Quasilinear subdivision schemes with applications to ENO interpolation

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Abstract

We analyze the convergence and smoothness of certain class of nonlinear subdivision schemes. We study the stability properties of these schemes and apply this analysis to the specific class based on ENO and weighted-ENO interpolation techniques. Our interest in these techniques is motivated by their application to signal and image processing.

Introduction

Subdivision schemes are a powerful tool for the fast generation of curves and surfaces in computer-aided geometric design. In such algorithms discrete data are recursively generated from coarse to fine scales by means of local rules. The stability and the convergence of such refinement process, as well as the smoothness properties of its limit function if it exists, have been the subject of active research in recent years. We refer to [6] and [19] for general surveys on subdivision algorithms, and e.g. to [14], [15], [22] for more specialized results on their convergence and smoothness.

An important motivation for the study of subdivision algorithms is their relation to multiresolution analysis and wavelets (see e.g.[11] or [13]). In particular, the contribution of a single wavelet coefficient in the representation of a discrete signal is precisely obtained by applying a subdivision scheme from the scale of the coefficient up to the signal discretization scale. Therefore, understanding the stability and smoothness of subdivision algorithms is fundamental in the context of applications of wavelets to data compression or signal denoising, in which certain coefficients are quantized or discarded.

In all those instances of subdivision schemes which have been analyzed so far, the refinement process is based on linear rules. The present work is concerned with the situation where this rule is nonlinear in the sense that the refinement operator depends itself on the data to be refined .

Our main motivation for such a study is the analysis of nonlinear multiresolution representations introduced by Ami Harten [23] in the context of the numerical simulation of conservation laws. As we recall in more details, these representations are based on nonlinear refinement rules which involve a data dependent stencil selection. The goal of this stencil selection is to make the refinement process more accurate in the presence of isolated singularities such as discontinuities. It is no surprise that these ideas have recently been applied to image compression. In this context, it is hoped that a better adapted treatment of the singularities corresponding to edges might improve the sparsity of the multiscale representations of images, and in turn the rate/distortion performance of compression algorithms based on such representations (see [12], [17],[21] and [26] for several results which relates the sparsity of the representation to concrete rate/distortion bounds). Some first numerical results, all based on tensor product techniques, which do confirm this intuition are available in [2], [3], [8], and [9].

From a mathematical point of view, edges are indeed the main limitation to the performance of wavelet based coding: this is reflected by the poor decay - $\mathcal{O}(N^{-1/2})$ - of the error of L^2 best wavelet N-term approximation for a "sketchy image function" $f = \chi_\Omega$, where Ω is a bounded domain with a smooth boundary. This reflects the fact that this type of approximation essentially provide local isotropic refinement near the edges. Improving on this rate through a better choice of the representation has motivated the recent development of *ridgelets* in [4] and of *curvelets* in [5] which are bases and frames having some anisotropic features, resulting in the better rate $\mathcal{O}(N^{-1})$.

Nonlinear multiscale representations are another possible track for such improvements, provided that one can overcome two difficulties: firstly, for a proper anisotropic adaptation to the edges, it is crucial to develop nonlinear methods which are not based on tensor products, and secondly, one needs to control the stability of these representations. This second point is crucial: since nonlinear multiscale representations cannot be thought as decompositions of the signal into a fixed wavelet basis, the error produced by thresholding or quantizing the coefficients is no more clearly understood: such perturbations might be greatly amplified by the iteration of the nonlinear refinement rules involved in the prediction process. In order to solve this problem, we essentially need to understand the behavior of the nonlinear subdivision schemes corresponding to these iterative refinements.

The objective of the present paper is to provide appropriate tools for analyzing the smoothness and stability of quasilinear subdivision schemes, and apply these tools in the particular case of the essentially non oscillatory (ENO) refinements introduced in [24].

The results of this paper represents the first step in the study of nonlinear multiscale representations. Using these results, our next perspective, is the analysis of data compression algorithms based on such nonlinear representations.

Our work is organized as follows. A quick overview of the framework introduced in [23] is given in Section 1, together with several relevant examples of quasilinear schemes. In Section 2 and 3, we prove several results concerning the smoothness and stability analysis of quasilinear subdivision schemes, in the uniform and Hölder metric. In Section 4, we apply these results to the particular example of the four points ENO and WENO refinement rules. Finally, an appendix is devoted to the generalization of the results of Section 2 and 3, to other smoothness and error measures, such as L_p, Sobolev or Besov norms.

Motivation and Background

The framework introduced by A. Harten [23] for the discrete multiresolution representations of data is based on two interscales discrete operators: *the projection and the prediction operators* .

The projection operator P_{j-1}^{j} acts from fine to coarse level of resolution. This operator extracts from v^j, the data string at the level j of discretization, the discrete information at the coarser level of resolution, $j-1$, i.e. v^{j-1}. The prediction operator P_j^{j-1}, acts from coarse to fine level of resolution. It yields an *approximation* of the the discrete vector v^j from the projected vector v^{j-1}. These two operators should in addition satisfy the property

$$P_{j-1}^{j} P_j^{j-1} = I, \tag{1}$$

i.e., the projection operator is a left inverse to the prediction operator.

The approximation build by P^j_{j-1} is defined as follows

$$\hat{v}^j := P^j_{j-1} v^{j-1}.$$

This gives the redundant representation of the vector v^j by its *approximation* $\hat{v}^j$ and the prediction error

$$e^j := \hat{v}^j - v^j.$$

From (1) , we have that P^{j-1}_j is onto, and that the prediction error belongs to the finite dimensional space W^{j-1}, defined as the null space of the projection operator. Therefore by decomposing e^j in terms of a basis of W^{j-1}, we can eliminate the redundant information in e^j. We denote by d^{j-1} the coordinate vector of the error vector in this basis of W^{j-1}. In analogy with the wavelet terminology we call d^{j-1} the detail vector. Since $\hat{v}^j = P^j_{j-1} v^j$ can be equivalently characterized by (v^{j-1}, d^{j-1}). By iteration we obtain a one to one correspondence between v^j and its multiresolution representation $(v^0, d^0, \ldots, d^{j-1})$.

If both discrete operators, projection and prediction are linear, then the corresponding multiresolution transform is equivalent to a biorthogonal wavelet transform.

Some of the prediction operators proposed by Harten [23] are nonlinearly data dependent since they are based on *essentially non-oscillatory* (ENO) prediction techniques. By using them, the corresponding multiresolution transforms cannot be thought as a change of basis, which make the analysis of these transforms more difficult.

The representations introduced by Harten are formulated for specific types of discretization, often used in computational applications (e.g., the point values and the cell averages discretization). The selection of the discretization depends on the problem under consideration, e.g., for the image modelisation by square integrable functions, an appropriate choice of the discretization is by the cell averages (instead of point values discretization, which does not make sense in this case). In the following, we briefly evoke the nonlinear prediction operators based on ENO, in the point value and cell averages context.

Example 1. Point Value Multiresolution
In this setting, we interpretate the discrete vector $v^j = (v^j_k)_{k\in\mathbb{Z}}$ as the

point values of a continuous function v on the grid $\Gamma^j := (2^{-j}k)_{k\in\mathbb{Z}}$, i.e. $v_k^j := v(2^{-j}k)$. This suggest the choice for P_{j-1}^j as the simple downsampling operator. For the prediction operator, we notice that the vector $\hat{v}^j$ should coincide with v^j on the coarse grid; then building prediction operator can be viewed as an interpolation problem. The details are defined by the restriction of the interpolation error e^j on $\Lambda^{j-1} := \Gamma^j \setminus \Gamma^{j-1}$, i.e.

$$d^{j-1} := (v_k^j - \hat{v}_k^j)_{k\in\Lambda^{j-1}}.$$

In the sequel, we present an important class of local predictors obtained by Lagrange interpolation.

At scale j we want to predict for each $k \in \mathbb{Z}$ the value $\hat{v}_{2k+1}^j$ from the values $(v_l^{j-1})_{l\in\mathbb{Z}}$. To such a k we associate a *prediction stencil* of length M

$$S_r(k) := \{(k-r)2^{-j+1}, \dots, (k-r+M-1)2^{-j+1}\},$$

with r an integer representing the position of the stencil with respect to k. Using the values $(v(\gamma))_{\gamma\in S_r(k)}$, we define $p^r \in \Pi_M$ as the unique polynomial of degree M which interpolates the values of v on $S_r(k)$. We then define the predicted value

$$\hat{v}_{2k+1,r}^j := p^r((2k+1)2^{-j+1}).$$

Note that M is exactly the order of accuracy of the prediction. If the parameter r is fixed independently of the data, we obtain a linear prediction operator, and the multiresolution transform is then equivalent to a biorthogonal interpolatory wavelet transform, for which the dual scaling function is the Dirac distribution.

The goal of ENO interpolation is to obtain a better adapted prediction near the singularities of the data. The idea is to select by some prescribed numerical criterion the polynomial p^r which is the *least oscillatory* in the neighborhood of k.

We give below the formulae for the third order accurate prediction ($M = 4$). The predicted values $\hat{v}_{2k+1,r}^j$ using $S_r(k)$, $r = 0, 1, 2$, are obtained respectively by

$$\left\{\begin{array}{lll} \hat{v}_{2k+1,0}^j & := & \frac{5}{16}v_k^{j-1} + \frac{15}{16}v_{k+1}^{j-1} - \frac{5}{16}v_{k+2}^{j-1} + \frac{1}{16}v_{k+3}^{j-1}, \\ \hat{v}_{2k+1,1}^j & := & -\frac{1}{16}v_{k-1}^{j-1} + \frac{9}{16}v_k^{j-1} + \frac{9}{16}v_{k+1}^{j-1} - \frac{1}{16}v_{k+2}^{j-1}, \\ \hat{v}_{2k+1,2}^j & := & \frac{1}{16}v_{k-2}^{j-1} - \frac{5}{16}v_{k-1}^{j-1} + \frac{15}{16}v_k^{j-1} + \frac{5}{16}v_{k+1}^{j-1}. \end{array}\right. \tag{2}$$

In the case of prediction by the value of the unique cubic polynomial that interpolates v^j on the centered stencil, the corresponding multiresolution transform is equivalent to the Dubuc–Deslaurier interpolatory wavelet transform (see [16] and [20]). For the properties of the interpolant as well as for the smoothness of the limit of this iterative process, we refer the reader to [14], [20] and [16].

Example 2. Cell average multiresolution

In the cell average context, $\mathbb{R}$ is partitioned in disjointed dyadic cells $\Gamma^j := \{\Gamma_k^j = [k2^{-j}, (k+1)2^{-j})\}_{k\in\mathbb{Z}}$. In this context, the discrete vector v^j is viewed as the average $\left(2^j \int_{\Gamma_k^j} v(t)\, dt\right)_{k\in\Gamma^j}$ of a locally integrable function.

As in the point values setting, this suggests to take for P_{j-1}^j the averaging operator. The construction of the prediction operator is similar to the prediction in the point values setting. To each Γ_k^{j-1}, we associate a stencil of cells

$$S_r(k) := \{[(k-r)2^{-j+1}, (k-r+1)2^{-j+1}], \ldots, [(k-r+M-1)2^{-j+1}, (k-r+M)2^{-j+1}]\}.$$

Using the averages within the stencil $S_r(k)$, we define $q^r \in \Pi_{M-1}$ as the unique polynomial of degree $M-1$ which interpolates these averages.

We then define the predicted averages as those of q^r on the half intervals $[2^{-j+1}2k, 2^{-j+1}(2k+1)]$ and $[2^{-j+1}(2k+1), 2^{-j+1}(2k+2)]$.

Notice that by using the averages of a local integrable function we can obtain the point values of its primitive function. This interpretation allows to obtain the polynomial used to make the prediction in cell averages context through a derivation of the prediction polynomial used in the point values setting for the primitive function.

The multiresolution decomposition based on cell averages is equivalent to the biorthogonal wavelet transform, for which the dual scaling function is the box function, [18].

We can also make the same remarks concerning the possibility of using ENO-type reconstructions. In the case of two order accurate prediction based on Lagrange interpolation, the predicted averages $\hat{v}_{2k,r}^j$ using $S_r(k)$, $r = 0, 1, 2$, are given by

$$\left\{\begin{array}{lcl} \hat{v}_{2k,0}^j & := & \frac{11}{8}v_k^{j-1} - \frac{1}{2}v_{k+1}^{j-1} + \frac{1}{8}v_{k+2}^{j-1}, \\ \hat{v}_{2k,1}^j & := & \frac{1}{8}v_{k-1}^{j-1} + v_k^{j-1} - \frac{1}{8}v_{k+1}^{j-1}, \\ \hat{v}_{2k,2}^j & := & -\frac{1}{8}v_{k-2}^{j-1} + \frac{1}{2}v_{k-1}^{j-1} + \frac{5}{8}v_k^{j-1}. \end{array}\right. \tag{3}$$

In both types of discretization, the details are defined as the prediction error at the odd samples.

Weighted-ENO interpolation . The weighted-ENO (WENO) interpolation developed in [7] is based on the ENO idea. In this technique, in contrast to ENO interpolation which uses only one of the candidate stencils to make the prediction, one consider a convex combination of the polynomials associated to these stencils, i.e.

$$\hat{v}_k := \sum_{r=0}^{M-1} \alpha_r \hat{v}_k^r,$$

with $\alpha_r \geq 0$ and $\sum_{r=0}^{M-1} \alpha_r = 1$. In ENO interpolation, a small round-off error perturbation of the data can result in changing the selected stencils. This situation is avoided in WENO interpolation which provides with a smooth transition between the stencils. A possible form of the weights is given in [7] by

$$\alpha_r := \frac{a_r}{\sum_{l=0}^{M-1} a_l}, \quad r = 0, \ldots, M-1,$$

where

$$a_r := \frac{d_r}{(\epsilon + b_r)^2}, \quad \text{and} \quad b_r := \sum_{l=1}^{M-1} 2^{-j(2l-1)} \int_{\Gamma_k^j} \Big(\frac{\partial^l p^r(x)}{\partial^l x}\Big)^2 \, dx. \tag{4}$$

The d_r are fixed positive constants. The b_r are defined by the sum of the squares of L^2 norms for all the derivatives of the interpolation polynomial p_r over the interval Γ_k^j. The factor $2^{-j(2l-1)}$ is introduced to remove any level dependency on the derivatives. Here ϵ is introduced in order to avoid the denominator to vanish, b_r are the so called "smoothness indicators" of the stencil $S_r(k)$: if the function $v(x)$ is smooth inside the stencil $S_r(k)$, then $b_r \sim \mathcal{O}(2^{-2j})$, else if the function has a discontinuity inside the stencil $S_r(k)$, then $b_r \sim \mathcal{O}(1)$.

The rational form of the weights is chosen in order to emulate the ENO idea and to be computationally efficient. If the stencil S_r is located in a smooth region, the smoothness indicator b_r is close to 0 and then the weight α_r is close to 1. In contrast, if the stencil contains a singularities the smoothness indicator b_r is larger and the weight α_r is closer to 0.

In the case of four point interpolatory schemes, we compute the predicted value as a convex combination of the predicted values by the three stencils, as follows

$$\hat{v}^j_{2k+1} := \alpha_0 \hat{v}^j_{2k+1,0} + \alpha_1 \hat{v}^j_{2k+1,1} + \alpha_2 \hat{v}^j_{2k+1,2} \tag{5}$$

where α_0, α_1 and α_2 represent the weights associated to the right, centered and left stencil, respectively. More precisely

$$\begin{array}{rl} \hat{v}^j_{2k+1} := & \frac{\alpha_2}{16} v^{j-1}_{k-2} - \frac{5\alpha_2+\alpha_1}{16} v^{j-1}_{k-1} + (1 + \frac{5\alpha_2+2\alpha_1}{8}) v^{j-1}_k \\ & +(1 + \frac{5\alpha_0+2\alpha_1}{8}) v^{j-1}_{k+1} - \frac{5\alpha_0+\alpha_1}{16} v^{j-1}_{k+2} + \frac{\alpha_0}{16} v^{j-1}_{k+3}, \end{array} \tag{6}$$

The weights associated to the three stencils are defined as in [25] and [7]. In this case ($M = 3$), (4) gives

$$\left\{ \begin{array}{ll} b_0 & := c_{0,0}(v^j_{k+2} - 2v^j_{k+1} + v^j_k)^2 + c_{0,1}(v^j_{k+2} - 4v^j_{k+1} + 3v^j_k)^2, \\ b_1 & := c_{1,0}(v^j_{k+1} - 2v^j_k + v^j_{k-1})^2 + c_{1,1}(v^j_{k+1} - v^j_{k-1})^2, \\ b_2 & := c_{2,0}(v^j_k - 2v^j_{k-1} + v^j_{k-2})^2 + c_{2,1}(3v^j_k - 4v^j_{k-1} + v^j_{k-2})^2, \end{array} \right. \tag{7}$$

where $c_{i,j}$, $i = 0,1,2, j = 0,1$, are fixed positive constants. Some possible choices of the constants are suggested in [7].

As we already explained, stability of the multiresolution transform is a key issue in applications where some coefficients are discarded (such as compression or denoising). In this paper, we limit our study to the nonlinear subdivision scheme corresponding to the iterative application of prediction operator, from coarse to fine scales, without adding any details. To begin with, we give some basic notations and definitions and recall some properties of the subdivision operators.

A subdivision scheme defines a function (called the limit function) as the limit of a subdivision process in which an initial finite set of points, called the control points, is recursively refined.

Definition 1 *A data dependent subdivision rule is an operator valued function S which associates to each $v \in \ell_\infty(\mathbb{Z})$ a linear operator*

$$S(v) : \ell_\infty(\mathbb{Z}) \to \ell_\infty(\mathbb{Z}),$$

defined by a rule of the type

$$(S(v)w)_k := \sum_l a_{k,l}(v) w_l, \tag{8}$$

where the coefficients $a_{k,l}(v)$ are zero if $|k-2l| > M$ for some fixed $M > 0$.

We define the associated quasilinear subdivision scheme as the recursive action of the quasilinear rule $Sv := S(v)v$ on an initial set of data v^0, according to

$$v^j := Sv^{j-1} = S(v^{j-1})v^{j-1},\ j \geq 1. \tag{9}$$

In the above definition, M typically represents the size of the stencil used in the subdivision rule. For linear subdivision schemes, the coefficients $a_{k,l}$ do not depend on the data v, i.e. $S(v) = S$ a fixed operator. For linear and uniform subdivision schemes, these coefficients have the form $a_{k,l} = a_{k-2l}$.

The analysis of a subdivision scheme consists of establishing conditions for the convergence of the scheme, and in characterizing the smoothness as well as the order of approximation of the set of limit functions. We refer the reader to [6],[19] and [22] for a general survey on this subject, in the linear and uniform case.

Definition 2 *A subdivision scheme, generating recursively the data $\{v^j : j \in \mathbb{Z}_+\}$, is called uniformly convergent if, for every set of initial control points $v^0 \in \ell_\infty(\mathbb{Z})$, there exists a continuous function $f \in C(\mathbb{R})$, called the limit function, such that*

$$\lim_{j\to+\infty} \sup_{k\in\mathbb{Z}} |v_k^j - f(2^{-j}k)| = 0, \tag{10}$$

and that f is non-trivial at least for one initial data v^0.

We also associate a function f^j to the data v^j as the piecewise affine interpolation to $\{(2^{-j}k, v^j) : j \in \mathbb{Z}_+\}$. Thus

$$f^j(x) := \sum_{k\in\mathbb{Z}} v_k^j \varphi(2^j x - k), \tag{11}$$

where $\varphi(x) := \max\{1-|x|, 0\}$ is the hat function. It is clear that the uniform convergence of the subdivision scheme is equivalent to

$$\lim_{j\to+\infty} \|f^j - f\|_{L_\infty} = 0.$$

The limit function f is denoted by $S^\infty v^0$. The following definition plays an important role in the analysis of subdivision schemes.

Definition 3 *Let $N \geq 0$ be a fixed integer. The data dependent subdivision rule has the property of polynomial reproduction of order N if for all $u \in \ell_\infty(\mathbb{Z})$ and $P \in \Pi_N$ there exists $\widetilde{P} \in \Pi_N$ with $P - \widetilde{P} \in \Pi_{N-1}$ such that $S(u)p = \widetilde{p}$ where p and $\widetilde{p}$ are defined by $p_k = P(k)$ and $\widetilde{p}_k = \widetilde{P}(\frac{k}{2})$.*

In particular, the ENO and WENO schemes discussed in the previous section satisfy such a property up to the order M for point values and $M-1$ for cell averages. We recall the definition of the n-th order forward finite difference operator,

$$(\Delta^n v)_k = \sum_{m=0}^{n} (-1)^m \binom{n}{m} v_{k+m}. \tag{12}$$

For the first order finite difference we omit the superscript 1. In the case of linear subdivision scheme, using a formalism based on Laurent polynomials [19], it has been proved that if the subdivision scheme has the property of polynomial reproduction up to the order N, then there exist similar schemes for the differences of order $n := 1, \cdots, N+1$

$$S_n : \ell_\infty(\mathbb{Z}) \to \ell_\infty(\mathbb{Z}), \quad \Delta^n(Sv) = S_n(\Delta^n v).$$

The convergence and smoothness properties of a subdivision schemes are then studied through the contraction properties of the schemes S_n. More precisely, denoting by $\rho_\infty(A)$ the spectral radius of an operator A in ℓ_∞, the uniform convergence of the linear subdivision is equivalent to the property $\rho_\infty(S_1) < 1$. Moreover, if for some $m \in \{1, \cdots, N+1\}$, we have $\rho_\infty(S_m) < 2^{-m+1}$, then the limit function is in C^s for all $s < s^* = -\frac{\log \rho_\infty(S_m)}{\log 2}$ (and therefore $m-1$ times differentiable since $s^* > m-1$).

In order to study quasilinear subdivision schemes, we need to introduce some additional definitions. We start with the boundedness property.

Definition 4 *A data dependent subdivision rule is called bounded if there exists a constant $B > 0$ such that for all $v \in \ell_\infty(\mathbb{Z})$,*

$$\|S(v)\|_{\ell_\infty} \leq B, \tag{13}$$

where the norm stands for the ℓ_∞ operator norm.

Clearly, this property can also be expressed by saying that the coefficients $\{a_{k,l}(v)\}$ are bounded independently of k, l and v. In the following, we always assume that the rules that we study are bounded.

We have already remarked that, in the WENO technique, the transition between two stencils is made in a continuous way. This property is crucial in the study of the stability of quasilinear subdivision schemes. This notion is expressed in the next definition.

Definition 5 *A data dependent subdivision rule is called continuously dependent on the data if for every $v, w \in \ell_\infty(\mathbb{Z})$, the associated operators $S(v)$ and $S(w)$ satisfy*

$$\|S(v) - S(w)\|_{\ell_\infty} \leq C\|v - w\|_{\ell_\infty}, \tag{14}$$

where C dependents in a non-decreasing way on $\max\{\|v\|_{\ell_\infty}, \|w\|_{\ell_\infty}\}$.

The fact that the constant C might grow with $\|v\|_{\ell_\infty}$ and $\|w\|_{\ell_\infty}$ is encountered in the practical examples that we have in mind such as WENO interpolation.

We finally introduce the notion of joint spectral radius associated to a data dependent subdivision rule.

Definition 6 *The joint spectral radius of a data dependent subdivision rule S is the number*

$$\rho_\infty(S) := \limsup_{j\to\infty} \sup_{(u^0,u^1,\ldots,u^{j-1})\in(\ell_\infty(\mathbb{Z}))^j} \|S(u^{j-1})\cdots S(u^0)\|_{\ell_\infty}^{\frac{1}{j}}.$$

In other words, $\rho_\infty(S)$ is the infimum of all $\rho > 0$ for which there exists $C > 0$ such that for all arbitrary $(u^j)_{j\geq 0}$ in ℓ_∞ and $v \in \ell_\infty$ one has

$$\|S(u^{j-1})\cdots S(u^0)v\|_{\ell_\infty} \leq C\rho^j\|v\|_{\ell_\infty}, \tag{15}$$

for all $j \geq 0$. Note that in the case of linear subdivision schemes, this is exactly the spectral radius of S in ℓ_∞.

Convergence and Smoothness analysis

In this section, we provide sufficient conditions for the convergence of quasilinear subdivision schemes and for the smoothness of the limit function. In fact the results in this section, but not those of the next section, apply to a wider class of subdivision schemes than the class of quasilinear subdivision schemes. In this class, a scheme is defined by a data dependent rule , S, and by given sequence of data $\{u^l : l \in \mathbb{Z}_+\}$ and initial data v^0 according to $v^j := S(u^{j-1}) \cdots S(u^0)v^0$. As in the linear case, the results of this section are obtained through the study of the associated schemes for the differences. The existence of the scheme for the differences is obtained by using the property of polynomial reproduction of the data dependent rule. This result is given in the next proposition.

Proposition 1 *Let S be a data dependent subdivision rule which reproduces polynomials up to degree N. Then for $1 \leq n \leq N+1$ there exists data dependent subdivision rule S_n with the property that for all $v, w \in \ell_\infty$,*

$$\Delta^n S(v)w := S_n(v)\Delta^n w.$$

Proof :
Let $1 \leq n \leq N+1$ and let $u := S(v)w$. Combining (12) and (8), we obtain

$$(\Delta^n u)_k = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \sum_{l \text{ s.t. } |k+m-2l| \leq M} a_{k+m,l}(v) w_l. \tag{16}$$

Therefore, $(\Delta^n u)_k$ can be written as a linear combination of the w_l

$$(\Delta^n u)_k = \sum_l b_{k,l}(v) w_l, \tag{17}$$

where $b_{k,l}(v) := \sum_{m=0}^{n} (-1)^m \binom{n}{m} a_{k+m,l}(v)$. Note that $b_{k,l}(v)$ is zero for $l < (k-M)/2$ and $l > (k+n+M)/2$. For each fixed k we thus have a finite vector $(b_{k,l}(v))_{l \in E_k}$ with $E_k := \{l : (k-M)/2 \leq l \leq (k+n+M)/2\}$.

Since the rule reproduces polynomials of degree up to N, we have

$$\sum_{|k-2l| \leq M} a_{k,l}(v) l^m = P_m(k), \quad 0 \leq m \leq n-1, \tag{18}$$

with $P_m \in \Pi_m$. Applying the n-th order finite difference operator Δ^n on this identity, we obtain

$$\sum_l b_{k,l}(v) l^m = 0, \quad m = 0, \ldots, n-1. \tag{19}$$

Therefore, for each k $(b_{k,l}(v))_{l\in E_k}$ is orthogonal to the vectors $(l^m)_{l\in E_k}$ for $m = 0, \ldots, n-1$. It follows that $(b_{k,l}(v))_{l\in E_k}$ can be written in terms of a basis of the orthogonal complement of $\mathbf{span}\{(l^m)_{l\in E_k} | m = 0, \ldots, n-1\}$. A natural choice for this basis is given by

$$\begin{array}{l} e_0(l) := \binom{n}{l}(-1)^{n+l}, \quad \text{if} \quad l = 0, \ldots n, \\ e_0(l) := 0, \quad \text{if} \quad l \notin \{0, \ldots, n\}, \end{array}$$

and taking $e_q(l) := e_0(l-q)$ with $(k-M)/2 \leq q \leq (k-n+M)/2$. Therefore, we have

$$b_{k,l}(v) := \sum_{(k-M)/2 \leq q \leq (k-n+M)/2} \beta_{k,q}(v) e_q(l), \tag{20}$$

from which we derive a subdivision rule for the n-the order differences of the type

$$(\Delta^n u)_k = \sum_l b_{k,l}(v) w_l = \sum_{|k-2q| \leq M} \beta_{k,q}(v)(\Delta^n w)_q. \tag{21}$$

□

Notice from the above proof that the the stencils used in S_n are always smaller than those used in S. Moreover, if S is bounded (resp. continuously dependent on the data), then S_n is also bounded (resp. continuously dependent on the data). The next result gives a relation between the joint spectral radius of these schemes.

Proposition 2 *For all $n = 0, \cdots, N$, one has $\rho_\infty(S_{n+1}) \geq \rho_\infty(S_n)/2$.*

Proof:
We shall prove that $\rho_\infty(S_1) \geq \rho_\infty(S)/2$, and the general result will follow by induction. Let $\rho > \rho_\infty(S_1)$, and $C > 0$ such that for all sequence $(u^l)_{l\geq 0}$ in ℓ_∞ and $v \in \ell_\infty$ one has

$$\|S_1(u^{j-1}) \cdots S_1(u^0)\Delta v\|_{\ell_\infty} \leq C\rho^j \|\Delta v\|_{\ell_\infty}, \tag{22}$$

for all $j > 0$. Defining

$$w^j := S(u^{j-1}) \cdots S(u^0)v, \tag{23}$$

it follows that

$$\|\Delta w^j\|_{\ell_\infty} \leq C\rho^j \|\Delta v\|_{\ell_\infty}. \tag{24}$$

We use the relation

$$\|w^j\|_{\ell_\infty} = \sup_{k\in\mathbb{Z}} \sup\{|w^j_l| : l \in [2^j k, 2^j(k+1))\}, \tag{25}$$

and exploit the fact that the scheme is local. The values of w^j_l for $l \in [2^j k, 2^j(k+1))$ only depend on those of v_l for $|l-k| \le M$. For a fixed k, we define $\tilde{v}$ by $\tilde{v}_l = v_l$ if $|l-k| \le M$ and $\tilde{v}_l = 0$ otherwise, and we let $\tilde{w}^j := S(u^{j-1})\cdots S(u^0)\tilde{v}$. It follows that $w^j_l = \tilde{w}^j_l$ for $l \in [2^j k, 2^j(k+1))$ and that $\tilde{w}^j_l = 0$ for $|l - 2^j k| > 2^j 2M$. In turn, we obtain that

$$\begin{aligned}
\sup_{l\in[2^j k,2^j(k+1))} |w^j_l| &= \sup_{l\in[2^j k,2^j(k+1))} |\tilde{w}^j_l| \\
&\le \sum_{|l-2^j k|<2^j 2M} |\Delta\tilde{w}^j_l| \\
&\le C2^j \|\Delta\tilde{w}^j_l\|_{\ell_\infty} \\
&\le C(2\rho)^j \|\Delta\tilde{v}\|_{\ell_\infty} \\
&\le 2C(2\rho)^j \|v\|_{\ell_\infty}
\end{aligned}$$

It follows that $\|w^j\|_{\ell_\infty} \le 2C(2\rho)^j \|v\|_{\ell_\infty}$, and thus $\rho_\infty(S) \le 2\rho$. Letting ρ tendto $\rho_\infty(S_1)$, we obtain the claimed result. □

Note that convergence of the subdivision scheme implies $\rho(S) \ge 1$ since otherwise $S^\infty v^0 = 0$ for all initial data v^0. Therefore, the above result shows that we always have

$$\rho_\infty(S_n) \ge 2^{-n}. \tag{26}$$

We are now ready to establish a sufficient condition for the convergence of quasilinear subdivision schemes and for the C^s smoothness of the limit function with $s < 1$.

Theorem 1 *Let S be a data dependent subdivision rule which reproduces constants. If the rule for the differences satisfies $\rho_\infty(S_1) < 1$, then the quasi-linear subdivision scheme based on S is uniformly convergent and the limit function $S^\infty v^0$ is C^s for all $s < -\frac{\log \rho_\infty(S_1)}{\log 2}$.*

Proof:
Let ρ be such that $\rho_\infty(S_1) < \rho < 1$. There exists a constant C such that for all initial data $v^0 \in \ell_\infty$ and $j \ge 0$,

$$\|\Delta v^j\|_{\ell_\infty} \le C\rho^j \|\Delta v^0\|_{\ell_\infty}. \tag{27}$$

Observe that

$$\|f^{j+1} - f^j\|_{L_\infty} \leq \sup_{k\in\mathbb{Z}} |v_{2k}^{j+1} - v_k^j|, |v_{2k+1}^{j+1} - \frac{v_k^j + v_{k+1}^j}{2}|. \tag{28}$$

We now write

$$v_{2k}^{j+1} - v_k^j = \sum_{l\in F_k} c_{k,l} v_l^j \tag{29}$$

and

$$v_{2k+1}^{j+1} - \frac{v_k^j + v_{k+1}^j}{2} = \sum_{l\in F_k} d_{k,l} v_l^j \tag{30}$$

where $F_k := \{l \ ; \ |k-l| \leq M\}$, $c_{k,l} := a_{2k,l} - \delta(k-l)$ and $d_{k,l} := a_{2k+1,l} - \frac{\delta(k-l)+\delta(k+1-l)}{2}$ Since our scheme reproduces constants, the vectors $(c_{k,l})_{l\in F_k}$ and $(d_{k,l})_{l\in F_k}$ are orthogonal to the constant vector. By the same reasoning as in the proof of Proposition 1, we conclude that both $v_{2k}^{j+1} - v_k^j$ and $v_{2k+1}^{j+1} - \frac{v_k^j+v_{k+1}^j}{2}$ are linear combinations of the finite differences Δv_l^j for $l = k - M, \cdots, k+M-1$ From this it follows that

$$\|f^{j+1} - f^j\|_{L_\infty} \leq C\|\Delta v^j\|_{\ell_\infty} \leq C\rho^j \|\Delta v^0\|_{\ell_\infty}. \tag{31}$$

Therefore the sequence f_j converges uniformly to a continuous limit $f = S^\infty v^0$. We also see that

$$\begin{aligned} \|f\|_{L_\infty} &\leq \|f^0\|_{L_\infty} + \textstyle\sum_{j\geq 0} \|f^{j+1} - f^j\|_{L_\infty} \\ &\leq C(\|v^0\|_{\ell_\infty} + \|\Delta v^0\|_{\ell_\infty}) \leq C\|v^0\|_{\ell_\infty}. \end{aligned}$$

In order to prove that $f \in C^s$ it suffices to evaluate $|f(x) - f(y)|$ for $|x-y| \leq 1$. Let j be such that $2^{-j-1} < |x-y| \leq 2^{-j}$. We then write

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f^j(x)| + |f(y) - f^j(y)| + |f^j(x) - f^j(y)| \\ &\leq 2\|f - f^j\|_{L_\infty} + |f^j(x) - f^j(y)| \\ &\leq C\rho^j\|\Delta v^0\|_{\ell_\infty} + 2^{-j}\|(f^j)'\|_{L_\infty} \\ &\leq C\rho^j\|\Delta v^0\|_{\ell_\infty} + \|\Delta v^j\|_{\ell_\infty} \\ &\leq C\rho^j\|\Delta v^0\|_{\ell_\infty} \\ &\leq C|x-y|^s\|\Delta v^0\|_{\ell_\infty}, \end{aligned}$$

with $s := -\log(\rho)/\log 2$. This concludes the proof. □

In the following, we give sufficient conditions for the C^s smoothness of the limit function for $s \geq 1$.

Theorem 2 *Let S be a data dependent subdivision rule which reproduces polynomials up to degree N. If the rule for the differences satisfies $\rho_\infty(S_{n+1}) < 2^{-n}$ for some $n \in \{0, \cdots N\}$, then the quasilinear subdivision scheme based on S is uniformly convergent and the limit function $S^\infty v^0$ is C^s for all $s < -\frac{\log \rho_\infty(S_{n+1})}{\log 2}$.*

Proof:
Notice that by Proposition 2, the assumption that $\rho_\infty(S_{n+1}) < 2^{-n}$ implies that $\rho_\infty(S_{m+1}) < 2^{-m}$ for $m = 0, 1, \cdots, n$. In particular $\rho_\infty(S_1) < 1$ and the scheme is convergent by Theorem 1.

We shall use induction on n to prove C^s smoothness. For $n = 0$, the result is proved by Theorem 1. For $n = 1$, we let $f = S^\infty v^0$ and we assume that $\rho_\infty(S_2) < 1/2$. Introducing

$$w^j := 2^j \Delta v^j = 2^j S_1(v^{j-1}) S_1(v^{j-2}) \cdots S_1(v^0) \Delta v^0, \tag{32}$$

we have

$$\Delta w^j := 2^j \Delta v^j = 2^j S_2(v^{j-1}) S_1(v^{j-2}) \cdots S_2(v^0) \Delta^2 v^0, \tag{33}$$

and therefore if ρ is such that $2\rho_\infty(S_2) < \rho < 1$, then

$$\|\Delta w^j\|_{\ell_\infty} := 2^j \Delta v^j = C \rho^j \|\Delta^2 v^0\|_{\ell_\infty}. \tag{34}$$

We obtain as in Theorem 1 that w^j uniformly converges to a continuous function g namely

$$\lim_{j\to\infty} \sup_k |w^j_k - g(2^{-j}k)| = 0.$$

Introducing the function $\tilde{\varphi} := \chi_{[0,1]}$ and the functions

$$g^j := \sum_{k\in\mathbb{Z}} w^j_k \tilde{\varphi}(2^j \cdot -k), \tag{35}$$

one easily check that $g^j = \frac{d}{dx} f^j$, where f^j is the affine function defined by (11), i.e.

$$\int_a^b g^j(x)dx = f^j(a) - f^j(b), \tag{36}$$

for all a and b. We know that $\lim_{j\to\infty} \|f^j - f\|_{L_\infty} = 0$, and we also have $\lim_{j\to\infty} \|g^j - g\|_{L_\infty} = 0$. It follows that

$$\int_a^b g(x)dx = f(a) - f(b), \tag{37}$$

for all a and b. Therefore, f is differentiable with $f' = g$. Moreover, as in Theorem 1, we obtain that $g \in C^t$ for all $t < -\frac{\log 2\rho_\infty(S_2)}{\log 2} < -1 - \frac{\log \rho_\infty(S_2)}{\log 2}$. Therefore $f \in C^s$ for all $s < -\frac{\log \rho_\infty(S_2)}{\log 2}$. Iterating this argument for $n > 1$, we obtain the general result. $\square$

Stability analysis

In this section, we study the stability of quasilinear subdivision schemes, e.g. properties of the type

$$\|S^\infty v^0 - S^\infty \tilde{v}^0\|_{L_\infty} \leq C\|v^0 - \tilde{v}^0\|_{\ell_\infty}. \tag{38}$$

In the linear case, this is a simple consequence of convergence, namely of $\|S^\infty v^0\|_{L_\infty} \leq C\|v^0\|_{\ell_\infty}$. In the nonlinear case, it requires a more specific study.

In our study of stability we need the additional assumption that there exists a linear left inverse operator of the subdivision operator (called *restriction* or *projection* operator by Harten). More precisely we assume that there exists coefficients $(\gamma_l)_{|l|<P}$ with $\sum_{|l|<P} \gamma_l = 1$ such that

$$v_k^{j-1} := \sum_{|l|<P} \gamma_l v_{2k-l}^j, \tag{39}$$

whenever $v^j := Sv^{j-1}$.

In many interesting case of linear or nonlinear subdivision algorithms, such an operator exists. In the point-value context $\gamma_l = \delta_{0,l}$, and in the cell-averages context $\gamma_0 = \gamma_{-1} = 1/2$, $\gamma_l = 0$ otherwise. In the following we always assume the existence of a restriction operator of the form (39) . In the next result we obtain the existence of a similar left-inverse for the subdivision schemes S_n associated to the finite differences.

Proposition 3 *Let S be a data dependent subdivision rule which reproduces polynomials of degree N. Then, for $n = 1, \cdots, N+1$ there exists coefficients $(\gamma_l^n)_{|l|<P}$ with $\sum_{|l|<P} \gamma_l^n = 2^n$ such that*

$$(\Delta^n v^{j-1})_k := \sum_{|l|<P+n} \gamma_l^n (\Delta^n v^j)_{2k-l}, \tag{40}$$

whenever $v^j := Sv^{j-1}$.

Proof:
Consider the case $n = 1$. Assuming (39), we can write

$$\begin{aligned}(\Delta v^{j-1})_k &= \sum_{|l|<P} \gamma_l (v^j_{2k+2-l} - v^j_{2k-l}) \\ &= \sum_{|l|<P} \gamma_l ((\Delta v^j)_{2k+1-l} + (\Delta v^j)_{2k-l}) \\ &= \sum_{|l|\leq P+1} \gamma^1_l (\Delta v^j)_{2k-l},\end{aligned}$$

with $\gamma^1_l := \gamma_l + \gamma_{l+1}$ which proves the result. The case $n > 1$ follows by induction. □

We use the restriction operators for the finite differences through the following lemma.

Lemma 1 *Let S be a data dependent subdivision rule which reproduces polynomials of degree N. Then there exists a constant $D > 0$, depending only on n, such that*

$$\|\Delta^n v^j\|_{\ell_\infty} \leq 2^{-n} \|\Delta^n v^{j-1}\|_{\ell_\infty} + D\|\Delta^{n+1} v^j\|_{\ell_\infty}, \quad 0 \leq n \leq N. \tag{41}$$

for all $j \geq 0$ and $v^0 \in \ell_\infty$.

Proof:
Since $(\Delta^n v^{j-1})_k = \sum_{|l|\leq P+n} \gamma^n_l (\Delta^n v^j)_{2k-l}$ with $\sum_l \gamma^n_l = 2^n$, we also have

$$(\Delta^n v^{j-1})_k = 2^n (\Delta^n v^j)_{2k} + \sum_{|l|<P+n} \gamma^n_l ((\Delta^n v^j)_{2k-l} - (\Delta^n v^j)_{2k}) \tag{42}$$

It follows that

$$(\Delta^n v^j)_{2j} := 2^{-n} [(\Delta^n v^{j-1})_k + \sum_{|l|<P+n} c_l (\Delta^{n+1} v^j)_{2k-l}], \tag{43}$$

with $c_l := \sum_{k=0}^{l-1} \gamma^n_k$. In a similar way, we obtain

$$(\Delta^n v^j)_{2k+1} := 2^{-n} [(\Delta^n v^{j-1})_k + \sum_{|l|<P+n} d_l (\Delta^{n+1} v^j)_{2k-l}]. \tag{44}$$

The claim follows with $D := 2^{-n} \max\{\sum_{|l|<P+n} |c_l|, \sum_{|l|<P+n} |d_l|\}$. □

Remark 1 *Note that, since the restriction operator is linear, we also have,*

$$\|\Delta^n v^j - \Delta^n \tilde{v}^j\|_{\ell_\infty} \leq 2^{-n} \|\Delta^n v^{j-1} - \Delta^n \tilde{v}^{j-1}\|_{\ell_\infty} + D\|\Delta^{n+1} v^j - \Delta^{n+1} \tilde{v}^j\|_{\ell_\infty}, \tag{45}$$

for $v^j = S(v^{j-1})v^{j-1}$ and $\tilde{v}^j = S(\tilde{v}^{j-1})\tilde{v}^{j-1}$.

The main ingredient for our analysis of the stability of quasilinear subdivision scheme is the following result.

Lemma 2 *Let S be a quasilinear subdivision rule, which reproduces polynomials up to the order N. Assume that S is continuously dependent on the data. Then for $n = 0, \ldots, N$, and $\rho > \rho_\infty(S_{n+1})$, we have*

$$\|\Delta^{n+1} v^j - \Delta^{n+1} \tilde{v}^j\|_{\ell_\infty} \leq C\rho^j \Big(\sum_{l=0}^{j-1} \|\Delta^n v^l - \Delta^n \tilde{v}^l\|_{\ell_\infty}\Big), \tag{46}$$

where C depends in a continuous non-decreasing way on $\Big(\max\{\|v^l\|_{\ell_\infty}, \|\tilde{v}^l\|_{\ell_\infty}; l = 0 \ldots j-1\}\Big)$.

Proof:
It is enough to give the proof for $n = 0$, since it is similar for larger values of n. If $\rho > \rho_\infty(S_1)$, there exists a constant K such that for all initial data v^0,

$$\|\Delta v^j\|_{\ell_\infty} \leq K\rho^j \|\Delta v^0\|_{\ell_\infty}. \tag{47}$$

Moreover there exists an integer L such that

$$\|\Delta v^j\|_{\ell_\infty} \leq \rho^L \|\Delta v^{j-L}\|_{\ell_\infty},\ j \geq L. \tag{48}$$

Assuming that $j \geq L$, we have

$$\begin{aligned}\|\Delta v^j - \Delta\tilde{v}^j\|_{\ell_\infty} &= \|S_1(v^{j-1})\cdots S_1(v^{j-L})\Delta v^{j-L} - S_1(\tilde{v}^{j-1})\cdots S_1(\tilde{v}^{j-L})\Delta\tilde{v}^{j-L}\|_{\ell_\infty}\\ &\leq A^j + B^j.\end{aligned}$$

where

$$A^j = \|S_1(v^{j-1})\cdots S_1(v^{j-L})(\Delta v^{j-L} - \Delta\tilde{v}^{j-L})\|_{\ell_\infty},$$

and

$$B^j = \|S_1(v^{j-1})\cdots S_1(v^{j-L})\Delta\tilde{v}^{j-L} - S_1(\tilde{v}^{j-1})\cdots S_1(\tilde{v}^{j-L})\Delta\tilde{v}^{j-L}\|_{\ell_\infty}.$$

By (48), we obtain

$$A^j \leq \rho^L \|\Delta v^{j-L} - \Delta\tilde{v}^{j-L}\|_{\ell_\infty}. \tag{49}$$

In order to estimate B^j, we define for $i > j - L$

$$G^i := S_1(v^{i-1})\cdots S_1(v^{j-L})\Delta\tilde{v}^{j-L} - S_1(\tilde{v}^{i-1})\cdots S_1(\tilde{v}^{j-L})\Delta\tilde{v}^{j-L},$$

and

$$K^i := S_1(v^{i-1})S_1(v^{i-2})\cdots S_1(v^{j-L})\Delta\tilde{v}^{j-L} - S_1(\tilde{v}^{i-1})S_1(v^{i-2})\cdots S_1(v^{j-L})\Delta\tilde{v}^{j-L},$$

$$L^i := S_1(\tilde{v}^{i-1})S_1(v^{i-2})\cdots S_1(v^{j-L})\Delta\tilde{v}^{j-L} - S_1(\tilde{v}^{i-1})S_1(\tilde{v}^{i-2})\cdots S_1(\tilde{v}^{j-L})\Delta\tilde{v}^{j-L}.$$

We thus have

$$B^j = \|G^j\|_{\ell_\infty} \le \|K^j\|_{\ell_\infty} + \|L^j\|_{\ell_\infty} \tag{50}$$

Recalling the boundedness and continuous dependency on the data of the scheme S_1, i.e.

$$\|S_1(v)\|_{\ell_\infty} \le B_1, \tag{51}$$

and

$$\|S_1(v) - S_1(\tilde{v})\|_{\ell_\infty} \le C_1 \|v - \tilde{v}\|_{\ell_\infty}, \tag{52}$$

where C_1 depends in a continuous non-decreasing way on $\max\{\|v\|_{\ell_\infty}, \|\tilde{v}\|_{\ell_\infty}\}$, we can estimate the first term according to

$$\|K^j\|_{\ell_\infty} \le C_1\, B_1^{L-1}\, \|v^{j-1} - \tilde{v}^{j-1}\|_{\ell_\infty} \|\Delta\tilde{v}^{j-L}\|_{\ell_\infty}, \tag{53}$$

where C_1 depends in a continuous non-decreasing way on $\max\{\|v^{j-1}\|_{\ell_\infty}, \|\tilde{v}^{j-1}\|_{\ell_\infty}\}$, and the second term by

$$\|L^j\|_{\ell_\infty} \le B_1 \|G^{j-1}\|_{\ell_\infty}. \tag{54}$$

Therefore, we obtain

$$\|G^j\|_{\ell_\infty} \le C_1 B_1^{L-1} \|v^{j-1} - \tilde{v}^{j-1}\|_{\ell_\infty} \|\Delta\tilde{v}^{j-L}\|_{\ell_\infty} + B_1 \|G^{j-1}\|_{\ell_\infty}$$

where C_1 depends in a continuous non-decreasing way on $\max\{\|v^{j-1}\|_{\ell_\infty}, \|\tilde{v}^{j-1}\|_{\ell_\infty}\}$, Similarly, we have

$$\|G^{j-1}\|_{\ell_\infty} \le C_1 B_1^{L-2} \|v^{j-2} - \tilde{v}^{j-2}\|_{\ell_\infty} \|\Delta\tilde{v}^{j-L}\|_{\ell_\infty} + B_1 \|G^{j-2}\|_{\ell_\infty},$$

where C_1 depends in a continuous non-decreasing way on $\max\{\|v^{j-2}\|_{\ell_\infty}, \|\tilde{v}^{j-2}\|_{\ell_\infty}\}$, and therefore

$$\|G^j\|_{\ell_\infty} \le C_1 B_1^{L-1} (\|v^{j-1} - \tilde{v}^{j-1}\|_{\ell_\infty} + \|v^{j-2} - \tilde{v}^{j-2}\|_{\ell_\infty}) \|\Delta\tilde{v}^{j-L}\|_{\ell_\infty} + B_1^2 \|G^{j-2}\|_{\ell_\infty},$$

where C_1 depends in a continuous non-decreasing way on

$$\max\{\|v^{j-1}\|_{\ell_\infty}, \|\tilde{v}^{j-1}\|_{\ell_\infty}, \|v^{j-2}\|_{\ell_\infty}, \|\tilde{v}^{j-2}\|_{\ell_\infty}\}.$$

By iteration, and since $G^{j-L} := \Delta\widetilde{v}^{j-L} - \Delta\widetilde{v}^{j-L} = 0$, we obtain

$$B^j \leq C_1 B_1^{L-1} \|\Delta\widetilde{v}^{j-L}\|_{\ell_\infty} \Big(\sum_{l=1}^{L} \|v^{j-l} - \widetilde{v}^{j-l}\|_{\ell_\infty}\Big), \tag{55}$$

where C_1 depends in a continuous non-decreasing way on $\max\{\|v^l\|_{\ell_\infty}, \|\widetilde{v}^l\|_{\ell_\infty}; l = 0\dots j-1\}$. Adding (49) and (55), we thus obtain

$$\|\Delta v^j - \Delta\widetilde{v}^j\|_{\ell_\infty} \leq \rho^L \|\Delta v^{j-L} - \Delta\widetilde{v}^{j-L}\|_{\ell_\infty} + C_1 B_1^{L-1} \|\Delta\widetilde{v}^{j-L}\|_{\ell_\infty} \Big(\sum_{l=1}^{L} \|v^{j-l} - \widetilde{v}^{j-l}\|_{\ell_\infty}\Big).$$

Combining this estimate with (47) gives

$$\|\Delta v^j - \Delta\widetilde{v}^j\|_{\ell_\infty} \leq \rho^L \|\Delta v^{j-L} - \Delta\widetilde{v}^{j-L}\|_{\ell_\infty} + C_2 \rho^{j-L} \Big(\sum_{l=1}^{L} \|v^{j-l} - \widetilde{v}^{j-l}\|_{\ell_\infty}\Big), \tag{56}$$

with $C_2 = 2C_1 K \|\widetilde{v}^0\|_{\ell_\infty}$. If $j - L \geq L$, we also have

$$\|\Delta v^{j-L} - \Delta\widetilde{v}^{j-L}\|_{\ell_\infty} \leq \rho^L \|\Delta v^{j-2L} - \Delta\widetilde{v}^{j-2L}\|_{\ell_\infty} + C_2 \rho^{j-2L} \Big(\sum_{l=L+1}^{2L} \|v^{j-l} - \widetilde{v}^{j-l}\|_{\ell_\infty}\Big),$$

and therefore

$$\|\Delta v^j - \Delta\widetilde{v}^j\|_{\ell_\infty} \leq \rho^{2L} \|\Delta v^{j-2L} - \Delta\widetilde{v}^{j-2L}\|_{\ell_\infty} + C_2 \rho^{j-L} \Big(\sum_{l=1}^{2L} \|v^{j-l} - \widetilde{v}^{j-l}\|_{\ell_\infty}\Big).$$

After $[\frac{j}{L}]$ iterations, we thus obtain

$$\|\Delta v^j - \Delta\widetilde{v}^j\|_{\ell_\infty} \leq \rho^{L[\frac{j}{L}]} \max_{0\leq l\leq L-1} \|\Delta v^l - \Delta\widetilde{v}^l\|_{\ell_\infty} + C_2 \rho^{j-L} \Big(\sum_{l=1}^{j} \|v^{j-l} - \widetilde{v}^{j-l}\|_{\ell_\infty}\Big).$$

For the values $l = 0, \cdots, L-1$, as well as in the case $0 \leq j < L$, we simply use $\|\Delta v^l - \Delta\widetilde{v}^l\|_{\ell_\infty} \leq 2\|v^l - \widetilde{v}^l\|_{\ell_\infty}$ it follows that

$$\|\Delta v^j - \Delta\widetilde{v}^j\|_{\ell_\infty} \leq C\rho^j \Big(\sum_{l=1}^{j} \|v^{j-l} - \widetilde{v}^{j-l}\|_{\ell_\infty}\Big), \tag{57}$$

where

$$C = 2\max\{1, \rho^{-L}\}(1 + KC_1)\|\widetilde{v}^0\|_{\ell_\infty}, \tag{58}$$

depends in a continuous non-decreasing way on $\max\{\|v^l\|_{\ell_\infty}, \|\tilde{v}^l\|_{\ell_\infty}; l = 0 \dots j-1\}$. $\square$

We are now ready to give condition for the stability of the quasilinear subdivision schemes for various norms measuring $S^\infty v - S^\infty \tilde{v}$. We begin with the uniform norm.

Theorem 3 *Let S be a quasilinear subdivision rule which reproduces constants. Assume that S is continuously dependent on the data and that $\rho_\infty(S_1) < 1$. Then for all data v^0 and $\tilde{v}^0$, we have*

$$\|S^\infty v^0 - S^\infty \tilde{v}^0\|_{L_\infty} < C\|v^0 - \tilde{v}^0\|_{\ell_\infty}, \tag{59}$$

where C depends in a continuous non-decreasing way on $\max\{\|v^0\|_{\ell_\infty}, \|\tilde{v}^0\|_{\ell_\infty}\}$. *Also for $s < -\log(\rho_\infty(S_1))/\log 2$ we have that*

$$\|\Delta v^j - \Delta\tilde{v}^j\|_{\ell_\infty} \leq C2^{-sj}\|v^0 - \tilde{v}^0\|_{\ell_\infty}. \tag{60}$$

Proof:
It suffices to prove that for all $j > 0$

$$\|v^j - \tilde{v}^j\|_{\ell_\infty} < C\|v^0 - \tilde{v}^0\|_{\ell_\infty}, \tag{61}$$

with C independent of j, since we then have

$$\|f^j - \tilde{f}^j\|_{L_\infty} < C\|v^0 - \tilde{v}^0\|_{\ell_\infty}, \tag{62}$$

and therefore (59) by letting j go to $+\infty$.

Let ρ be such that $\rho_\infty(S_1) < \rho < 1$. Let us denote $\alpha^j := \|v^j - \tilde{v}^j\|_{\ell_\infty}$ and $\beta^j := \|\Delta v^j - \Delta\tilde{v}^j\|_{\ell_\infty}$. By Remark 1 and Lemma 2, these sequences satisfy the following inequalities

$$\begin{cases} \alpha^j & \leq \alpha^{j-1} + D\beta^j, \\ \beta^j & \leq C\rho^j(\alpha^{j-1} + \cdots + \alpha^0), \end{cases}$$

where C depends in a continuous non-decreasing way on $\max\{\|v^l\|_{\ell_\infty}, \|\tilde{v}^l\|_{\ell_\infty}; l = 0 \dots j-1\}$. However, we remark that since $\rho_\infty(S_1) < 1$, we have $\|S^j v\|_{\ell_\infty} \leq K\|v\|_{\ell_\infty}$ with K a constant independent of j and v, and therefore we have that C depends in a continuous non-decreasing way on $\max\{\|v^0\|_{\ell_\infty}, \|\tilde{v}^0\|_{\ell_\infty}\}$.

If we now consider the positives non-decreasing sequences $\bar{\alpha}^j$ and $\bar{\beta}^j$ defined by $\bar{\alpha}^0 = \alpha^0$, $\bar{\beta}^0 = \beta^0$ and satisfying

$$\left\{ \begin{array}{ll} \bar{\alpha}^j & = \bar{\alpha}^{j-1} + D\bar{\beta}^j, \\ \bar{\beta}^j & = C\rho^j(\bar{\alpha}^{j-1} + \cdots + \bar{\alpha}^0), \end{array} \right. \tag{63}$$

we clearly have $\alpha^j \leq \bar{\alpha}^j$ and $\beta^j \leq \bar{\beta}^j$. Using the last equality from (63) and the fact that $\bar{\alpha}^j$ is increasing, we get

$$\bar{\beta}^j \leq Cj\rho^j\bar{\alpha}^{j-1}. \tag{64}$$

Combining this with the first equality in (63) , we obtain

$$\bar{\alpha}^j \leq (1 + CDj\rho^j)\bar{\alpha}^{j-1}, \tag{65}$$

and therefore

$$\bar{\alpha}^j = \prod_{l=0}^{j}(1 + CDl\rho^l)\alpha^0. \tag{66}$$

Clearly the product $\prod_{l=0}^{\infty}(1 + CDl\rho^l)$ is convergent, and by taking its logarithm, one easily check that its limit is bounded by $CD\frac{\rho}{(1-\rho)^2}$. Therefore, we obtain

$$\|v^j - \tilde{v}^j\|_{\ell_\infty} = \alpha^j \leq CD\frac{\rho}{(1-\rho)^2}\alpha^0 = CD\frac{\rho}{(1-\rho)^2}\|v^0 - \tilde{v}^0\|_{\ell_\infty}, \tag{67}$$

which proves our first claim since the constant C of Lemma 2 depends in a continuous non-decreasing way on $\max\{\|v^0\|_{\ell_\infty}, \|\tilde{v}^0\|_{\ell_\infty}\}$. For the second claim we note that

$$\beta^j \leq \bar{\beta}^j \leq Cj\rho^j\bar{\alpha}^{j-1} \leq C^2D\frac{\rho}{(1-\rho)^2}j\rho^j\|v^0 - \tilde{v}^0\|_{\ell_\infty} \leq C2^{-sj}\|v^0 - \tilde{v}^0\|_{\ell_\infty},$$

with the last constant C depends in a continuous non-decreasing way on $\max\{\|v^0\|_{\ell_\infty}, \|\tilde{v}^0\|_{\ell_\infty}\}$. □

We next address the stability in Hölder norm C^s for $0 < s < 1$.

Theorem 4 *Under the assumptions of Theorem 3, we have*

$$\|S^\infty v^0 - S^\infty\tilde{v}^0\|_{C^s} < C\|v^0 - \tilde{v}^0\|_{\ell_\infty}, \tag{68}$$

for all $s > 0$ such that $s < -\frac{\log(\rho_\infty(S_1))}{\log 2}$, where C depends in a continuous non-decreasing way on $\max\{\|v^0\|_{\ell_\infty}, \|\tilde{v}^0\|_{\ell_\infty}\}$.

Proof:
Let ρ be such that $s < -\frac{\log \rho}{\log 2} < -\frac{\log(\rho_\infty(S_1))}{\log 2}$, i.e. $\rho_\infty(S_1) < \rho < 2^{-s} < 1$. Let us define $f = S^\infty v^0$, $\widetilde{f} = S^\infty \widetilde{v}^0$, and $F = f - \widetilde{f}$. We also recall f^j and $\widetilde{f}^j$ defined by the interpolation of v^j and $\widetilde{v}^j$ according to (11), and we define $F^j = f^j - \widetilde{f}^j$. As in the proof of Theorem 1, we can write

$$\|F^{j+1} - F^j\|_{L_\infty} \leq C\|\Delta v^j - \Delta \widetilde{v}^j\|_{\ell_\infty}. \tag{69}$$

From Theorem 3, we thus obtain

$$\|F^{j+1} - F^j\|_{L_\infty} \leq C2^{-sj}\|v^0 - \widetilde{v}^0\|_{\ell_\infty}, \tag{70}$$

where C depends in a continuous non-decreasing way on $\max\{\|v^0\|_{\ell_\infty}, \|\widetilde{v}^0\|_{\ell_\infty}\}$. It follows that

$$\|F - F^j\|_{L_\infty} \leq C2^{-sj}\|v^0 - \widetilde{v}^0\|_{\ell_\infty}. \tag{71}$$

For $|x-y| \leq 1$ and j such that $2^{-j-1} < |x-y| \leq 2^{-j}$,

$$\begin{aligned} |F(x) - F(y)| &\leq |F(x) - F^j(x)| + |F(y) - F^j(y)| + |F^j(x) - F^j(y)| \\ &\leq 2\|F - F^j\|_{L_\infty} + |F^j(x) - F^j(y)| \\ &\leq C2^{-sj}\|v^0 - \widetilde{v}^0\|_{\ell_\infty} + 2^{-j}\|(F^j)'\|_{L_\infty} \\ &\leq C2^{-sj}\|v^0 - \widetilde{v}^0\|_{\ell_\infty} + \|\Delta v^j - \Delta \widetilde{v}^j\|_{\ell_\infty} \\ &\leq C2^{-sj}\|v^0 - \widetilde{v}^0\|_{\ell_\infty} \\ &\leq C|x-y|^s\|v^0 - \widetilde{v}^0\|_{\ell_\infty}, \end{aligned}$$

up to a multiplicative change in C. This concludes the proof. □

Finally, we address stability in the Hölder norm C^s for $s > 1$.

Theorem 5 *Let S be a quasilinear subdivision rule which reproduces polynomials up to degree N. Assume that S is continuously dependent of the data, and that $\rho_\infty(S_{n+1}) < 2^{-n}$ for some $n \in \{0, \cdots, N\}$. We then have*

$$\|S^\infty v^0 - S^\infty \widetilde{v}^0\|_{C^s} < C\|v^0 - \widetilde{v}^0\|_{\ell_\infty}, \tag{72}$$

for all $s > 0$ such that $s < -\frac{\log(\rho_\infty(S_{n+1}))}{\log 2}$, where C depends in a continuous non-decreasing way on $\max\{\|v^0\|_{\ell_\infty}, \|\widetilde{v}^0\|_{\ell_\infty}\}$.

Proof:
We shall use induction on n in a similar way as in the proof of Theorem 2.

For $n = 0$, the result is proved by Theorem 4. For $n = 1$, we assume that $\rho(S_2) < 1/2$. We define f, $\tilde{f}$, F, f^j, $\tilde{f}^j$ and F^j as in the proof of Theorem 4. We recall the sequences $w^j := 2^j \Delta v^j$ and $\tilde{w}^j := 2^j \Delta \tilde{v}^j$, and the functions $g^j := \sum_{k \in \mathbb{Z}} w_k^j \tilde{\varphi}(2^j \cdot -k)$ and $\tilde{g}^j := \sum_{k \in \mathbb{Z}} \tilde{w}_k^j \tilde{\varphi}(2^j \cdot -k)$.

We already know from the proof of Theorem 2 that g^j and $\tilde{g}^j$ uniformly converge to $g = f'$ and $\tilde{g} = \tilde{f}'$. Therefore $G^j := g^j - \tilde{g}^j$ converges to $G = F'$. Since $s < -\frac{\log(\rho_\infty(\tilde{S}_2))}{\log 2}$, we obtain by similar arguments as in the proof of Theorem 3 that

$$\|\Delta w^j - \Delta \tilde{w}^j\|_{\ell_\infty} \leq C 2^{(1-s)j} \|w^0 - \tilde{w}^0\|_{\ell_\infty}. \tag{73}$$

Note that, we use the fact that, according to Remark 1, we also have the inequality

$$\|w^j - \tilde{w}^j\|_{\ell_\infty} \leq \|w^{j-1} - \tilde{w}^{j-1}\|_{\ell_\infty} + D \|\Delta w^j - \Delta \tilde{w}^j\|_{\ell_\infty}, \tag{74}$$

with constant 1 for the first term. We then use the same type of arguments as in the proof of Theorem 4 to derive that

$$|G(x) - G(y)| \leq C|x-y|^{s-1} \|w^0 - \tilde{w}^0\|_{\ell_\infty} \leq 2C|x-y|^{s-1} \|v^0 - \tilde{v}^0\|_{\ell_\infty},$$

which gives the desired result. Iterating this argument for $n > 1$, we obtain the general result. □

Application

In this section we apply the results of the previous section to quasilinear subdivision schemes based on ENO and WENO interpolation techniques in the point values setting as described in Example 1 of Section 2. Remark that the smoothness of the limit functions based on ENO interpolation techniques is inherently limited in the following sense: if the data v_k^0 are such that the stencil selection always avoids a singularity point on the coarse grid, then the limit function will not be differentiable at this point. Similarly, we cannot expect continuity in the ENO cell-average setting.

We treat here the particular case of 4 point interpolation, i.e. $M = 4$. The associated scheme S_1 is defined by a rule of the type

$$(S_1(v)\Delta w)_k := \sum_{|k-2l| \leq 4} b_{k,l}(v) \Delta w_l, \tag{75}$$

where $b_{k,l}$, are the coefficients associated to the interval $\Gamma_l^j := [(k-l)2^{-j}, (k-l+1)2^{-j}]$. In the particular case of four point ENO interpolation, the differences are calculated with one of the following rules:

$$\left\{ \begin{array}{lll} \Delta v_{2k,0}^{j+1} & := & \frac{11}{16}\Delta v_k^j \ - \frac{1}{4}\Delta v_{k+1}^j + \frac{1}{16}\Delta v_{k+2}^j, \\ \Delta v_{2k,1}^{j+1} & := & \frac{1}{16}\Delta v_{k-1}^j + \frac{1}{2}\Delta v_k^j \ - \frac{1}{16}\Delta v_{k+1}^j, \\ \Delta v_{2k,2}^{j+1} & := & -\frac{1}{16}\Delta v_{k-2}^j + \frac{1}{4}\Delta v_{k-1}^j + \frac{5}{16}\Delta v_k^j, \end{array} \right. \tag{76}$$

obtained respectively from each case of (2) . By symmetry, we can also write the rule for the odd differences

$$\left\{ \begin{array}{lll} \Delta v_{2k+1,0}^{j+1} & := & \frac{5}{16}\Delta v_k^j \ + \frac{1}{4}\Delta v_{k+1}^j - \frac{1}{16}\Delta v_{k+2}^j, \\ \Delta v_{2k+1,1}^{j+1} & := & -\frac{1}{16}\Delta v_{k-1}^j + \frac{1}{2}\Delta v_k^j \ + \frac{1}{16}\Delta v_{k+1}^j, \\ \Delta v_{2k+1,2}^{j+1} & := & \frac{1}{16}\Delta v_{k-2}^j - \frac{1}{4}\Delta v_{k-1}^j + \frac{11}{16}\Delta v_k^j, \end{array} \right. \tag{77}$$

These rules allow us to estimate the joint spectral radius of S_1, according to the following result.

Lemma 3 *In the case of ENO four point subdivision scheme, one has*

$$\sup_{u,w\in\ell_\infty} \|S_1(u)S_1(w)\|_{\ell_\infty} < 1 \tag{78}$$

and therefore $\rho_\infty(S_1) < 1$.

Proof:
Notice first that the ℓ_∞ norm of the operator defined in (76) and (77) satisfies

$$\|S_1(v)\|_{\ell_\infty} = \sup_k \sum_l |b_{k,l}(v)| = \frac{11}{16} + \frac{1}{4} + \frac{1}{16} = 1. \tag{79}$$

For fixed $u, w \in \ell_\infty(\mathbb{Z})$, we have that

$$\Big(S_1(u)S_1(w)\Big)_{k,l} := \sum_{k'\in\mathbb{Z}} (S_1(u))_{k,k'}(S_1(w))_{k',l}. \tag{80}$$

and therefore $\|S_1(u)S_1(w)\|_{\ell_\infty}$ is estimated by

$$\begin{array}{ll} \sup_k \sum_l |(S_1(u)S_1(w))_{k,l}| & \leq \sup_k \sum_l \sum_{k'} |(S_1(u))_{k,k'}||(S_1(w))_{k',l}| \\ & \leq \sup_k \sum_{k'\in S(k)} \Big(|b_{k,k'}(u)| \sum_l |b_{k',l}(w)|\Big), \end{array}$$

where $S(k)$ is the selected stencils for k. Since $S(k)$ includes three consecutive integers, it always include a pair $(2m, 2m+1)$. From (76) and (77) , we notice that either $\sum_l |b_{2m,l}(w)| = 5/8$ or $\sum_l |b_{2m+1,l}(w)| = 5/8$. Therefore, there exist $k_0 \in S(k)$ such that

$$\sum_l |b_{k_0,l}(w)| = 5/8. \tag{81}$$

Since $k_0 \in S(k)$, we also have $|b_{k,k_0}(u)| \geq \frac{1}{16}$. It follows that

$$\begin{array}{ll} \sum_{k'} \Big(|b_{k,k'}(u) \sum_l |b_{k',l}(w)|\Big) & = \frac{5}{8}|b_{k,k_0}(u)| + \sum_{k' \neq k_0} |b_{k,k'}(u)| \sum_l |b_{k',l}(w)| \\ & \leq \frac{5}{8}|b_{k,k_0}(u)| + \sum_{k' \neq k_0} |b_{k,k'}(u)| \\ & \leq 1 + (\frac{5}{8} - 1)|b_{k,k_0}(u)| \\ & \leq 1 - \frac{3}{8}\frac{1}{16} = \frac{125}{128} < 1. \end{array}$$

□

A more precise estimation of $\|S_1(u)S_1(w)\|_{\ell_\infty}$ can be obtained by an explicit computation for each different stencil combinations. This leads to the sharper bound

$$\rho_\infty(S_1) \leq \sup_{u,w \in \ell_\infty} \|S_1(u)S_1(w)\|_{\ell_\infty}^{1/2} = \frac{9}{16}\sqrt{2}. \tag{82}$$

As a consequence of Theorem 1 and (82) , we obtain the following smoothness result of the limit function, in the particular case of four point ENO interpolation. subdivision.

Theorem 6 *In the case of ENO four point interpolation, the limit function of the subdivision scheme is bounded and belongs to C^s for all $s < -\frac{\log(\frac{9}{16}\sqrt{2})}{\log 2} \approx 0.6601499$.*

We finally turn to WENO interpolation defined in Section 1. The scheme S_1 is defined by a rule of the type

$$(S_1(v)\Delta w)_k := \sum_{|k-2l| \leq 6} b_{k,l}(v)\Delta w_l. \tag{83}$$

The rule for the differences has the form of a convex combination of the rules (76) , namely

$$\begin{array}{ll} \Delta v_{2k}^j := & \frac{-\alpha_2}{16}\Delta v_{k-2}^{j-1} + \frac{4\alpha_2+\alpha_1}{16}\Delta v_{k-1}^{j-1} + \frac{11\alpha_0+8\alpha_1+5\alpha_2}{16}\Delta v_k^{j-1} \\ & + \frac{-4\alpha_0-\alpha_1}{16}\Delta v_{k+1}^{j-1} + \frac{\alpha_0}{16}\Delta v_{k+2}^{j-1}, \end{array} \tag{84}$$

By symmetry, we can also write the rule for the odd differences

$$\begin{array}{rl}\Delta v^j_{2k+1} := & \frac{\alpha_2}{16}\Delta v^{j-1}_{k-2} + \frac{-4\alpha_2-\alpha_1}{16}\Delta v^{j-1}_{k-1} + \frac{11\alpha_2+8\alpha_1+5\alpha_0}{16}\Delta v^{j-1}_{k} \\ & + \frac{4\alpha_0+\alpha_1}{16}\Delta v^{j-1}_{k+1} - \frac{\alpha_0}{16}\Delta v^{j-1}_{k+2}.\end{array} \tag{85}$$

Note that in both formulas, α_0, α_1 and α_2 vary with k. We then have the following result for the joint spectral radius of S_1.

Lemma 4 *In the case of WENO interpolation, one has*

$$\sup_{u,w\in\ell_\infty} \|S_1(u)S_1(w)\|_{\ell_\infty} < 1 \tag{86}$$

and therefore $\rho_\infty(S_1) < 1$.

Proof:
From (84) and (85) we have that

$$\sum_l |b_{2k,l}(v)| \leq \alpha_0 + \frac{5}{8}(\alpha_1+\alpha_2) \leq 1, \tag{87}$$

and

$$\sum_l |b_{2k+1,l}(v)| \leq \alpha_2 + \frac{5}{8}(\alpha_1+\alpha_0) \leq 1, \tag{88}$$

and therefore $\|S_1(v)\|_{l_\infty} \leq 1$. For fixed $u, w \in \ell_\infty(\mathbb{Z})$, we have that

$$\Big(S_1(u)S_1(w)\Big)_{k,l} := \sum_{k'\in\mathbb{Z}} (S_1(u))_{k,k'}(S_1(w))_{k',l}. \tag{89}$$

We recall that $\|S_1(u)S_1(w)\|_{\ell_\infty}$ is estimated by

$$\begin{array}{rl}\sup_k \sum_l |(S_1(u)S_1(w))_{k,l}| & \leq \sup_k \sum_l \sum_{k'} |(S_1(u))_{k,k'}||(S_1(w))_{k',l}| \\ & \leq \sup_k \sum_{k' \text{ s.t. } |k-2k'|\leq 6} \Big(|b_{k,k'}(u)| \sum_l |b_{k',l}(w)|\Big),\end{array}$$

We remark that the set $\{k' \; s.t \; |k-2k'| \leq 6\}$ includes five consecutive integers, and then it always include a quadruplet $(2m, 2m+1, 2m+2, 2m+3)$. We then again remark that one of the rules (84) or (85) for the differences is contractive, since we have

$$\sum_l |b_{2k,l}(v)| + \sum_l |b_{2k+1,l}(v)| = \frac{5}{4}\alpha_1 + \frac{13}{8}(\alpha_0+\alpha_2) \leq \frac{13}{8} < 2. \tag{90}$$

Consequently, there exists p and q in $\{0,1\}$ such that $\sum_l |b_{2m+p,l}(w)| \leq \frac{13}{16} < 1$ and $\sum_l |b_{2m+2+q,l}(w)| \leq \frac{13}{16} < 1$. We also derive from the rules (84) or (85) that we always have

$$\min\{|b_{k,2m+p}(u)|, |b_{k,2m+2+q}(u)|\} \geq 1/16. \tag{91}$$

Therefore, there exist k_0 such that $\sum_l |b_{k_0,l}(w)| \leq \frac{13}{16}$ and $|b_{k,k_0}(u)| \geq 1/16$. It follows that

$$\begin{aligned}
\sum_{k'} \Big(|b_{k,k'}(u) \sum_l |b_{k',l}(w)|\Big) &= \tfrac{13}{16}|b_{k,k_0}(u)| + \textstyle\sum_{k' \neq k_0} |b_{k,k'}(u)| \sum_l |b_{k',l}(w)| \\
&\leq \tfrac{13}{16}|b_{k,k_0}(u)| + \textstyle\sum_{k' \neq k_0} |b_{k,k'}(u)| \\
&\leq 1 + (\tfrac{13}{16} - 1)|b_{k,k_0}(u)| \\
&\leq 1 - \tfrac{3}{16}\tfrac{1}{16} = \tfrac{253}{256} < 1.
\end{aligned}$$

□

A more precise estimation of $\|S_1(u)S_1(w)\|_{\ell_\infty}$ can be obtained by an explicit computation for each different stencil combinations. This leads to the same sharper bound as in ENO case

$$\rho_\infty(S_1) \leq \sup_{u,w \in \ell_\infty} \|S_1(u)S_1(w)\|_{\ell_\infty}^{1/2} = \frac{9}{16}\sqrt{2}. \tag{92}$$

As a consequence of Theorem 1 and (92) , we obtain the following smoothness result of the limit function of the subdivision process, based on WENO interpolation:

Theorem 7 *In the case of WENO interpolation, the limit function of the subdivision scheme is bounded and belongs to C^s for all $s < -\frac{\log(\frac{9}{16}\sqrt{2})}{\log 2} \approx$ 0.6601499.*

Although they are bounded, the nonlinear operators based on ENO techniques are unstable. The ENO techniques use a numerical criterion in the selection process of the stencil. If the two terms in the numerical criterion are close to zero, then a small change at the round off level would change the direction in the numerical criterion and hence the stencil. In this situation, there is no hope to have stability. In contrast, WENO interpolation based on the weights introduced in [7] is stable.

Proposition 4 *In the case of WENO interpolation, the subdivision operator given in* (2) *with the weights defined in* (7) *is continuous with respect to the data.*

Proof:
Let $u, \tilde{u} \in \ell_\infty(\mathbb{Z})$. From the definition of the subdivision operator operator we have

$$\|S(u) - S(\tilde{u})\|_{\ell_\infty} = \sup_k \sum_l |a_{k,l}(u) - a_{k,l}(\tilde{u})|. \tag{93}$$

In the particular case of WENO interpolation we obtain

$$\|S(u) - S(\tilde{u})\|_{\ell_\infty} \leq |\alpha_0 - \tilde{\alpha}_0| + |\alpha_1 - \tilde{\alpha}_1| + |\alpha_2 - \tilde{\alpha}_2|.$$

where $\alpha_2, \alpha_1, \alpha_0, \tilde{\alpha}_2, \tilde{\alpha}_1, \tilde{\alpha}_0$ represent the weights of the left and of the right stencil for u and $\tilde{u}$. From the definition of the weights in § 2, we have

$$\begin{aligned} |\alpha_i - \tilde{\alpha}_i| &= \left|\tfrac{a_i}{a_0+a_1+a_2} - \tfrac{\tilde{a}_i}{\tilde{a}_0+\tilde{a}_1+\tilde{a}_2}\right| \\ &\leq \left|\tfrac{a_i-\tilde{a}_i}{a_0+a_1+a_2}\right| + \left|\tilde{a}_i\left(\tfrac{1}{a_0+a_1+a_2} - \tfrac{1}{\tilde{a}_0+\tilde{a}_1+\tilde{a}_2}\right)\right| \\ &\leq \tfrac{1}{a_0+a_1+a_2}\left[2|a_i - \tilde{a}_i| + \textstyle\sum_{j\neq i} |a_j - \tilde{a}_j|\right], \end{aligned}$$

and therefore

$$\|S(u) - S(\tilde{u})\|_{\ell_\infty} \leq \frac{4}{a_0 + a_1 + a_2} \sum_i |a_i - \tilde{a}_i|.$$

From (7) we have that $|b_i| \leq C_0 \|u\|^2_{\ell_\infty}$ where $C_0 > 0$, constant independent on u and w. It follows that

$$\begin{aligned} a_0 + a_1 + a_2 &= \tfrac{d_0}{(\epsilon+b_0)^2} + \tfrac{d_1}{(\epsilon+b_1)^2} + \tfrac{d_2}{(\epsilon+b_2)^2} \\ &\geq \textstyle\sum_{i=0}^2 \tfrac{d_i}{\epsilon + C_0\|u\|^2_{\ell_\infty}} = \tfrac{3}{\epsilon + C_0\|u\|^2_{\ell_\infty}} \end{aligned} \tag{94}$$

Using straightforward computations we also obtain

$$\begin{aligned} |a_i - \tilde{a}_i| &= d_i \tfrac{2\epsilon + b_i + \tilde{b}_i}{(\epsilon+b_i)^2(\epsilon+\tilde{b}_i)^2} |b_i - \tilde{b}_i| \\ &\leq 2\tfrac{d_i}{\epsilon^3}|b_i - \tilde{b}_i|. \end{aligned} \tag{95}$$

From (7) , we obtain

$$|b_i - \tilde{b}_i| \leq C_1 \|u + \tilde{u}\|_{\ell_\infty} \|u - \tilde{u}\|_{\ell_\infty}, \tag{96}$$

where $C_1 > 0$, constant independent on u and $\tilde{u}$, and therefore

$$\sum_i |a_i - \tilde{a}_i| \leq \frac{6C_1}{\epsilon^3}(\|u\|_{\ell_\infty} + \|\tilde{u}\|_{\ell_\infty})\|u - \tilde{u}\|_{\ell_\infty}. \tag{97}$$

Combining (94) and (97) , we therefore obtain

$$\|S(u) - S(\tilde{u})\|_{\ell_\infty} \leq [\frac{2C_1}{\epsilon^3}(\|u\|_{\ell_\infty} + \|\tilde{u}\|_{\ell_\infty})(\epsilon + C_0\|u\|_{\ell_\infty}^2)]\|u - \tilde{u}\|_{\ell_\infty}, \tag{98}$$

which concludes the proof.

□

We can thus apply the results of § 4 to derive the following result.

Theorem 8 *In the case of WENO four point interpolatory techniques, defined in (2) , with the weights satisfying (7) , the subdivision scheme is L_∞ stable and C^s stable for all $s < -\frac{\log(\rho_\infty(S_1))}{\log 2} \approx 0.6601499$.*

Appendix

We shall briefly sketch some smoothness and stability results in the spaces L_p and $B^s_{p,q}$ which generalize those obtained in § 3 and § 4. The Besov spaces $B^s_{p,q}$ roughly represent the functions with s derivatives in L_p. They can be defined through the n-th order L_p modulus of of f,

$$\omega_n(f,t)_{L_p} = \sup_{|h|\leq t} \|\Delta_h^n f\|_{L_p}, \tag{99}$$

where $\Delta_h^n f$ is the usual $n-$th order finite difference operator

$$\Delta_h^n f = \sum_{m=0}^{n} (-1)^m \binom{n}{m} f(\cdot + hm).$$

For $p, q \geq 1, s > 0$, the Besov spaces $B^s_{p,q}$ consists of the functions $f \in L_p$ such that

$$\left(2^{sj}\omega_n(f, 2^{-j})_{L_p}\right)_{j\geq 0} \in \ell_q. \tag{100}$$

Here n is an integer strictly larger than s. A natural norm for such a space is then given by

$$\|f\|_{B^s_{p,q}} := \|f\|_{L_p} + \|\left(2^{sj}\omega_n(f, 2^{-j})_{L_p}\right)_{j\geq 0}\|_{\ell_q}.$$

Remark 2 *For $q = \infty$,* (100) *simply means that $\|\Delta_h^n f\|_{L_p} \leq Ch^s$. In particular, one has $C^s = B^s_{\infty,\infty}$ when s is not an integer. More generally, one has $W^{s,p} = B^s_{p,p}$ if s is not an integer and $H^s = W^{s,2} = B^s_{2,2}$ for all s.*

We can study the convergence of quasilinear subdivision schemes in L_p according to the following natural definition.

Definition 7 *A subdivision scheme is called L_p convergent if, for every finite set of initial control points $v_0 \in \ell_p(\mathbb{Z})$, there exists a function $f \in L_p$, called the limit function, such that*

$$\lim_{j\to\infty} \|f^j - f\|_{L_p} = 0, \tag{101}$$

where f^j is the function defined in (11) *.*

One easily check that we have

$$\|f^j\|_{L_p} \leq 2^{-j/p}\|v^j\|_{\ell_p}. \tag{102}$$

Therefore, similar convergence and smoothness results can be obtained, based on the ℓ_p study of the S_n. We assume boundedness of S in the ℓ_p sense which means that for all $v \in \ell_p(\mathbb{Z})$,

$$\|S(v)\|_{\ell_p} \leq B, \tag{103}$$

where $\|A\|_{\ell_p} := \sup\{\|Aw\|_{\ell_p}\ ;\ \|w\|_{\ell_p} = 1\}$, and we define the ℓ_p joint spectral radius

$$\rho_p(S) := \limsup_{j\to\infty} \sup_{(u^0,\cdots,u^{j-1})\in(\ell_p(\mathbb{Z}))^j} \|S(u^{j-1}),\ldots,S(u^0)\|_{\ell_p}^{\frac{1}{j}}. \tag{104}$$

It can easily be checked that Proposition 2 extends to the ℓ_p joint spectral radius, i.e. $\rho_p(S_{n+1}) \geq \rho_p(S_n)/2$. Note that convergence of the subdivision scheme implies $\rho_p(S) \geq 2^{1/p}$ since otherwise $S^\infty v^0 = 0$ for all initial data v^0 in view of (102) . Therefore, the above result shows that we always have

$$\rho_p(S_n) \geq 2^{1/p-n}. \tag{105}$$

With such definitions, we have the following results, similar to Theorem 1 and Theorem 2

Theorem 9 *Let S be a quasilinear subdivision rule which reproduces constants. If $\rho_p(S_1) < 2^{\frac{1}{p}}$ then S is L_p-convergent. Moreover, the limit function f belong to $B^s_{p,q}$ for all $s < -\frac{\log(\rho_p(S_1))}{\log 2} + 1/p$.*

Proof:
By similar arguments as in the proof of Theorem 1, we establish that

$$\|f^{j+1} - f^j\|_{L_p} \le C 2^{-j/p} \|\Delta v^j\|_{\ell_p} \le C\rho^j 2^{-j/p} \|\Delta v^0\|_{\ell_p}, \tag{106}$$

for ρ such that $\rho_p(S_1) < \rho < 2^{\frac{1}{p}}$, from which we obtain the L_p convergence of f^j to some $f \in L^p$. If $|h| \le 1$ and j is such that $2^{-j-1} < |h| \le 2^{-j}$, we have

$$\begin{aligned} \|f - f(\cdot + h)\|_{L_p} &\le 2\|f - f^j\|_{L_p} + \|f^j - f^j(\cdot + h)\|_{L_p} \\ &\le C\rho^j 2^{-j/p} \|\Delta v^0\|_{\ell_p} + 2^{-j} \|(f^j)'\|_{L_p} \\ &\le C(\rho^j 2^{-j/p} \|\Delta v^0\|_{\ell_p} + 2^{-j/p} \|\Delta v^j\|_{\ell_p}) \\ &\le C\rho^j 2^{-j/p} \|\Delta v^0\|_{\ell_p} \\ &\le C|h|^s \|\Delta v^0\|_{\ell_p}, \end{aligned}$$

with $s = -\frac{\log \rho}{\log 2} + 1/p$. Therefore $f \in B^s_{p,\infty}$ for all $s < -\frac{\log(\rho_p(S_1))}{\log 2} + 1/p$. Since $B^t_{p,\infty} \subset B^s_{p,q}$ when $t > s$, it follows that we also have $f \in B^s_{p,q}$ for all $s < -\frac{\log(\rho_p(S_1))}{\log 2} + 1/p$. $\square$

Theorem 10 *Let S be a quasilinear subdivision rule which reproduces polynomials up to the order N. If $\rho_p(S_{n+1}) < 2^{\frac{1}{p}-n}$ for some $n \le N$, the limit function f is in $B^s_{p,q}$ for all $s < -\frac{\log(\rho_p(S_{n+1}))}{\log 2} + 1/p$.*

Proof:
We use exactly the same arguments as those used in the proof of Theorem 2. For $n = 0$, the result is proved by Theorem 9. For $n = 1$, we recall the sequence $w^j := 2^j \Delta v^j$ and the function $g^j := \sum_{k\in\mathbb{Z}} w^j_k \widetilde{\varphi}(2^j \cdot - k)$. We get that $g := \widetilde{S}^\infty \Delta v^0$ belongs to $B^s_{p,q}$ for $s < -\frac{\log(\rho_p(\widetilde{S}_1))}{\log 2} + 1/p$ and satisfies $f' = g$. Therefore $f \in B^s_{p,q}$ for all $s < -\frac{\log \rho_\infty(S_2)}{\log 2}$. Iterating this argument for $n > 1$, we obtain the general result. $\square$

We finally want to generalize the stability results given in Theorem 3 and Theorem 4 to the L_p norm and $B^s_{p,\infty}$ norm. A first possibility is to proceed in a similar way as in the proof of these results, replacing the assumptions on the spectral radius of S_1 or S_n in ℓ_∞ by assumptions of their spectral radius in ℓ_p similar to those in theorems 9 and 10, and to assume continuous dependency with respect to the data in the sense where

$$\|S(v^j) - S(\tilde{v}^j)\|_{\ell_p} \leq C\|f^j - \tilde{f}^j\|_{L_p} = C2^{-j/p}\|v^j - \tilde{v}^j\|_{\ell_p}.$$

However this last assumption is too restrictive in view of the factor $2^{-j/p}$. In particular, it is not fulfilled by the WENO point value subdivision scheme. In the following, we show that L_p (resp. $B^s_{p,\infty}$) stability can be obtained by combining the L_∞ (resp. C^s) stability with the fact that the subdivision scheme is local.

Theorem 11 *Let S be a quasilinear subdivision scheme which reproduces constants and which is continuously dependent on the data in the sense of (14) . Assume that $\rho_\infty(S_1) < 1$. Then we have*

$$\|S^\infty v^0 - S^\infty \tilde{v}^0\|_{L_p} < C\|v^0 - \tilde{v}^0\|_{\ell_p}, \tag{107}$$

where C depends in a continuous non-decreasing way on $\max\{\|v^0\|_{\ell_\infty}, \|\tilde{v}^0\|_{\ell_\infty}\}$.

Proof:
For all $j > 0$, we have

$$\|f^j - \tilde{f}^j\|^p_{L_p} \leq 2^{-j}\|v^j - \tilde{v}^j\|^p_{\ell_p} = 2^{-j}\sum_{k\in\mathbb{Z}} \|v^j - \tilde{v}^j\|^p_{\ell_p(\mathbb{Z}\cap[2^jk,2^j(k+1)))}. \tag{108}$$

We also have

$$2^{-j}\|v^j - \tilde{v}^j\|^p_{\ell_p(\mathbb{Z}\cap[2^jk,2^j(k+1)))} \leq \|v^j - \tilde{v}^j\|^p_{\ell_\infty(\mathbb{Z}\cap[2^jk,2^j(k+1)))}. \tag{109}$$

Using the L_∞ stability result established in Theorem 3, together with the fact that our scheme is local, we obtain that

$$\|v^j - \tilde{v}^j\|_{\ell_\infty(\mathbb{Z}\cap[2^jk,2^j(k+1)))} \leq C\|v^0 - \tilde{v}^0\|_{\ell_\infty(\mathbb{Z}\cap[k-2M,k+2M])} \tag{110}$$

where C depends in a continuous non-decreasing way on $\max\{\|v^0\|_{\ell_\infty}, \|\tilde{v}^0\|_{\ell_\infty}\}$. Elevating this last estimate to the power p and summing on k, we thus obtain from (108) that

$$\|f^j - \tilde{f}^j\|_{L_p} \leq C\|v^0 - \tilde{v}^0\|_{\ell_p}, \tag{111}$$

where C depends in a continuous non-decreasing way on $\max\{\|v^0\|_{\ell_\infty}, \|\tilde{v}^0\|_{\ell_\infty}\}$. The claim follows by letting j tend to $+\infty$ in the above inequality. $\Box$

We finally give a stability result in Besov norms.

Theorem 12 *Let S be a quasilinear subdivision rule which reproduces polynomials up to the order N, which is continuously dependent of the data in the sense of (14). Assume that $\rho_\infty(S_1) < 1$ and that for some $n \leq N$, $\rho_p(S_{n+1}) < 2^{1/p-n}$. Then we have*

$$\|S^\infty v^0 - S^\infty \tilde{v}^0\|_{B^s_{p,q}} < C\|v^0 - \tilde{v}^0\|_{\ell_p}, \tag{112}$$

for all $s < -\frac{\log(\rho_p(S_{n+1}))}{\log 2} + 1/p$, where C depends in a continuous non-decreasing way on $\max\{\|v^0\|_{\ell_\infty}, \|\tilde{v}^0\|_{\ell_\infty}\}$.

Proof:
For $n = 0$, we proceed as in the proof of Theorem 4. Let ρ be such that $s < -\frac{\log \rho}{\log 2} + 1/p < -\frac{\log(\rho_p(S_1))}{\log 2} + 1/p$, i.e. $\rho_p(S_1) < \rho < 2^{1/p-s} < 2^{1/p}$. Recalling $F^j := f^j - \tilde{f}^j$ and its L_p limit $F = f - \tilde{f}$, we first establish

$$\|F^{j+1} - F^j\|^p_{L_p} \leq C2^{-j}\|\Delta v^j - \Delta\tilde{v}^j\|^p_{\ell_p}, \tag{113}$$

where C depends in a continuous non-decreasing way on $\max\{\|v^0\|_{\ell_\infty}, \|\tilde{v}^0\|_{\ell_\infty}\}$, by the same technique as in the proof of Theorem 1, In order to estimate the right hand side, we use the same localization technique as in the proof of Theorem 10, i.e.

$$\begin{aligned}
2^{-j}\|\Delta v^j - \Delta\tilde{v}^j\|^p_{\ell_p} &= 2^{-j}\textstyle\sum_{k\in\mathbb{Z}} \|\Delta v^j - \Delta\tilde{v}^j\|^p_{\ell_p(\mathbb{Z}\cap[2^j k, 2^j(k+1)))} \\
&\leq \textstyle\sum_{k\in\mathbb{Z}} \|\Delta v^j - \Delta\tilde{v}^j\|^p_{\ell_\infty(\mathbb{Z}\cap[2^j k, 2^j(k+1)))} \\
&\leq \textstyle\sum_{k\in\mathbb{Z}} \|\Delta v^j - \Delta\tilde{v}^j\|^p_{\ell_\infty(\mathbb{Z}\cap[2^j k, 2^j(k+1)))} \\
&\leq C\textstyle\sum_{k\in\mathbb{Z}} 2^{-spj}\|v^0 - \tilde{v}^0\|^p_{\ell_\infty(\mathbb{Z}\cap[k-2M, k+2M])} \\
&\leq C2^{-spj}\|v^0 - \tilde{v}^0\|^p_{\ell_p}.
\end{aligned}$$

In the third inequality, we have used the local version of the estimate $\|\Delta v^j - \Delta\tilde{v}^j\|_{\ell_\infty} \leq 2^{-sj}\|v^0 - \tilde{v}^0\|_{\ell_\infty}$ used in the proof of Theorem 4. It follows that

$$\|F - F^j\|^p_{L_p} \leq C2^{-sj}\|v^0 - \tilde{v}^0\|_{\ell_p}, \tag{114}$$

where C depends in a continuous non-decreasing way on $\max\{\|v^0\|_{\ell_\infty}, \|\tilde{v}^0\|_{\ell_\infty}\}$. For $|h| \leq 1$ and j such that $2^{-j-1} < |h| \leq 2^{-j}$, we then write

$$\begin{aligned}
\|\Delta_h F\|_{L_p} &\leq 4\|F - F^j\|_{L_p} + \|\Delta_h F^j\|_{L_p} \\
&\leq C2^{-sj}\|v^0 - \tilde{v}^0\|_{\ell_p} + 2^{-j}\|(F^j)'\|_{L_p} \\
&\leq C2^{-sj}\|v^0 - \tilde{v}^0\|_{\ell_\infty} + 2^{-j/p}\|\Delta v^j - \Delta\tilde{v}^j\|_{\ell_p} \\
&\leq C2^{-sj}\|v^0 - \tilde{v}^0\|_{\ell_p} \\
&\leq C|h|^s\|v^0 - \tilde{v}^0\|_{\ell_p},
\end{aligned}$$

which proves the result for $q = \infty$ and therefore for all q since $B^t_{p,\infty} \subset B^s_{p,q}$ when $t > s$. For $n > 0$ we use exactly the same argument as in Theorem 5.

□

The results of this Appendix can be applied to the L_p analysis of ENO and WENO subdivision schemes in a similar way as in Section 5. We end this Appendix with a smoothness result in the cell averages setting. We consider the prediction operator defined in Example 2 of Section 1. An estimation of $\|S_1(u)S_1(v)S_1(w)\|_{\ell_1}$ can be obtained by an explicit computation for each different stencil combinations. This leads to the same bound for ENO and WENO interpolation:

$$\rho_1(S_1) \leq \sup_{u,v,w\in\ell_1} \|S_1(u)S_1(v)S_1(w)\|_{\ell_1}^{1/3} = 1.2365. \tag{115}$$

As a consequence of Theorem 9, the following result holds:

Theorem 13 *In the case of three cell averages ENO interpolation and in the case of three cell averages WENO interpolation the quasilinear subdivision operator S is L_1-convergent. Moreover, in both situations, the limit function, belong to $B^s_{1,q}$ for all $s < -\frac{\log(1.2365)}{\log 2} + 1 \approx 0.69371$.*

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