Stability and Rank Properties of Matrix Subdivision Schemes

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Abstract. Subdivision schemes with matrix masks are a natural extension of the well studied case of subdivision schemes with scalar masks. Such schemes arise in the analysis of multivariate scalar schemes, in subdivision processes corresponding to shift-invariant spaces generated by more than one function, in geometric modeling where each component of the curve/surface is designed by a different linear combination of the control points, and in the case of schemes which interpolate function and derivatives values simultaneously. The limit of a matrix subdivision scheme can be expressed as a combination of shifts of a refinable matrix function Φ. It is shown that if Φ is stable in the sense of a new stability notion for matrix valued functions, then the scheme is uniformly convergent. Also it is shown that the stability of a maximal submatrix of Φ is related with the linear dependence of its rows, and hence of any vector valued function generated by the subdivision scheme. Finally it is shown that by proper renormalization of the process relative to the vanishing rows of Φ, it is possible to generate vector limit functions with components, which are the first derivative of certain linear combinations of the other components. The same approach allows to analyze the smoothness of Φ.

§1 Introduction

Matrix subdivision schemes play an important role in the analysis of multivariate subdivision schemes [7], in the construction of multiple-knot splines and in Hermite type subdivision [11]. They also have strong connections with multiresolution approximation of multiplicity more than 1 ([15],[16],[19],[20]) and multi-wavelets ([17],[18],[21]).

The purpose of this work is to understand the characteristics of the different types of matrix subdivision schemes.

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A uniform stationary matrix subdivision scheme is defined by a set of real $n\times n$ matrix coefficients $\{A_j : j\in \mathbb{Z}\}$, with a finite number of non-zero A_j's, generating control points in $\mathbb{R}^n$ $f^k=\{f_j^k \ : j\in\mathbb{Z}\}$, $k\geq 0$, recursively by

$$f_i^{k+1}=\sum_{j\in\mathbb{Z}} A_{i-2j} f_j^k \ , \ i\in\mathbb{Z} \ . \tag{1.1}$$

In (1.1) we find two rules, or masks, for even and for odd i, involving the non-zero matrix coefficients A_j with even and with odd j's respectively.

We say, that $\mathcal{S}$ is a convergent subdivision scheme, or that $\mathcal{S}$ is C^0, if for every set of control points $f^0=\{f_j^0\in\mathbb{R}^n : \ j\in\mathbb{Z}\}$ there is a continuous function $f : \mathbb{R}\rightarrow\mathbb{R}^n$ such that

$$\lim_{k\to\infty}\sup_{j\in\mathbb{Z}}\|(\mathcal{S}^k f^0)_j - f(2^{-k}j)\| \ = 0 \ , \tag{1.2}$$

and $f\not\equiv 0$ for at least one initial data f^0. We denote the above function f by $\mathcal{S}^\infty f^0$, and call it a limit function of $\mathcal{S}$ or a function generated by $\mathcal{S}$. In matrix subdivision schemes theory it is also convenient to consider $n\times n$ matrix valued functions, generated by applying the scheme to sets of matrix control points $F^0=\{F_j^0\in\mathbb{R}^n\times\mathbb{R}^n : \ j\in\mathbb{Z}\}$. The basic limit vector functions of a convergent matrix subdivision scheme are the rows of $\Phi=\mathcal{S}^\infty\Delta_0$ where $\Delta_i=\{\delta_{j,i}I\}_{j\in\mathbb{Z}}$, $I=I_{n\times n}$.

As in scalar subdivision schemes, the matrix function Φ satisfies a two-scale refinement equation

$$\Phi(x)=\sum_{j\in\mathbb{Z}}\Phi(2x-j)A_j \ . \tag{1.3}$$

Hence, each row of Φ, i.e., each basic limit vector function satisfies a refinement equation with the A_j's as coefficients. By the finite support of the 'mask' $\{A_j : j\in\mathbb{Z}\}$ it follows that elements of Φ are functions of compact support. The interest in the basic limit vector functions is due to the fact that the vector limit functions of the subdivision process are in the span of the integer shifts of the columns of Φ. That is, if we start the subdivision process (1.1) with an initial set $f^0=\{f_i^0\in\mathbb{R}^n : \ i\in\mathbb{Z}\}$ we get

$$f=\mathcal{S}^\infty f^0=\mathcal{S}^\infty\sum_{i\in\mathbb{Z}}\Delta_i f_i^0=\sum_{i\in\mathbb{Z}}\Phi(\cdot-i)f_i^0 \ . \tag{1.4}$$

Thus the k-th component of the limit vector function f is in the span of the integer shifts the k-th basic limit vector function,

$$f_k(x)=\sum_{i\in\mathbb{Z}}\sum_{\ell=1}^{n}\Phi_{k,\ell}(x-i)(f_i^0)_\ell \ . \tag{1.5}$$

Therefore, it is important to know if

$$U_k=\{\Phi_{k,\ell}(\cdot-i) : 1\leq\ell\leq n \ , \ i\in\mathbb{Z}\} \ , \tag{1.6}$$

is a stable basis, say in $L^2(I\!R)$ or $L^\infty(I\!R)$. Another interesting issue is the inter-dependence of the coordinates in the limit vector function f, which plays an important role when the coordinates of the vector subdivision are intertwined as in certain signal processing applications [21]: assuming that the limit function is not identically zero, at least for some initial data f^0, we would like to know when are the functions f_k linearly dependent.

In §2, we show that the number of inter-dependence relations between the coordinates of f is related to a generalized stability notion for the matrix function Φ. We also show that this stability property implies convergence of the subdivision scheme.

We restrict to the study of convergence and stability in the L^∞ sense. More general L^p results of the same type, as well as specific criterions on the mask that ensure stability and regularity of Φ, will appear in [22].

Note that the relations between stability and rank property have been studied in the L^p framework in [15], but in the case of stable refinable vectors that corresponds, as we shall see, to having inter-dependence between every pair of coordinates of f. Also in the case of refinable vectors, conditions on the mask for stability have been obtained in [23].

Inter-dependence can also be expressed by the fact that a coordinate in the limit vector function is uniformly zero. We analyze such a situation in §3 and show that a proper renormalization of the process relative to the vanishing rows of Φ generates vector limit functions with components related to the first derivative of certain linear combinations of the other components.

§2 Relations between inter-dependence and stability

We start with the assumption that the scheme (1.1) is C^0, i.e., convergent. The relation (1.1) represents two rules, one generating the even entries $\{f_{2i}^{k+1}\}_{i\in\mathbb{Z}}$, using the even coefficients $\{A_{2i}\}_{i\in\mathbb{Z}}$, and the other generating the odd entries $\{f_{2i+1}^{k+1}\}_{i\in\mathbb{Z}}$, using the odd coefficients $\{A_{2i+1}\}_{i\in\mathbb{Z}}$. Important to this investigation are the matrices

$$B_0 = \sum_{i\in\mathbb{Z}} A_{2i}\ , \quad B_1 = \sum_{i\in\mathbb{Z}} A_{2i+1}\ , \quad M(0) = \tfrac{1}{2}(B_0 + B_1)\ , \tag{2.1}$$

where $M(\omega) = \frac{1}{2}\sum_{j\in\mathbb{Z}} A_j e^{-ij\omega}$.

The following two Theorems are the matrix analogue of the necessary condition for scalar C^0 schemes, namely, that the sum of the even coefficients and the sum of the odd coefficients are both 1. The proof of the first Theorem is as the proof of the scalar case (Proposition 2.2 in [7]).

Theorem 2.1 *If $\mathcal{S}$ is C^0 then B_0 and B_1 have a common eigenvector with an eigenvalue 1.*

Let W denote the space of all common eigenvectors of B_0 and B_1 corresponding to the eigenvalue 1, and let m be its dimension. Let us also introduce the matrix $V = \left(v^{(1)}...v^{(m)}\right)$, where $\{v^{(1)}, ..., v^{(m)}\}$ is a basis of W.

Theorem 2.2 *If $\mathcal{S}$ is C^0 then W is also the space of eigenvectors corresponding to the eigenvalue 1 of the matrix $M(0)$. Moreover, $M(0)$ has $n-m$ eigenvalues with modulus less than 1 .*

Proof: Clearly, W is included in the space $\tilde{W}$ of eigenvectors with eigenvalue 1 of $M(0)$. Now we observe that

$$J_k = 2^{-k} \sum_{i \in \mathbb{Z}} (\mathcal{S}^k \Delta_0)_i \ \longrightarrow J = \int_{\mathbb{R}} \Phi(x) dx \ ,$$

where $\Phi = \mathcal{S}^\infty \Delta_0$, and that $2^k J_k$ is the sum of the matrices of the k-iterated scheme. Thus $J_k = M(0)^k$ and $M(0)^k \to J$. Since $M(0)^\infty$ exists, its columns are in $\tilde{W}$ while those of J are in W. Therefore $\tilde{W} = W$ and $M(0)$ has $n-m$ eigenvalues with modulus less than 1 . ∎

In the following we assume that W is the eigenspace of $M(0)$ corresponding to the eigenvalue 1 .

Theorem 2.3 *Let $m < n$, then any $m+1$ elements of $f = \mathcal{S}^\infty f^0$ are linearly dependent.*

Proof: By the continuity of the limit it follows from (1.1) that $B_0 f(x) = B_1 f(x) = f(x)$ for any $x \in \mathbb{R}$. Thus $f(x) \in W$. The matrix V is of rank m, hence any $m+1$ rows of V are linearly dependent. Since $f(x) = Vu(x)$ with $u(x) \in \mathbb{R}^m$, any $m+1$ elements of f are linearly dependent. ∎

Definition 2.1 We say that $h_1, ..., h_r$ constitute a stable basis if their integer shifts constitute a Riesz basis of $span\{h_j(\cdot - i) : 1 \le j \le r \ , \ i \in \mathbb{Z}\}$.

Theorem 2.4

A. *The 1-periodic matrix function $\tilde{\Phi} = \sum_{i \in \mathbb{Z}} \Phi(\cdot - i)$ satisfies $\tilde{\Phi} V = V$ where $\Phi = \mathcal{S}^\infty \Delta_0$.*

B. *Let $dim W = m$, $2 \le m \le n$, then within each basic limit vector function of $\mathcal{S}$ there exist $n-m+2$ elements which do not constitute a stable basis.*

Proof: The property $\tilde{\Phi} V = V$ follows directly from (1.4) when taking the constant initial data $f^0 = v^{(j)}$, $0 \le j \le m$, since by (2.1) $\mathcal{S}^\infty v^{(j)} = v^{(j)}$. For part B let us assume, w.l.o.g., that the upper $m \times m$ part of the $n \times m$ matrix V is $I = I_{m \times m}$, i.e., $V = \left(I \mid U\right)^t$. Viewing the equality $\tilde{\Phi} V = V$ we have

$$\sum_{i \in \mathbb{Z}} \sum_{\ell=1}^{n} \Phi_{k,\ell}(x-i) V_{\ell,j} = V_{k,j} \ , \ 1 \le k \le n \ , \ 1 \le j \le m \ , \tag{2.2}$$

where for each k only $n-m+1$ elements of the kth row of Φ take part in the sum. Now, if $V_{k,j}=0$ then obviously those $n-m+1$ elements cannot constitute a stable basis. If $V_{k,j}\neq 0$ for $1\leq j\leq m$ we take a linear combination of the equalities (2.2) for $j=1,2$ to get a zero in the r.h.s., using $n-m+2$ elements of the kth row of Φ. ∎

Corollary 2.5 *If one basic limit vector function is stable then $m=1$ and any limit function of the subdivision is of the form $f(x)=h(x)v$ with $v\in \mathbb{R}^n$.*

Since by Corollary 2.5 only in the special case $m=1$ a basic limit vector function can constitute a stable basis, we suggest to consider the stability of Φ as a matrix function.

Definition 2.2 We say that an $m\times n$ matrix function $G(x)$ is stable if its integer shifts constitute a stable basis (in L_∞) of $span\{\Phi(\cdot - i):\ i\in \mathbb{Z}\}$. i.e., for any $\{q_k\}\in \ell_{\infty,\mathbb{R}^n}$

$$c_1 \sup_{k\in\mathbb{Z}}\{\|q_k\|\}\leq \sup_{x\in\mathbb{R}}\{\|\sum_{k\in\mathbb{Z}} G(x-k)q_k\|\}\leq c_2 \sup_{k\in\mathbb{Z}}\{\|q_k\|\}\ . \tag{2.3}$$

It should be noted that a matrix function is stable if a matrix consisting of a subset of its rows is stable. With this definition at hand, the proof of Theorem 2.4 can be further exploited to give:

Theorem 2.6 *Any matrix function consisting of $k<m$ rows of $\Phi=\mathcal{S}^\infty\Delta_0$ is not stable.*

As a direct consequence of Theorems 2.3 and 2.6 we get,

Corollary 2.7 *Let $\Phi=\mathcal{S}^\infty\Delta_0$ be stable, and let m be the minimal number of rows of Φ which constitute a stable matrix. Then any $m+1$ components in any vector valued function generated by $\mathcal{S}$ are linearly dependent.*

In the rest of this section we consider refinement equations of the type (1.3), with masks satisfying the requirements of Theorems 2.1 and 2.2, and prove that if a compactly supported continuous solution of such a refinement equation is stable, then the corresponding subdivision scheme is uniformly convergent. This result extends a known result of a similar nature for scalar schemes [2].

Proposition 2.8 *Let Ψ be any compactly supported $C(\mathbb{R})$ solution of the refinement equation (1.3), then $\Psi(x)=\widehat{\Psi}(0)VN(x)$ where N is an $m\times n$ matrix function.*

Proof: Applying the Fourier transform to (1.3) we obtain

$$\widehat{\Psi}(\omega)=\widehat{\Psi}(\omega/2)M(\omega/2), \tag{2.4}$$

and by iteration,

$$\widehat{\Psi}(\omega) = \widehat{\Psi}(0)[\prod_{k=1}^{\infty} M(2^{-k}\omega)]. \tag{2.5}$$

The convergence of

$$P(\omega) = \prod_{k=1}^{\infty} M(2^{-k}\omega) = \lim_{n\to\infty}[M(2^{-n}\omega)\cdots M(\omega/2)],$$

is ensured by the spectral properties of $M(0)$. Note that $P(0) = M(0)^{\infty}$ is a projection operator onto the subspace W. Moreover, by definition,

$$P(\omega) = \lim_{n\to\infty}[P(2^{-n}\omega)M(2^{-n}\omega)\cdots M(\omega/2)] = P(0)P(\omega),$$

and hence by (2.5)

$$\widehat{\Psi}(\omega) = \widehat{\Psi}(0)P(0)P(\omega). \tag{2.6}$$

We conclude by noting that $P(0)P(\omega)$ can be written in a unique way as $VQ(\omega)$, where $Q(\omega)$ is an $m \times n$ matrix. ∎

Proposition 2.9 *Let Ψ be a compactly supported stable $C(I\!R)$ solution of the refinement equation (1.3). Then there exists a non-singular $n \times n$ matrix C, such that $\Phi = C\Psi$ is a stable solution of (1.3) satisfying $\widetilde{\Phi}(x)V = V$ and $\widehat{\Phi}(0)V = V$. We call Φ a normalized solution.*

Proof: ¿From the refinement equation (1.3)

$$\widetilde{\Psi}(x) = \sum_{i\in\mathbb{Z}} \Psi(x-i) = \sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \Psi(2x-2i-j)A_j$$

$$= \sum_{\ell\in\mathbb{Z}} \Psi(2x-\ell)\sum_{i\in\mathbb{Z}} A_{\ell-2i} \ .$$

Therefore, $\widetilde{\Psi}(x)V = \widetilde{\Psi}(2x)V$, implying that $\widetilde{\Psi}(x)V = \widetilde{\Psi}(0)V$, by the continuity of $\widetilde{\Psi}$. Now, if Ψ is stable, then for any vector u

$$\sum_{i\in\mathbb{Z}} \Psi(x-i)Vu = \widetilde{\Psi}(0)Vu \neq 0 \ .$$

Hence, $rank\big(\widetilde{\Psi}(0)V\big) = m$ and there exists a non-singular martix C such that $C\widetilde{\Psi}(0)V = V$, and hence $C\widetilde{\Psi}(x)V = V$. The stability of $C\Psi$ follows from that of Ψ and the non-singularity of C. Moreover, since $\widetilde{\Psi}(x)V$ is a periodic matrix function which is constant, then $\widetilde{\Psi}(x)V = \widehat{\Psi}(0)V$, which completes the proof of the proposition. ∎

Theorem 2.10 *Let Φ be a continuous compactly supported stable solution of the refinement equation (1.3). If the necessary conditions in Theorems 2.1-2.2 are satisfied then the corresponding subdivision scheme (1.1) is convergent.*

Proof: We can assume that Φ is a normalized stable solution of the refinement equation. It is enough to check the convergence with the initial condition $F^0 = \Delta_0$. Using (1.3) we find out that

$$\Phi(x) = \sum_{i\in \mathbb{Z}} \Phi(2^k x - i)(\mathcal{S}^k \Delta_0)_i \ . \tag{2.4}$$

Also, using $\Phi(x) = VN(x)$ and $\widetilde{\Phi}V = V$, we have

$$\sum_{i\in \mathbb{Z}} \Phi(2^k x - i)\Phi(x) = \widetilde{\Phi}(2^k x)VN(x) = VN(x) = \Phi(x) \ . \tag{2.5}$$

Thus it follows that

$$\begin{aligned} 0 &= \sum_{i\in \mathbb{Z}} \Phi(2^k x - i)[(\mathcal{S}^k \Delta_0)_i - \Phi(x)] \\ &= \sum_{i\in \mathbb{Z}} \Phi(2^k x - i)[(\mathcal{S}^k \Delta_0)_i - \Phi(2^{-k} i)] \\ &\quad + \sum_{i\in \mathbb{Z}} \Phi(2^k x - i)[\Phi(2^{-k} i) - \Phi(x)] \ . \end{aligned} \tag{2.6}$$

By the continuity of Φ and its compact support, for any $\varepsilon > 0$ $\exists K$ s.t. for $k > K$

$$\| \sum_{i\in \mathbb{Z}} \Phi(2^k x - i)[\Phi(2^{-k} i) - \Phi(x)] \|_\infty < \varepsilon \ . \tag{2.7}$$

Now, from (2.6) and (2.7)

$$\| \sum_{i\in \mathbb{Z}} \Phi(2^k x - i)[(\mathcal{S}^k \Delta_0)_i - \Phi(2^{-k} i)] \|_\infty \le \varepsilon \quad \textit{for } k > K \ , \tag{2.8}$$

and by the stability of Φ

$$\| \mathcal{S}^k \Delta_0 - \Phi(2^{-k} \cdot) \|_\infty \le \varepsilon \quad \textit{for } k > K \ . \tag{2.9}$$

Hence $\mathcal{S}$ is convergent in the sense of (1.2) and $\mathcal{S}^\infty \Delta_0 = \Phi$. ∎

§3 Analysis of smoothness and differential inter-dependence

It is convenient for the analysis of matrix subdivision schemes to introduce the following transformation:

Let Q be an $n \times (n-m)$ matrix such that $T = \big(V \mid Q\big)$ is a non-singular $n \times n$ matrix. Define the matrix mask

$$\overline{A}_j = T^{-1} A_j T \ , \quad j \in \mathbb{Z} \ , \tag{3.1}$$

and consider the subdivision scheme $\overline{S}$ and the refinement equation corresponding to this matrix mask. Now the matrices

$$\overline{B}_0 = \sum_{i\in\mathbb{Z}} \overline{A}_{2i} \quad , \quad \overline{B}_1 = \sum_{i\in\mathbb{Z}} \overline{A}_{2i+1} \ , \tag{3.2}$$

have the standard unit vectors $e^{(j)}$, $1 \le j \le m$ as eigenvectors with eigenvalue 1, and correspondingly we define $\overline{V} = \big(e^{(1)}...e^{(m)}\big)$. Thus

$$\overline{B}_i = \big(e^{(1)}...e^{(m)}, E_i\big) \ , \quad i = 0, 1 \ . \tag{3.3}$$

An appropriate choice of the matrix Q is that consisting of the $n-m$ eigenvectors and possibly generalized eigenvectors of $M(0)$ corresponding to eigenvalues with modulus less than 1 . With this choice of Q , $\overline{M}(0)$ is a diagonal matrix or a Jordan matrix.

Let Ψ be a solution of (1.3), then $\overline{\Psi} = T^{-1}\Psi T$ satisfies

$$\overline{\Psi}(x) = \sum_{j\in\mathbb{Z}} \overline{\Psi}(2x-j)\overline{A}_j \ , \tag{3.4}$$

and if Ψ is stable then $\overline{\Psi}$ is stable. Also, if we perform the subdivision

$$\overline{f}_i^{k+1} = \sum_{j\in\mathbb{Z}} \overline{A}_{i-2j} \overline{f}_j^k \ , \ i \in \mathbb{Z} \ . \tag{3.5}$$

starting with the initial vector $\overline{f}^0 = T^{-1} f^0$ then

$$T\overline{S}^\infty \overline{f}^0 = S^\infty f^0 \ . \tag{3.6}$$

Thus, to simplify notations, we can assume that the original mask is such that $V = \big(e^{(1)}...e^{(m)}\big)$. In this case, the last $n-m$ rows of Φ are identically zero. With this special form of V we can easily derive appropriate difference and divided difference schemes for the scheme (1.1) for analyzing its convergence and the smoothness of its limit vector functions. As in [11] we use a corresponding matrix-valued Laurent polynomial

$$A(z) = \sum_{i \in \mathbb{Z}} A_i z^i \ ,$$

to represent the matrix subdivision scheme (1.1). Now we define the Laurent polynomial

$$D(z) = 2E(z)A(z)E(z^2)^{-1} = \sum_{i \in \mathbb{Z}} D_i z^i \ , \tag{3.7}$$

where $E(z) = \big((z^{-1}-1)e^{(1)} \ldots (z^{-1}-1)e^{(m)}, e^{(m+1)} \ldots e^{(n)}\big)$. The corresponding mask $\{D_i\}$ is of finite support by (3.3). It generates a stationary matrix subdivision scheme $\mathcal{S}_D$ transforming the values $\{g_i^k\}$,

$$g_i^k = 2^k \big((f_{i+1}^k)_1 - (f_i^k)_1, \ldots, (f_{i+1}^k)_m - (f_i^k)_m, (f_i^k)_{m+1}, \ldots, (f_i^k)_n\big)^t \tag{3.8}$$

into the corresponding values at level $k+1$, namely $\{g_i^{k+1}\}$.

As in the scalar case we have the following relation between $\mathcal{S}$ and $\mathcal{S}_D$.

Theorem 3.1 *If $\frac{1}{2}\mathcal{S}_D$ is contractive then $\mathcal{S}$ is C^0 . If $\mathcal{S}_D$ is C^0 then $\mathcal{S}$ is C^1. Moreover with the initial data G^0 consisting of the first m rows of $\Delta_{-1} - \Delta_0$ and the last $n-m$ rows of Δ_0 , the first m rows of $\mathcal{S}_D^\infty G^0$ are the derivatives of the corresponding elements in the first m rows of Φ . Also, $\Phi_D = \mathcal{S}_D^\infty \Delta_0$ is related to the non-trivial part of Φ by*

$$\Phi_{i,j} = (\Phi_D)_{i,j} * \chi_{[0,1]} \ , \quad 1 \le i,j \le m \ , \tag{3.9a}$$

$$\Phi'_{i,j} = (\Phi_D)_{i,j} \ , \quad 1 \le i \le m \ , \quad m+1 \le j \le n \ , \tag{3.9b}$$

where $\chi_{[0,1]}$ is the characteristic function of the interval $[0,1]$.

In the following we analyze properties of $\mathcal{S}_D$ determined by those of $\mathcal{S}$.

Theorem 3.2 *Let m' denote the dimension of the common eigenspace W_D of $B_0^D = \sum_{i \in \mathbb{Z}} D_{2i}$ and $B_1^D = \sum_{i\in\mathbb{Z}} D_{2i+1}$ corresponding to the eigenvalue 1. If $\mathcal{S}_D$ is C^0 , then $m' \ge m$, and the first m rows of any basis V_D of W_D are linearly independent.*

Proof: Starting from the initial data G^0 as defined in Theorem 3.1, we get by Theorem 3.1 that $\mathcal{S}_D^\infty G^0$ is a matrix function whose first m rows are the derivatives of the first m rows of $\Phi = \mathcal{S}^\infty \Delta_0$. Suppose the first m rows of $\mathcal{S}_D^\infty G^0$ are linearly dependent then we get

$$\sum_{i=1}^m \alpha_i \Phi'_{i,j} = 0 \ , \quad 1 \le j \le n \ . \tag{3.10}$$

Thus $\sum_{i=1}^m \alpha_i \Phi_{i,j} = c_j$, $1 \le j \le n$, and since these functions are of compact support, it is possible only if all are identically zero, which contradicts the form of V. Thus the first m rows in a limit matrix function generated by $\mathcal{S}_D$ are linearly independent, implying that $m' \ge m$, and that the first m rows in any basis V_D of W_D are linearly independent. ∎

Theorem 3.3 *Assume $\mathcal{S}$ is C^1, and that Φ is stable. Then $\mathcal{S}_D$ is C^0 and the first m rows of Φ_D constitute a stable matrix. Moreover $m' = m$.*

Proof: To prove that $\mathcal{S}_D$ is C^0 , it is sufficient by Theorem 2.10 to construct a continuous compactly supported stable solution of the corresponding functional equation. Let us define the first m rows of Φ_D by (3.9), and denote it by $\Phi_D^{[m]}$. It is easy to see that since Φ is of compact support (3.9a) can be rewritten as

$$(\Phi_D)_{i,j} = \sum_{\ell=0}^{\infty} \Phi'_{i,j}(\cdot - \ell) \ . \tag{3.11}$$

Thus, the elements of $\Phi_D^{[m]}$ are in $C^0(\mathbb{R})$. To show the stability of $\Phi_D^{[m]}$ assume that it is not stable. Then there is $\alpha = \{\alpha_j \in \mathbb{R}^n \ : \ j \in \mathbb{Z}\}$, satisfying $0 < \sup_{j\in\mathbb{Z}} \|\alpha_j\| < \infty$, such that

$$\sum_{j\in\mathbb{Z}} \Phi_D^{[m]}(\cdot - j)\alpha_j = 0 \ . \tag{3.12}$$

In case all the first m components of all the vectors $\{\alpha_i \ : \ i \in \mathbb{Z}\}$ are identically zero we get by integrating (3.12) form $-\infty$ to x that

$$\sum_{j\in\mathbb{Z}} \Phi^{[m]}(\cdot - j)\alpha_j = c \ , \tag{3.13}$$

where $\Phi^{[m]}$ stands for the first m rows of Φ . But since Φ is of compact support $c = 0$ in (3.13) in contradiction to the stability of Φ . If some of the first m components of the vectors $\{\alpha_j \ : \ j \in \mathbb{Z}\}$ are non-zero, we integrate (3.12) from $x-1$ to x and get

$$\sum_{j\in\mathbb{Z}} \Phi^{[m]}(\cdot - j)\gamma_j = 0 \ , \tag{3.14}$$

with γ_j having the same m components of α_j and the last $n-m$ components of $\alpha_j - \alpha_{j-1}$. It is clear that $0 < \sup_{j\in\mathbb{Z}} \|\gamma_j\|$, and hence (3.14) contradicts the stability of Φ . Thus $\Phi_D^{[m]}$ is stable.

In the following we show that $\Phi_D^{[m]}$ solves the functional equation corresponding to $\mathcal{S}_D$. Since Φ satisfies (1.3) we obtain that

$$\hat{\Phi}(\omega) = \hat{\Phi}(\omega/2)M(\omega/2) \ . \tag{3.15}$$

In view of the definition of the mask of $\mathcal{S}_D$ by (3.7), we obtain that $\omega\hat{\Phi}^{[m]}(\omega)\mathcal{E}(\omega)^{-1}$ satisfies a functional equation of the type (3.15) with $M_D(\omega) = 2\mathcal{E}(\omega)M(w)\mathcal{E}(2\omega)^{-1}$, where $\mathcal{E}(\omega) = E(e^{-i\omega})$. It follows from (3.9) and the form of $E(z)$ that

$$\omega\hat{\Phi}^{[m]}(\omega)\mathcal{E}(2\omega)^{-1} = \hat{\Phi}_D^{[m]}(\omega) \ , \tag{3.16}$$

and hence $\Phi_D^{[m]}$ solves the functional equation corresponding to $\mathcal{S}_D$. Now, defining $\Phi_D = V_D \Phi_D^{[m]}$, we obtain a stable continuous compactly supported solution of the functional equation corresponding to $\mathcal{S}_D$, with the first m rows of the solution being a stable matrix function. This together with Theorems 2.6 and 3.2 implies that $m' = m$. ∎

As a consequence of the above three theorems we get

Corollary 3.4 *Assume $\mathcal{S}$ is C^1, and that Φ is stable. Let*

$$h^k = T_k \mathcal{S}^k f^0 \ , \quad k \in \mathbb{Z}_+ \ ,$$

where T_k is an $n \times n$ diagonal matrix with elements

$$(T_k)_{i,i} = 1 \ , \quad 1 \le i \le m \ , \quad (T_k)_{i,i} = 2^k \ , \quad m+1 \le i \le n \ . \tag{3.17}$$

Then the sequence $\{h^k\}_{k \in \mathbb{Z}_+}$ converges uniformly in the sense of (1.2) to a continuous limit vector function h. The first m components of h are in $C^1(\mathbb{R})$. Moreover there is a matrix $\beta \in \mathbb{R}^{n-m} \times \mathbb{R}^m$, dependent on $\mathcal{S}$ only, such that

$$h_i = \sum_{k=1}^{m} \beta_{i,k} h'_k \ , \quad m+1 \le i \le n \ . \tag{3.18}$$

Proof: By Theorem 3.3 $\mathcal{S}_D$ is C^0 . Let G^0 be as in Theorem 3.1, then $\mathcal{S}_D^\infty G^0$ exists and is continuous. Also we get in view of (3.8), that the elements of the first m rows of $\mathcal{S}_D^\infty G^0$ are the derivatives of the corresponding elements in the limit matrix function H generated by the sequence $H^k = T_k \mathcal{S}^k \Delta_0$, $k \in \mathbb{Z}_+$, while the last $n-m$ rows of $\mathcal{S}_D^\infty G^0$ consist of the corresponding rows of H . This together with Theorem 3.2 and Theorem 3.3 guarantees the existence of a matrix $\beta \in \mathbb{R}^{n-m} \times \mathbb{R}^m$, such that

$$H_{i,j} = \sum_{k=1}^{m} \beta_{i,k} H'_{k,j} \ , \quad m+1 \le i \le n \ , \quad 1 \le j \le n \ , \tag{3.19}$$

implying (3.18). ∎

One should note that this analysis can be pushed further: if the renormalized scheme is also C^1 with one of its coordinates converging to zero, a proper renormalization will make this coordinate converge to a combination of the second derivatives of the non-zero components of the original scheme, and the same property will hold for higher derivatives. Also higher smoothness of the non-zero components of the original scheme can be derived, by a repeated application of (3.7), with appropriate matrices $A(z)$ and $E(z)$, corresponding to the structure of the derived scheme in the previous step.

Specific criterions on the masks that ensure the smoothness of the limit function are given in [16] and [23] in the case of refinable vectors and in [22] for refinable matrices.

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