Nonstationary subdivision schemes and multiresolution analysis

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Abstract

Nonstationary subdivision schemes consist of recursive refinements of an initial sparse sequence with the use of masks that may vary from one scale to the next finer one. This paper is concerned both with the convergence of nonstationary subdivision schemes and with the properties of their limit functions. We first establish a general result on the convergence of such schemes to C^∞ compactly supported functions. We show that these limit functions allow to define a multiresolution analysis that has the property of spectral approximation. Finally, we use these general results to construct C^∞ compactly supported cardinal interpolants and also C^∞ compactly supported orthonormal wavelet bases that constitute Riesz bases for Sobolev spaces of any order.

Key-words : Subdivision schemes, multiresolution analysis, spectral approximation, dyadic interpolation, wavelets.

I.Introduction

Subdivision schemes constitute a useful tool for the fast generation of smooth curves and surfaces from a set of control points by means of iterative refinements. In the most often considered binary univariate case, one starts from a sequence $s_0(k)$ and obtains at step j a sequence $s_j(2^{-j}k)$, generated from the previous one by linear rules :

$$s_j(2^{-j}k) = 2 \sum_{n \in k+2\mathbb{Z}} c_{j,k}(n) s_{j-1}(2^{-j}(k-n)). \qquad (1.1)$$

The masks $c_{j,k} = \{c_{j,k}(n)\}_{n \in \mathbb{Z}}$ are in general finite sequences, a property that is clearly useful for the practical implementation of (1.1).

A natural problem is then to study the convergence of such an algorithm to a limit function. In particular, the scheme is said to be strongly convergent if and only if there exists a continuous function $f(x)$ such that $\lim_{j \to +\infty}(\sup_k |s_j(2^{-j}k) - f(2^{-j}k)|) = 0$. One can study more general types of convergence with the use of a smooth function g that is well localized in space (for example compactly supported) and satisfies the interpolation property $g(k) = \delta_k$. One can then define $f_j(x) = \sum_k s_j(2^{-j}k)g(2^j x - k)$ and study the convergence in a functional sense of f_j to f.

A subdivision scheme is said to be stationary and uniform when the masks $c_{j,k}(n) = c_n$ are independant of the parameters j and k. In that case, one can rewrite (1.1) as

$$s_j(2^{-j}k) = 2 \sum_n c_{k-2n} s_{j-1}(2^{-j+1}n). \qquad (1.2)$$

Note that (1.2) is equivalent in filling the sequence s_{j-1} with zeros at the intermediate points $2^{-j}(2k+1)$ and applying a discrete convolution with the sequence (c_k). Detailed reviews of stationary subdivision have been done by Cavaretta, Dahmen and Micchelli (1991) and Dyn (1992).

These algorithms apply in a natural way to computer aided geometric design. Moreover, the interest in stationary subdivision schemes has grown in the digital image processing and numerical analysis communities since they have been connected to multiresolution analysis and wavelet bases.

A multiresolution analysis consists of a nested sequence of approximation subspaces

$$\{0\} \to \dots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \dots \to L^2(\mathbb{R}), \tag{1.3}$$

that are generated by a "scaling function" $\varphi \in V_0$ in the sense that the set $\{\varphi(2^j x - k)\}_{k\in\mathbb{Z}}$ constitutes a Riesz basis for V_j. By $V_j \to L^2(\mathbb{R})$, we mean here that for any f in $L^2(\mathbb{R})$, $\lim_{j\to+\infty} \|P_j f - f\|_0 = 0$ where $P_j f$ is the L^2-projection of f onto V_j and $\|\cdot\|_0$ is the L^2 norm (we shall use the notation $\|\cdot\|_s$ for the Sobolev $H^s = W_2^s$ norm). Here again, many generalizations are possible (see Meyer (1990) or Daubechies (1992) for a detailed review of this concept).

Since the spaces V_j are embedded, the scaling function satisfies an equation of the type

$$\varphi(x) = 2\sum_n c_n \varphi(2x - n). \tag{1.4}$$

We shall assume here that φ is compactly supported so that the c_n's are finite in number. In that case, φ is also an L^1 function and by taking the Fourier transform of (1.4), we have

$$\hat{\varphi}(\omega) = m(\omega/2)\hat{\varphi}(\omega/2) \tag{1.5}$$

where $m(\omega) = \sum_{n\in\mathbb{Z}} c_n e^{-in\omega}$. Assuming that φ is normalized in the sense that $\int \varphi = \hat{\varphi}(0) = 1$, one obtains by iterating (1.5),

$$\hat{\varphi}(\omega) = \prod_{k=1}^{+\infty} m(2^{-k}\omega). \tag{1.6}$$

This last formula indicates that φ is the limit, in the weak (or distribution) sense, of a stationary subdivision scheme since it represents, in the Fourier domain, the refinement of an initial Dirac sequence by iterative convolutions with c_n. Note also that the support of φ is contained in the convex hull of the support of the mask (c_k). Conversly, any refinable function, i.e. weak limit of such a scheme, satisfies a "refinement equation" of the type described above and is a potential candidate to generate a multiresolution analysis (see also Derfel, Dyn and Levin (1992)).

Given a stationary subdivision scheme, we see here that two questions are relevant :

- Is the scheme convergent and in what sense ?
- What are the properties of the limit functions ?

By the last question, we mean in particular the approximation properties of the spaces V_j (can we approximate in other norms than L^2, in particular in Sobolev spaces H^s, with specific rates...), the exact regularity of the scaling function and other properties of φ such as cardinal interpolation or orthonormality of its integer shifts.

Numerous contributions have been made on these two problems. The convergence of the subdivision and the approximation properties of the multiresolution spaces are strongly linked : in particular, one can prove (see Dyn and Levin (1990), Cavaretta, Dahmen and Micchelli (1992), Daubechies and Lagarias (1991)) that both the convergence of the subdivision scheme to a C^r function for some $r \geq 0$ and the property that $\lim_{j\to+\infty} 2^{js}\|P_j f - f\|_0 = 0$ for all $f \in H^s$ $(s \leq r)$ imply that the scaling function satisfies the Strang-Fix conditions of order N, where N is an integer such that $N \leq r < N+1$. These conditions can be expressed by three equivalent statements :

- Any polynomial of degree not exceedding N can be expressed as a combination of the integer shifts of φ.
- For all $p \leq N$ and $m \in \mathbb{Z} - \{0\}$, $(\frac{d}{d\omega})^p \hat{\varphi}(2m\pi) = 0$, $\hat{\varphi}(0) = 1$.
- For all $p \leq N$, $(\frac{d}{d\omega})^p m(\pi) = 0$ or equivalently $\sum_m (-1)^m m^p c_m = 0$.

Note that this last statement reveals that $m(\omega)$ can be written as

$$m(\omega) = (\frac{1+e^{-i\omega}}{2})^{N+1} q(\omega), \tag{1.7}$$

where $q(\omega)$ is a trigonometric polynomial. (1.7) implies that there are at least $N+2$ nonzero c_n, and thus the support length of φ is at least $N+1$. This leads to the observation that very good approximation rates for regular functions, as well as convergence of the subdivision in a smooth norm, can only be achieved if one accepts to loose some space localization (in particular, one cannot build a refinable function that is both compactly supported and in C^∞).

More recently, attention has been given to subdivision schemes that are nonstationary in scale, i.e. for which the masks may vary from one step of the refinement process to the next one. A model case is the scheme that uses at step k the mask $c_n^k = \binom{k}{n} 2^{-k+1}$ $(0 \leq n \leq k)$, that gives rise in the stationary subdivision case to B-splines of degree $k-1$. It was proved by Dyn, Levin and Derfel (1992) that such a scheme converges strongly to the "up-function" introduced by Rvachev (1971) (see also Rvachev (1990)). The limit function can thus be written in the Fourier domain as

$$\hat{\varphi}(\omega) = \prod_{k=1}^{+\infty} (\frac{1+e^{-i2^{-k}\omega}}{2})^k. \tag{1.8}$$

The length of its support is given by $L = \sum_{k>0} k2^{-k} = 2 < +\infty$. Such a function cannot satisfy a refinement equation of the type (1.4). However, note that the product (1.8) can also be written as

$$\begin{aligned} \textstyle\prod_{k=1}^{+\infty} (\frac{1+e^{-i2^{-k}\omega}}{2})^k &= \textstyle\prod_{n=0}^{+\infty} \prod_{k=n+1}^{+\infty} \frac{1+e^{-i2^{-k}\omega}}{2} \\ &= \textstyle\prod_{n=0}^{+\infty} \frac{1+e^{-i2^{-n}\omega}}{-i2^{-n}\omega} \\ &= \textstyle\prod_{n=0}^{+\infty} \hat{\chi}_{[0,1]}(2^{-n}\omega). \end{aligned}$$

It follows that

$$\varphi = \chi_{[0,1]} * 2\chi_{[0,1/2]} * \cdots * 2^j \chi_{[0,2^{-j}]} * \cdots$$

is a C^∞ function that satisfies a "continuous refinement equation" of the type

$$\varphi(x) = 2\chi_{[0,1]} * \varphi(2\cdot) = \int_0^2 \varphi(2x-y)dy. \tag{1.9}$$

By letting the masks grow linearly, it is thus possible to obtain a C^∞ function while preserving the compact support property. It was also shown by Dyn and Ron (1993) that a "half-multiresolution analysis" can be derived by defining, for all $j \geq 0$, $V_j = \mathrm{Span}\{\varphi_j(2^j x - k)\}_{k \in \mathbb{Z}}$ with

$$\hat{\varphi}_j(\omega) = \prod_{k=1}^{+\infty} (\frac{1+e^{-i2^{-k}\omega}}{2})^{k+j}, \tag{1.10}$$

and that these spaces have the property of spectral approximation in L^2 : for any $r \geq 0$ and for all $f \in H^r$, $\lim_{j\to+\infty} 2^{jr}\|P_j f - f\|_0 = 0$.

Our goal in this paper is to generalize these results to a large class of nonstationary subdivision schemes.

Assuming that such a scheme converges at least in the sense of tempered distributions, the general form of its limit function will be given in the Fourier domain by

$$\hat{\varphi}(\omega) = \prod_{k=1}^{+\infty} m_k(2^{-k}\omega), \tag{1.11}$$

where m_k is the sequence of trigonometric polynomials associated with the masks of the subdivision. Note that, since we do not assume any particular form for m_k, the function φ will not in general satisfy any type of refinement equation, discrete or continuous, making thus more difficult the analysis of its smoothness and approximation properties.

What is the interest of such a generalization ? An important remark is that the approximation properties of the up-function and its associated multiresolution analysis, very attractive from the theoretical point of view, suffer from a major numerical disadvantage : the computation of the L^2 projection onto V_j is difficult to manage at high scales since the Gram matrix of the basis $\{\varphi^j(2^j x - k)\}_{k\in\mathbb{Z}}$ becomes ill-conditionned. More precisely, its condition number $C(j)$ grows exponentially with j :

$$(\sqrt{2})^{j+1} \leq C(j) \leq \text{const}(\frac{\pi}{2})^j. \tag{1.12}$$

The upper bound is taken from Dyn and Ron (1993) and the lower bound is obtained here :

$$\begin{aligned} C(j)^2 &= (\sup_\omega \textstyle\sum_k |\hat{\varphi}^j(\omega + 2k\pi)|^2)(\inf_\omega \sum_k |\hat{\varphi}^j(\omega + 2k\pi)|^2)^{-1} \\ &\geq (\inf_\omega \textstyle\sum_k |\hat{\varphi}^j(\omega + 2k\pi)|^2)^{-1} \\ &\geq (\textstyle\sum_k |\hat{\varphi}^j((2k+1)\pi)|^2)^{-1} \\ &= (\textstyle\sum_k |\cos^{j+1}(\pi/4)\hat{\varphi}^{j+1}((2k+1)\pi/2)|^2)^{-1} \\ &\geq [\cos(\pi/4)]^{-2j-2} = 2^{j+1}. \end{aligned}$$

The same problem occurs when one wants to interpolate data on the grid $2^{-j}\mathbb{Z}$ by a function in V_j for j odd : one checks from a similar computation that the condition number $D(j)$ of the system grows exponentially.

In a more general setting, it is possible to keep these condition numbers bounded as j grows. One can even fix one of them to 1 by imposing constraints on the trigonometric polynomials m_k so that the limit functions have orthonormality or cardinal interpolation properties (see §4).

Finally, an important property of multiresolution analysis is the equivalence

$$\|f\|_r^2 \approx \|P_0 f\|_0^2 + \sum_{j>0} 2^{2jr} \|P_{j+1} f - P_j f\|_0^2, \tag{1.12}$$

that is the key to multilevel preconditionning techniques (see Dahmen and Kunoth (1992)) and that can also be expressed in terms of wavelet coefficients. So far, we could only prove this equivalence in the orthonormal case, for all $r > 0$ (see §4).

Our paper is organized as follows : in §2, we give a general result on the convergence of a nonstationary subdivision scheme in C^∞ under very mild conditions on the masks. We study the approximation properties of the associated multiresolution spaces in §3 and prove that spectral approximation can be achieved for all Sobolev norms. Finally we apply these results in §4 to dyadic interpolation and to orthonormal wavelets that constitute Riesz bases for all Sobolev spaces. This particular wavelet basis has been recently introduced in a paper by Berkolaiko and Novikov (1992) which was concerned with the existence of a multiscale orthonomal basis of compactly supported C^∞ functions.

For the sake of simplicity, we limit ourselves to the one dimensional setting and our results are stated in the case where the length of the masks grows at least linearly. We show in an appendix how this can be extended to more general growth rates of the mask length.

II.Nonstationary subdivision schemes

Let $\{m_k\}_{k>0}$ be a sequence of finite masks, i.e. $m_k(n) = 0$ if $|n| > d(k)$. We denote by $m_k(\omega) = \sum_n m_k(n)e^{-in\omega}$ their representation in the Fourier domain, i.e. a sequence of trigonometric polynomials of degree $d(k)$. Let us consider the nonstationary subdivision scheme that is associated with this sequence of masks, i.e. $s_j(2^{-j}k) = 2\sum_n m_j(k-2n)s_{j-1}(2^{-j+1}n)$. If the input is a Dirac sequence $\delta_{m,0}$, one obtains after n steps a sequence of samples on the grid $2^{-n}\mathbb{Z}$, that can be interpolated in a unique way by a function $\varphi^{[n]}$ that is band-limited on $[-2^n\pi, 2^n\pi]$. This function is defined by

$$\hat{\varphi}^{[n]}(\omega) = \prod_{k=1}^{n} m_k(2^{-k}\omega)\chi_{[-\pi,\pi]}(2^{-n}\omega). \tag{2.1}$$

Note that the functions $\varphi^{[n]}$ are analytic and thus not compactly supported. We shall use these particular interpolants in order to study the convergence of the subdivision scheme to the limit function defined (if this is possible) by

$$\hat{\varphi}(\omega) = \prod_{k=1}^{+\infty} m_k(2^{-k}\omega). \tag{2.2}$$

After n steps, the result of the subdivision in the space domain is supported in $[-L(n), L(n)]$, with $L(n) = \sum_{k=1}^{n} 2^{-k}d(k)$. A natural condition for compactly supported limit function is thus

$$L = \sum_{k=1}^{+\infty} 2^{-k}d(k) < +\infty. \tag{2.3}$$

Our first result shows that this condition is also instrumental in the derivation of the convergence, in the sense of tempered distributions, of the subdivision scheme.

Theorem 2.1 *Assume that $r_k = 2^{-k}d(k)$ and $s_k = |m_k(0) - 1|$ are both summable sequences, and that the functions $|m_k(\omega)|$ are uniformly bounded by some constant $M > 0$. Then $\hat{\varphi}^{[n]}$ converges uniformly on any compact set to $\hat{\varphi}$ and $\varphi^{[n]}$ converges to φ in the sense of tempered distributions. The tempered distribution φ is compactly supported in $[-L, L]$ with $L = \sum_{k>0} r_k$.*

Proof We first study the convergence of the infinite product (2.2). For a fixed ω, we have to check the summability in k of $t_k(\omega) = |m_k(2^{-k}\omega) - 1|$. If in addition, $\sum_{k>0} t_k(\omega)$ is uniformly bounded on every compact set, then (2.2) will also converge uniformly on every compact set.

We can write

$$\begin{aligned} t_k(\omega) &\leq |m_k(2^{-k}\omega) - m_k(0)| + s_k \\ &\leq 2^{-k}|\omega| \sup_\omega |\tfrac{\mathrm{d}}{\mathrm{d}\omega} m_k| + s_k. \end{aligned}$$

Using Bernstein's unequality, we obtain the estimate

$$\sup_\omega |\frac{\mathrm{d}}{\mathrm{d}\omega} m_k| \leq M d(k), \tag{2.4}$$

and thus

$$t_k(\omega) \leq M|\omega| r_k + s_k, \tag{2.5}$$

which proves the uniform convergence of (2.2) on every compact set.

The same argument shows that for any $n > p \geq 0$, the products

$$P_p^n(\omega) = \prod_{k=p+1}^{n} m_k(2^{-k}\omega), \tag{2.6}$$

are uniformly bounded on $[-2^{p+1}, 2^{p+1}]$ by the same $B > 0$. We can define these products to be equal to 1 whenever $n \leq p$ so that this statement makes sense for all $n, p > 0$. This applies in particular to $\hat{\varphi}^{[n]} = P_0^n$ and $\hat{\varphi} = P_0^\infty$ which are thus uniformly bounded on $[-2, 2]$. For $2^p \leq |\omega| \leq 2^{p+1}$ with $p \geq 0$, we can write

$$\begin{aligned} |\hat{\varphi}^{[n]}(\omega)| &= |P_p^n(\omega) \textstyle\prod_{k=1}^p m_k(2^{-k}\omega)| \\ &\leq BM^p \leq BM^{\log_2 |\omega|} \leq B|\omega|^b, \end{aligned}$$

with $b = \log_2(M)$ (we have assumed here, without loss of generality, that $M \geq 1$). For all $\omega \in \mathbb{R}$, we thus have the estimate

$$|\hat{\varphi}^{[n]}(\omega)| \leq B(1 + |\omega|)^b \tag{2.7}$$

where the constant B does not depend on n. Consequently, it also holds for the pointwise limit $\hat{\varphi}$.

Take now any test function $g(\omega)$ in the Schwartz class $S(\mathbb{R})$. For any $\varepsilon > 0$ there exists $A > 0$ such that

$$B \int_{|\omega|>A} g(\omega)(1+|\omega|)^b d\omega \ < \ \varepsilon/2. \tag{2.8}$$

By the uniform convergence of $\hat{\varphi}^{[n]}$ to $\hat{\varphi}$ on every compact, there exists N such that for all $n > N$,

$$|\int_{|\omega|<A} g(\omega)(\hat{\varphi}(\omega) - \hat{\varphi}^{[n]}(\omega))d\omega| \ < \ \varepsilon/2. \tag{2.9}$$

Combining (2.7), (2.8) and (2.9), we immediately obtain the convergence of $\langle \hat{\varphi}^{[n]}|g\rangle$ to $\langle \hat{\varphi}|g\rangle$. □

We are now interested in finding additional hypotheses for stronger convergence of the subdivision scheme to a C^∞ compactly supported function φ. Note that, in contrast to its approximants $\varphi^{[n]}$, the function φ cannot be analytic. Our next result states general conditions for the uniform convergence of $\varphi^{[n]}$ and all its derivatives.

Theorem 2.2 *Assume that the hypotheses of Theorem 2.1 are satisfied and that we have the estimate*

$$|m_k(\omega)| \leq (1+\alpha_k)|m(\omega)|^k, \tag{2.10}$$

with $\sum_k |\alpha_k| < +\infty$ *and* $m(\omega) = \cos^\beta(\omega/2)\tilde{m}(\omega)$*, for some* $\beta \geq 0$ *(not necessarily integer), where the function* $\tilde{m}(\omega)$ *is bounded, Hölder continuous at the origin and satisfies* $\tilde{m}(0) = 1$ *and* $\sigma_i = \sup_\omega |\prod_{k=1}^i \tilde{m}(2^k\omega)| < 2^{\beta i}$ *for some fixed integer* $i > 0$.

Then φ *is a* C^∞ *compactly supported function and, for all* $s \in \mathbb{Z}_+$, $(\frac{\mathrm{d}}{\mathrm{dx}})^s\varphi^{[n]}$ *converges uniformly to* $(\frac{\mathrm{d}}{\mathrm{dx}})^s\varphi$.

Proof It is sufficient to show that for all $s \in \mathbb{Z}_+$, the functions $|\omega|^s|\hat{\varphi}^{[n]}(\omega)|$ are majorized by an L^1 function $f_s(\omega)$ that does not depend on n : by dominated convergence this implies

$$\lim_{n\to+\infty} \int |\omega|^s|\hat{\varphi}(\omega) - \hat{\varphi}^{[n]}(\omega)|d\omega \ = \ 0, \tag{2.11}$$

and thus the uniform convergence of all the derivatives of $\varphi^{[n]}$ in the space domain. We shall construct these majorizing functions, using the additional hypotheses that we have made on the functions $m_k(\omega)$.

First, we need a technical estimate that will be useful : for any $q \geq 0$, there exists $C_q > 0$ such that, for any sequence $\{a_k\}_{k>0}$ with $0 \leq a_k \leq 1$ and any $n \geq p \geq 0$,

$$\prod_{k=p}^{n} |m_{q+k}(2^{-k}\omega)|^{a_k} \leq C_q(1+|\omega|)^b, \tag{2.12}$$

with $b = \log_2(M)$ (as in the previous theorem, we assume, without loss of generality, that $M \geq 1$). Indeed, using the same argument (Bernstein's inequality) as in the proof of Theorem 2.1, we observe that $\prod_{k=p}^{n} |m_{q+k}(2^{-k}\omega)|^{a_k}$ is uniformly bounded in $[-1,1]$ by a constant C_q that does not depend on a_k, p and n, since we have

$$\begin{aligned} |1 - |m_{q+k}(2^{-k}\omega)|^{a_k}| &\leq |m_{q+k}(2^{-k}\omega) - 1| \\ &\leq |m_{q+k}(2^{-k}\omega) - m_{q+k}(0)| + |m_{q+k}(0) - 1| \\ &\leq 2^q M|\omega| r_{q+k} + s_{q+k} \leq 2^q M r_{q+k} + s_{q+k}. \end{aligned}$$

For $2^l \leq |\omega| \leq 2^{l+1}$ with $p \leq l < n$, we now derive

$$\begin{aligned} \prod_{k=p}^{n} |m_{q+k}(2^{-k}\omega)|^{a_k} &= \prod_{k=p}^{l} |m_{q+k}(2^{-k}\omega)|^{a_k} \prod_{k=l+1}^{n} |m_{q+k}(2^{-k}\omega)|^{a_k} \\ &\leq M^l \prod_{k=l+1}^{n} |m_{q+k}(2^{-k}\omega)|^{a_k} \\ &\leq C_q(M)^{\log_2|\omega|} = C_q|\omega|^b. \end{aligned}$$

In the cases where $l \geq n$, this estimate still holds since $M^n \leq M^l$, while for $l < p$ the bound is C_q. This proves (2.12) for all $\omega \in \mathbb{R}$.

We are now ready to build the majorizing functions $f_s(\omega)$. For fixed $s \geq 0$, choose $p \in \mathbb{N}$ such that $p(\frac{\log_2\sigma_i}{i} - \beta) + s + b < -1$ (this is always possible since we have assumed $\frac{\log_2\sigma_i}{i} < \beta$). For $n \geq p$, we can estimate $\hat{\varphi}^{[n]}(\omega)$ on $[-2^n\pi, 2^n\pi]$ by

$$\begin{aligned} |\hat{\varphi}^{[n]}(\omega)| &= \prod_{k=1}^{n} |m_k(2^{-k}\omega)| \\ &\leq M^{p-1} \prod_{k=p}^{n} |m_k(2^{-k}\omega)| \\ &= M^{p-1} \prod_{k=p}^{n} |m_k(2^{-k}\omega)|^{\frac{p}{k}} \prod_{k=p}^{n} |m_k(2^{-k}\omega)|^{\frac{k-p}{k}}. \end{aligned}$$

Using the estimate (2.12) and the hypothesis (2.10), we thus obtain

$$\begin{aligned}|\hat{\varphi}^{[n]}(\omega)| &\leq M^{p-1}C_p(1+|\omega|)^b \prod_{k=p}^n |m(2^{-k}\omega)|^p \\ &= M^{p-1}C_p(1+|\omega|)^b \prod_{k=p}^n |\cos(2^{-k-1}\omega)|^{\beta p}|\tilde{m}(2^{-k}\omega)|^p \\ &= M^{p-1}C_p(1+|\omega|)^b \frac{|\mathrm{sinc}(2^{-p}\omega)|^{\beta p}}{|\mathrm{sinc}(2^{-n-1}\omega)|^{\beta p}} \prod_{k=p}^n |\tilde{m}(2^{-k}\omega)|^p \\ &\leq A_p(1+|\omega|)^{b-\beta p} \prod_{k=p}^n |\tilde{m}(2^{-k}\omega)|^p,\end{aligned}$$

where A_p only depends on p, since $|\mathrm{sinc}(2^{-n-1}\omega)|^{\beta p}$ is bounded below away from 0 on $[-2^n\pi, 2^n\pi]$ by a constant that does not depend on n but only on p. To estimate the remaining product, we remark that, since $\tilde{m}(\omega)$ is bounded, Hölder continuous at the origin and $\tilde{m}(0)=1$, then for all $n \geq p \geq 0$, the products $\prod_{k=p}^n |\tilde{m}(2^{-k}\omega)|^p$ are uniformly bounded on $[-1,1]$ by a constant B_p that is independant of n. For $2^l \leq |\omega| \leq 2^{l+1}$ with $p \leq l < n$, using the hypothesis on $\tilde{m}$, we obtain

$$\begin{aligned}\prod_{k=p}^n |\tilde{m}(2^{-k}\omega)|^p &= \prod_{k=p}^l |\tilde{m}(2^{-k}\omega)|^p \prod_{k=l+1}^n |\tilde{m}(2^{-k}\omega)|^p \\ &= \prod_{k=p}^l |\tilde{m}(2^{-k}\omega)|^p \prod_{k=1}^{n-l} |\tilde{m}(2^{-k-l}\omega)|^p \\ &\leq B_p \prod_{k=p}^l |\tilde{m}(2^{-k}\omega)|^p \\ &\leq B_p \sigma_i^{[\frac{l-p}{i}]p} (\sup_\omega |\tilde{m}(\omega)|)^{(i-1)p} \\ &= D_p \sigma_i^{lp/i} \\ &\leq D_p \sigma_i^{(p/i)\log_2|\omega|} = A_p|\omega|^{(p/i)\log_2 \sigma_i},\end{aligned}$$

where D_p depends only on p (again, in the cases where $l \geq n$ or $l < p$, this still holds by replacing the product which does not make sense by 1). Combining, with the previous estimate, we obtain

$$|\hat{\varphi}^{[n]}(\omega)| \leq K_p(1+|\omega|)^{b+p(\frac{\log_2 \sigma_i}{i}-\beta)}, \tag{2.13}$$

where K_p depends only on p, and thus

$$|\omega|^s|\hat{\varphi}^{[n]}(\omega)| \leq K_p(1+|\omega|)^{b+s+p(\frac{\log_2 \sigma_i}{i}-\beta)}. \tag{2.14}$$

This also holds trivially for $|\omega| > 2^n\pi$. Since we have assumed $b+s+p(\frac{\log_2 \sigma_i}{i}-\beta) < -1$, this gives us the desired uniform L^1 estimate. This concludes the proof of the theorem. □

Remarks

The hypotheses of Theorem 2.2 imply in particular that the degree $d(k)$ of m_k grows at least linearly (m_k has a zero of order βk at $\omega = \pi$). This is not strictly necessary : we show in the appendix that it is possible to obtain strongly converging subdivision schemes with a C^∞ limit function as soon as $d(k)$ tends to $+\infty$ without any assumption on its asymptotic behaviour (but with the assumption $|m(\omega)| \leq 1$ that removes a lot of technicalities).

These hypotheses can also be weakened by assuming that the estimate (2.10) is satisfied only for k sufficiently large : the limit behaviour of the subdivision does not depend on the first iterations.

III. Multiresolution approximation

Let $\{m_k\}_{k>0}$ be a sequence of finite masks that satisfy the hypotheses of Theorem 2.2. We define a sequence of C^∞ compactly supported functions by

$$\hat{\varphi}_j(\omega) = \prod_{k=1}^{+\infty} m_{k+j}(2^{-k}\omega),\ j \geq 0. \tag{3.1}$$

We see that $\varphi_0 = \varphi$ and that φ_j is obtained as the limit of the same subdivision algorithm by cancelling the first j iterations. It follows that φ_j is also in C_0^∞. Since $\hat{\varphi}_j(\omega) = m_{j+1}(\omega/2)\hat{\varphi}_{j+1}(\omega/2)$, we see that this sequence of functions satisfies a serie of recursive refinement equations :

$$\varphi_j(x) = \sum_{|n| \leq d(j+1)} m_{j+1}(n)\varphi_{j+1}(2x - n). \tag{3.2}$$

It is thus natural to define a "half multiresolution analysis" $\{V_j\}_{j\geq 0}$ by $V_j = \text{Span}\{\varphi_j(2^j x - k)\}_{k\in\mathbb{Z}}$. The inclusion $V_j \subset V_{j+1}$ comes from (3.2).

We shall now study the approximation properties of theses spaces in Sobolev spaces. Given a function $f \in H^r$, we can define for $s \leq r$

$$d(f, V_j)_s = \inf_{g\in V_j} \|f - g\|_s, \tag{3.3}$$

where $\|\cdot\|_s$ is the H^s norm. We are concerned here with the behaviour of $d(f, V_j)$ as j goes to $+\infty$. By definition, the spaces V_j have approximation order (resp. density order) r in H^s if $2^{(r-s)j} d(f, V_j)_s$ is bounded (resp. goes to 0) as $j \to +\infty$.

We shall first establish a general result, using a technique introduced in a paper of de Boor, DeVore and Ron (1992). In this paper, the authors are concerned with approximation in the L^2 norm, from shift-invariant spaces. Here, we adapt their technique to the derivation of density orders in Sobolev norms. Approximation orders in Sobolev norms by shift invariant spaces are studied in Zao (1993) and Ron (1993).

Theorem 3.1 *Let $\{\varphi^j\}_{j\geq 0}$ be a sequence of compactly supported functions in H^s for some $s \geq 0$ and define $V_j = \mathrm{Span}\{\varphi_j(2^j x - k)\}_{k\in\mathbb{Z}}$. For $r \geq s$, assume that there exists $t \in]0,\pi]$ such that, for all $0 \leq v \leq s$,*

$$\sup_{|\omega|<t}(|\omega|^{-2r}\frac{\sum_{n\neq 0}|\omega+2n\pi|^{2v}|\hat{\varphi}_j(\omega+2n\pi)|^2}{|\hat{\varphi}_j(\omega)|^2}) \to 0, \text{ as } j \to +\infty. \tag{3.4}$$

Then the spaces V_j have density order r in H^s. More precisely, let P_j be the L^2 projection P_j and S_j the operator defined by $\mathcal{F}S_j f(\omega) = \hat{f}(\omega)\chi_{[-t,t]}(2^{-j}\omega)$, where $\mathcal{F}$ represents the Fourier transform operator ($\mathcal{F}f(\omega) = \hat{f}(\omega)$). Then, for all $f \in H^r$, one has $d(f,V_j) \leq \|P_j S_j f - f\|_s \leq C2^{j(s-r)}\|f\|_r \varepsilon(f,j)$, with $0 \leq \varepsilon(f,j) \leq 1$ and $\lim_{j\to+\infty}\varepsilon(f,j) = 0$.

Proof First, observe that one can always associate with φ_j a function ϕ^j defined by

$$\hat{\phi}_j(\omega) = \frac{\hat{\varphi}_j(\omega)}{(\sum_{n\in\mathbb{Z}}|\hat{\varphi}_j(\omega+2n\pi)|^2)^{1/2}}, \tag{3.5}$$

such that $\{2^{j/2}\phi_j(2^j x - k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of V_j : in the case where $\sum_{n\in\mathbb{Z}}|\hat{\varphi}_j(\omega+2n\pi)|^2 = \sum_k\langle\varphi^j(\cdot)|\varphi^j(\cdot-k)\rangle e^{-ik\omega}$ vanishes at some isolated point, one easily checks that ϕ_j is still the L^2 limit when $\varepsilon \to 0$ of $\phi_{j,\varepsilon}$ defined by

$$\hat{\phi}_{j,\varepsilon}(\omega) = \frac{\hat{\varphi}_j(\omega)}{(\varepsilon + \sum_{n\in\mathbb{Z}}|\hat{\varphi}_j(\omega+2n\pi)|^2)^{1/2}}, \tag{3.6}$$

and that $\phi_{j,\varepsilon}$ is an ℓ^2 combination of $\varphi_j(x-k)$.

Consequently, we can write, for any $f \in L^2$,

$$P_j f(x) = 2^j \sum_{k\in\mathbb{Z}}\langle f|\phi_j(2^j\cdot - k)\rangle\phi_j(2^j x - k). \tag{3.7}$$

For all $j \geq 0$, we define $Q_j = I - P_j$ and $T_j = I - S_j$. We can thus estimate the approximation error in the following way :

$$\begin{aligned} \|P_j S_j f - f\|_s &\leq \|T_j f\|_s + \|P_j S_j f - S_j f\|_s \\ &\leq \|T_j f\|_s + \|S_j P_j S_j f - S_j f\|_s + \|T_j P_j S_j f\|_s \\ &\leq \|T_j f\|_s + \|S_j Q_j S_j f\|_s + \|T_j P_j S_j f\|_s. \end{aligned}$$

Let f be in H^r, i.e. $\|f\|_r^2 = (2\pi)^{-1} \int |\hat{f}(\omega)|^2 (1 + |\omega|^{2r}) d\omega < +\infty$. We shall examine separately these three quantities and prove that they all satisfy the estimate that we want for $d(V_j, f)_s$.

The "truncation error" $\|T_j f\|_s$ is independant of the approximating subspaces V_j. It is clear that we have

$$\begin{aligned} \|T_j f\|_s^2 &= (2\pi)^{-1} \int_{|\omega|>2^j t} |\hat{f}(\omega)|^2 (1 + |\omega|^{2s}) d\omega \\ &\leq (2\pi)^{-1} 2^{2j(s-r)} t^{2(s-r)} \int_{|\omega|>2^j t} |\hat{f}(\omega)|^2 (1 + |\omega|^{2r}) d\omega \\ &\leq C 2^{2j(s-r)} \|f\|_r^2 \varepsilon(f, j), \end{aligned}$$

with $0 \leq \varepsilon(f, j) \leq 1$ and $\varepsilon(f, j) \to 0$ as $j \to +\infty$.

For the second term, we have

$$\begin{aligned} \|S_j Q_j S_j f\|_s^2 &= (2\pi)^{-1} \int_{|\omega|<2^j t} |\mathcal{F} Q_j S_j f(\omega)|^2 (1 + |\omega|^{2s}) d\omega \\ &\leq (2\pi)^{-1} (1 + 2^{2js} t^{2s}) \int_{|\omega|<2^j t} |\mathcal{F} Q_j S_j f(\omega)|^2 d\omega \\ &\leq C 2^{2js} \|S_j Q_j S_j f\|_0^2. \end{aligned}$$

To estimate $\|S_j Q_j S_j f\|_0^2$, we note that

$$\begin{aligned} \mathcal{F} P_j S_j f(\omega) &= \hat{\phi}_j(2^{-j}\omega) \sum_{k\in\mathbb{Z}} \langle S_j f | \phi_j(2^j \cdot - k)\rangle e^{-i2^{-j}k\omega} \\ &= (2^{j+1}\pi)^{-1} \hat{\phi}_j(2^{-j}\omega) \sum_{k\in\mathbb{Z}} \langle \mathcal{F} S_j f(\cdot) | \hat{\phi}_j(2^{-j} \cdot) e^{i2^{-j}k\cdot}\rangle e^{-i2^{-j}k\omega}. \end{aligned} \tag{3.8}$$

Since the above sum defines a $2^{j+1}\pi$-periodic function, which coincides on $[-2^j\pi, 2^j\pi]$ with $\hat{f}(\omega)\chi_{[-t,t]}(2^{-j}\omega)\overline{\hat{\phi}(2^{-j}\omega)}$, it follows that, on the interval $[-2^j\pi, 2^j\pi]$,

$$\mathcal{F} P_j S_j f(\omega) = |\hat{\phi}_j(2^{-j}\omega)|^2 \hat{f}(\omega) \chi_{[-t,t]}(2^{-j}\omega).$$