The *lexicographic path ordering* $\succsim_{\text{lpo}}$ induced by a quasi-ordering $\succsim$, is defined as follows:

$$\frac{s_i \succsim_{\text{lpo}} t}{f(s_1,\ldots,s_m) \succ_{\text{lpo}} t}$$

$$\frac{f \succ g \qquad f(s_1,\ldots,s_m) \succ_{\text{lpo}} t_1,\ldots,t_n}{f(s_1,\ldots,s_m) \succ_{\text{lpo}} g(t_1,\ldots,t_n)}$$

$$\frac{f \approx g \qquad \langle s_1,\ldots,s_m\rangle \approx_{\text{lex}} \langle t_1,\ldots,t_n\rangle}{f(s_1,\ldots,s_m) \approx_{\text{lpo}} g(t_1,\ldots,t_n)}$$

$$\frac{f \approx g \qquad \langle s_1,\ldots,s_m\rangle \succ_{\text{lex}} \langle t_1,\ldots,t_n\rangle \qquad f(s_1,\ldots,s_m) \succ_{\text{lpo}} t_2,\ldots,t_n}{f(s_1,\ldots,s_m) \succ_{\text{lpo}} g(t_1,\ldots,t_n)}$$

where $\succ_{\text{lex}}$ and $\approx_{\text{lex}}$ are the lexicographic extensions of $\succ_{\text{lpo}}$ and $\approx_{\text{lpo}}$, respectively.

Suppose $\times > +$. Then we can show that $x \times (y+z) \succ_{\text{lpo}} x \times y + x \times z$, for any $x, y, z$, as follows:

$$\dfrac{\dfrac{\dfrac{x \approx_{\text{lpo}} x \quad \dfrac{y \approx_{\text{lpo}} y}{y+z >_{\text{lpo}} y}}{\langle x, y+z\rangle >_{\text{lex}} \langle x, y\rangle} \quad \dfrac{\dfrac{y \approx_{\text{lpo}} y}{y+z \succsim_{\text{lpo}} y}}{x \times (y+z) >_{\text{lpo}} y}}{x \times (y+z) >_{\text{lpo}} x \times y} \qquad \times > + \qquad \dfrac{\dfrac{\dfrac{x \approx_{\text{lpo}} x \quad \dfrac{z \approx_{\text{lpo}} z}{y+z >_{\text{lpo}} z}}{\langle x, y+z\rangle >_{\text{lex}} \langle x, z\rangle} \quad \dfrac{\dfrac{z \approx_{\text{lpo}} z}{y+z \succsim_{\text{lpo}} z}}{x \times (y+z) >_{\text{lpo}} z}}{x \times (y+z) >_{\text{lpo}} x \times z}}{x \times (y+z) >_{\text{lpo}} x \times y + x \times z}$$

The multiset path ordering $\succsim_{\text{mpo}}$, induced by a quasi-ordering $\succsim$, is defined as follows:

$$\frac{s_i \succsim_{\text{mpo}} t}{f(s_1,\ldots,s_m) \succ_{\text{mpo}} t}$$

$$\frac{f \succ g \qquad f(s_1,\ldots,s_m) \succ_{\text{mpo}} t_1,\ldots,t_n}{f(s_1,\ldots,s_m) \succ_{\text{mpo}} g(t_1,\ldots,t_n)}$$

$$\frac{f \approx g \qquad [s_1,\ldots,s_m] \approx_{\text{bag}} [t_1,\ldots,t_n]}{f(s_1,\ldots,s_m) \approx_{\text{mpo}} g(t_1,\ldots,t_n)}$$

$$\frac{f \approx g \qquad [s_1,\ldots,s_m] \succ_{\text{bag}} [t_1,\ldots,t_n]}{f(s_1,\ldots,s_m) \succ_{\text{mpo}} g(t_1,\ldots,t_n)}$$

where $\succsim_{\text{bag}}$ is the bag extension of $\succsim_{\text{mpo}}$.

To show that permuted distributivity is decreasing for this ordering, we have:

$$\dfrac{\dfrac{\dfrac{x \approx_{\text{mpo}} x \quad \dfrac{y \approx_{\text{mpo}} y}{y+z >_{\text{mpo}} y}}{[x, y+z] >_{\text{bag}} [x, y]}}{x \times (y+z) >_{\text{mpo}} x \times y} \quad \dfrac{\dfrac{x \approx_{\text{mpo}} x \quad \dfrac{z \approx_{\text{mpo}} z}{y+z >_{\text{mpo}} z}}{[x, y+z] >_{\text{bag}} [z, x]}}{x \times (y+z) >_{\text{mpo}} x \times z} \qquad \times > +}{x \times (y+z) >_{\text{mpo}} x \times y + z \times x}$$