen, 1908; Feferman, 1968; Schmidt, 1976]. Less natural orderings on (nonbinary) ordered trees correspond to much larger ordinals in that hierarchy. In particular, some orderings based on Kruskal's Tree Theorem [Kruskal, 1960] correspond to the first impredicative ordinal, $,_0$, and even to larger ones [Friedman, 19??; Simpson, 1985; Smoryński, 1986; Dershowitz, 1987; Gallier, 1991]. The significance of $,_0$ for computer science is discussed in [Gallier, 1991].

5 Conclusions

It has been argued [Gries, 1979] that the natural numbers suffice for termination proofs, since the (maximum) number of iterations of any terminating deterministic (or bounded nondeterministic) program loop is fixed, depending only on the values of the variables and inputs when the loop is begun. This begs the issue, however, since the *proof* that such a function exists may require transfinite induction with much larger ordinals than ω. As we have seen, the termination of the problem given in the introduction *requires* induction up to $\phi_2(0)$. As phrased, the "function" f makes nondeterministic choices, but (like the Battle of Hercules and Hydra) can be made deterministic by adding to the recursion an integer argument k, which increases by a fixed amount with each recursive call, and which determines the number of repetitions. Though one can define an integer-valued function $\tau(x)$ that counts how many steps it takes to reduce x to a, proving that τ acts as a termination ("variant" [Dijkstra, 1976]) function, decreasing with each recursive call, requires a much stronger principle of induction than provided by the Peano Axioms.

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