Spline Subdivision Schemes for Convex Compact Sets

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The application of spline subdivision schemes to data consisting of convex compact sets, with addition replaced by Minkowski sums of sets, is investigated. These methods generate in the limit set-valued functions, which can be expressed explicitly in terms of linear combinations of integer shifts of B-splines with the initial data as coefficients. The subdivision techniques are used to conclude that these limit set-valued spline functions have shape preserving properties similar to those of the usual spline functions. This extension of subdivision methods from the scalar setting to the set-valued case has application in the approximate reconstruction of 3-D bodies from finite collections of their parallel cross-sections.

Keywords: convex sets, support functions, Minkowski addition, set-valued functions, spline subdivision, shape preservation, approximation.

1 Introduction

Subdivision schemes are recursive methods for the generation of smooth functions from discrete data. By these methods at each recursion step, new discrete values on a finer grid are computed by weighted sums of the already existing discrete values. In the limit of the recursive process, data is defined on a dense set of points. Considering this data as function values, under certain conditions, a limit continuous function is defined by this process. The theory of subdivision processes is presented in e.g. [4], [6].

In this work we apply a class of subdivision methods with positive weights to data consisting of convex compact sets, replacing the addition by Minkowski sums of sets, and obtain in the limit set-valued functions.

This extension of subdivision methods from the scalar setting to the set-valued case has application in the approximate reconstruction of 3-D bodies from finite collections of their parallel cross-sections.

The class of subdivision methods considered here consists of methods which generate spline functions in the scalar setting. These methods have shape preserving properties and approximation properties in the set-valued case, similar to those in the scalar case.

The proof of the shape preserving properties of the limit set-valued functions relies on the subdivision technique. Yet the limit multifunction (set-valued function) has a simple explicit form in terms of the initial sets and integer shifts of B-splines, and there is no need to compute it recursively. This explicit form also yields the smoothness and the approximation properties of the limit set-valued function.

Among the spline subdivision schemes, there is a scheme which generates in the limit piecewise linear interpolants to the given data of sets. In [11] piecewise linear approximation to convex set-valued functions is studied. This approximation consists of a piecewiswe linear interpolant to samples of a multifunction, with addition replaced by Minikowski sums. Thus, one can regard the present paper as an extension of [11].

The mathematical tools used for analysing set-valued functions include the support function technique for describing convex compact sets (see e.g. [10]) and methods of embedding the cone of convex compact subsets of $\mathbb{R}^n$ in a linear normed space, with addition defined as the Minkowski sum of sets ([2], [7], [8], [9]). In any such linear normed space, we introduce a partial order generated by the set inclusion order in the cone of convex compact sets. Thus a multifunction with convex compact images from $\mathbb{R}$ to $\mathbb{R}^n$ is considered as an abstract function with values in a partially ordered normed linear space. Monotonicity and convexity of such abstract functions are easily expressed by the positivity of their first and second finite differences.

The paper is organised as follows. In section 2 basic facts about convex compact sets, their support functions and their embedding in a normed linear space with a partial order, are presented. Section 3 presents a simple example of a shape-preserving set-valued subdivision scheme. The main results about spline subdivision methods applied to convex compact sets are derived in Section 4. The explicit form of the limit multivalued spline function is obtained together with its approximation and shape-preserving properties.

2 Preliminaries

Denote by $\mathcal{C}(\mathbb{R}^n)$ the cone of nonempty convex compact subsets of $\mathbb{R}^n$. Recall the definitions of Minkowski sum and multiplication by scalars of sets $A, B \in \mathcal{C}(\mathbb{R}^n)$:

$$A + B = \{ \ a + b \mid a \in A, \ b \in B \ \}, \qquad \lambda A = \ \{ \ \lambda a \mid a \in A \ \}.$$

Since the technique of support functions is central to this text, we recall the definition and the basic properties of these functions (see e.g. [7], [10]). For a set $A \in \mathcal{C}(\mathbb{R}^n)$ its support function $\delta^*(A, \cdot) : \mathbb{R}^n \to \mathbb{R}$ is defined as follows:

$$\delta^*(A, l) = \max_{a \in A} \langle l, a \rangle, \quad l \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. Note that for any fixed $l \in \mathbb{R}^n$, $\delta^*(A, l)$ is finite.

The following properties of δ^* are well known ([10]):

1^o. $\delta^*(A, \lambda l) = \lambda \delta^*(A, l), \quad \lambda \geq 0.$

2^o. $\delta^*(A, l_1 + l_2) \leq \delta^*(A, l_1) + \delta^*(A, l_2).$

3^o. $\delta^*(A, \cdot)$ is Lipschitz continuous. with a constant $\|A\| = \max_{a \in A} \|a\|$, where $\| \cdot \|$ is the Euclidean norm, namely $|\delta^*(A, l_1) - \delta^*(A, l_2)| \leq \|A\| \|l_1 - l_2\|$, for $l_1, l_2 \in \mathbb{R}^n$.

4^o. A scalar function $\delta^* : \mathbb{R}^n \to \mathbb{R}$ is a support function of a convex compact set iff it satisfies $1^o, 2^o$ (see e.g. [10], Theorem 13.2 and its corollaries).

5^o. $\delta^*(A + B, \cdot) = \delta^*(A, \cdot) + \delta^*(B, \cdot).$

6^o. $\delta^*(\lambda A, \cdot) = \lambda \delta^*(A, \cdot), \quad \lambda \geq 0,$

7^o. $A \subseteq B \iff \delta^*(A, l) \leq \delta^*(B, l)$ for each $l \in \mathbb{R}^n$.

8°. The Hausdorff distance between two sets $A, B \in \mathcal{C}(\mathbb{R}^n)$ is given, in terms of the support functions of these two sets, by

$$\text{haus}(A,B) = \max_{l \in S_{n-1}} |\delta^*(A,l) - \delta^*(B,l)|,$$

where S_{n-1} is the unit sphere in $\mathbb{R}^n$.

Note that by 1° and 2°, $\delta^*(A,\cdot)$ is positively homogeneous and convex. Also, functions satisfying 1°, 2° are called sublinear. As we noted in 3°, (see e.g. [10]), sublinear functions defined on all $\mathbb{R}^n$ are Lipschitz continuous. The next proposition will be used in what follows.

Proposition 2.1 *Let $A_1, A_2, B_1, B_2 \in \mathcal{C}(\mathbb{R}^n)$ and $A_1 \supseteq B_1$. Then the equality $A_1 + B_2 = A_2 + B_1$ implies $A_2 \supseteq B_2$.*

Proof: The proof follows from properties 5°, 7°.

$$A_1 + B_2 = A_2 + B_1 \iff \delta^*(A_1,\cdot) + \delta^*(B_2,\cdot) = \delta^*(A_2,\cdot) + \delta^*(B_1,\cdot).$$

From $A_1 \supseteq B_1$ follows $\delta^*(A_1,\cdot) \geq \delta^*(B_1,\cdot)$. This, combined with the above equality implies $\delta^*(B_2,\cdot) \leq \delta^*(A_2,\cdot) \Longrightarrow B_2 \subseteq A_2$. ■

There are various ways to construct a **linear normed vector space** $\mathcal{D}(\mathbb{R}^n)$ relative to the Minkowski sum, in which the cone $\mathcal{C}(\mathbb{R}^n)$ is embedded by an embedding $J : \mathcal{C}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$ with the following properties(see e.g. [7], [9], [8]):

(i) $J(A+B) = J(A) + J(B)$.

(ii) $J(\lambda A) = \lambda J(A), \quad \lambda \geq 0$.

(iii) $J(A) = J(B) \iff A = B$.

(iv) $\|J(A) - J(B)\| = \text{haus}(A,B)$.

These properties imply $J(\{0\}) = \theta$, where θ is the zero element of $\mathcal{D}(\mathbb{R}^n)$.

A simple embedding is $J(A) = \delta^*(A,\cdot)$. It is easy to check , using the above stated properties of support functions, that this embedding has the required four properties.

Having an embedding of the cone $\mathcal{C}(\mathbb{R}^n)$ into a linear normed vector space $\mathcal{D}(\mathbb{R}^n)$, we can introduce a partial order in $\mathcal{D}(\mathbb{R}^n)$ and therefore in $\mathcal{C}(\mathbb{R}^n)$. For that, we define the following cone in $\mathcal{D}(\mathbb{R}^n)$:

$$\mathcal{K} = \left\{ C \in \mathcal{D}(\mathbb{R}^n) |\ C = J(A) - J(B),\ A, B \in \mathcal{C}(\mathbb{R}^n),\ A \supseteq B \right\} \tag{1}$$

The cone $\mathcal{K}$ determines the following partial order in $\mathcal{D}(\mathbb{R}^n)$:

$$\text{For } \ A, B \in \mathcal{D}(\mathbb{R}^n) \qquad A \leq B \iff B - A \in \mathcal{K}. \tag{2}$$

Remark 2.1 *Here are some observations regarding the partial order introduced above:*

1. *By (i) and (ii) $\mathcal{K}$ is a convex cone.*

2. *Note that (1) does not depend on the choice of the sets A and B in the following sense: if $C = J(A_1) - J(B_1) = J(A_2) - J(B_2)$, where $A_1 \supseteq B_1$, then by (i) and (iii) we get $A_1 + B_2 = A_2 + B_1$ and by Proposition 2.1 $A_2 \supseteq B_2$.*

3. *By the previous observation, the order defined in $\mathcal{D}(\mathbb{R}^n)$ by (2) induces the regular inclusion order in $\mathcal{C}(\mathbb{R}^n)$, namely for $A, B \in \mathcal{C}(\mathbb{R}^n)$*

$$A \subseteq B \iff J(A) \leq J(B).$$

   Thus we denote for $A, B \in \mathcal{C}(\mathbb{R}^n)$ $A \leq B$ iff $A \subseteq B$.

4. *Our definition of the positive cone $\mathcal{K}$ in $\mathcal{D}(\mathbb{R}^n)$ coincides with the positive cone in some concrete linear spaces in which $\mathcal{C}(\mathbb{R}^n)$ is embedded, like the space of the pairs of convex compact sets and the space of differences of support functions (see e.g. [8]).*

Remark 2.2 *It follows from the facts that $\mathcal{K}$ is a cone and $\mathcal{D}(\mathbb{R}^n)$ is a linear space that for $A \leq B$*

(a) $-A \geq -B$,

(b) $A + C \leq B + C$ *for every* $C \in \mathcal{D}(\mathbb{R}^n)$,

(c) If $C \leq D$, *then* $\alpha A + \beta C \leq \alpha B + \beta D$ *for* $\alpha, \beta \geq 0$.

The previous remark justifies the notions of positive and negative elements of $\mathcal{D}(\mathbb{R}^n)$. The element $A \in \mathcal{D}(\mathbb{R}^n)$ is called **nonnegative** when $\theta \leq A$, i.e. $A \in \mathcal{K}$. If $A \geq \theta$, $A \neq \theta$, then A is called **positive**. The element $B \in \mathcal{D}(\mathbb{R}^n)$ is called **nonpositive** if $-B$ is nonnegative and B is **negative** when $-B$ is positive.

We will call a convex compact set A nonnegative when $J(A) \geq \theta$, i.e. $0 \in A$. A is positive iff $0 \in A$ and $A \neq \{0\}$.

We are interested in this work in set-valued mappings from $\mathbb{R}$ to $\mathcal{C}(\mathbb{R}^n)$ (called also multimaps or multifunctions), and in particular in multimaps of the form

$$F(t) = \sum_{i=1}^{N} A_i f_i(t), \tag{3}$$

where $A_i \in \mathcal{C}(\mathbb{R}^n)$, $f_i : \mathbb{R} \mapsto \mathbb{R}$, $f_i(t) \geq 0$ for all $t \in \mathbb{R}$. Let $\mathcal{S}$ be the cone of maps of the form (3). We say that $F \in \mathcal{S}$ is C^k if in (3) $f_i \in C^k$ for $i = 1, ..., N$.

Definition 2.1 *A mapping $F : \mathbb{R} \to \mathcal{D}(\mathbb{R}^n)$ is called*

(A) **monotone increasing** *if $t_1 \leq t_2$ implies $F(t_1) \leq F(t_2)$ (i.e. $F(t_2) - F(t_1) \in \mathcal{K}$).*

(B) **monotone decreasing** *if $F(-t)$ is monotone increasing.*

(C) **convex** *if*

$$F(\alpha t_1 + (1-\alpha)t_2) \geq \alpha F(t_1) + (1-\alpha)F(t_2) \quad \text{for each } \alpha \in [0,1], \quad t_1, t_2 \in \mathbb{R}. \tag{4}$$

(D) **concave** *if* $F(-t)$ *is convex.*

Definition 2.2 *Define for a given function* $F : \mathbb{R} \to \mathcal{D}(\mathbb{R}^n)$ *the k-th forward finite difference at the point t with a step* $h > 0$

$$\Delta_h^k F(t) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} F(t + jh).$$

For a sequence $\{F_i\}_i$ *we define*

$$(\Delta^k F)_i = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} F_{i+j}.$$

In case $\Delta^1 F$ *is nonnegative for all* $t, h > 0$ *(i), the function (sequence) is monotone increasing. If* $\Delta^2 F$ *is nonpositive for all* $t, h > 0$ *(i), the function (sequence) is called convex.*

Remark 2.3 *For the sake of simplicity we sometimes identify the set* $A \in \mathcal{C}(\mathbb{R}^n)$ *with its embedded image* $J(A)$ *and for sets* $A, B \in \mathcal{C}(\mathbb{R}^n)$ *we denote by* $A - B$ *the difference* $J(A) - J(B)$*. Geometrically, a monotone increasing set-valued map* $F : \mathbb{R} \to \mathcal{C}(\mathbb{R}^n)$ *has a growing image as the argument t increases. Formally, F is monotone when for a given* $h > 0$ *the first difference* $\Delta_h^1(t) = F(t + h) - F(t)$ *is of a constant sign. Similarly, the convexity means that the second difference*

$$\Delta_h^2 F(t) = F(t) + F(t + 2h) - 2F(t + h)$$

is nonpositive (i.e. $F(t) + F(t + 2h) \subseteq 2F(t + h)$*). The inequality (4) is opposite to the common definition of convex scalar functions. We choose it this way in order to ensure the convexity of the graph of F. Clearly, in the case of a convex map* $F : \mathbb{R} \to \mathcal{C}(\mathbb{R}^n)$

$$F(\alpha t_1 + (1 - \alpha) t_2) \supseteq \alpha F(t_1) + (1 - \alpha) F(t_2) \ \ \textit{for each } \alpha \in [0, 1], \quad t_1, t_2 \in \mathbb{R},$$

which means that the graph of F is a convex set in $\mathbb{R}^{n+1}$*. The last inclusion and property* 7° *of support functions imply that for each given direction l the support function* $\delta^*(F(\cdot), l)$ *of the convex multimap F is a concave scalar function.*

3 Chaikin Subdivision Scheme for Convex Compact Sets

Let $F_i^0, \quad i = 0, \ldots, N$ be convex compact sets in $\mathbb{R}^n$. We seek a set-valued function in $\mathcal{S}$ which has a similar structure to the the piecewise linear multifunction $F : \mathbb{R} \to \mathbb{R}^n$ defined by

$$F(i + \xi) = (1 - \xi) F_i^0 + \xi F_{i+1}^0, \qquad 0 \le \xi \le 1, i = 0, 1, ..., N - 1,$$

but is smoother in the sense of (3). Note that every piecewise linear multifunction is in $\mathcal{S}$ and is C^0.

Consider the following iterative procedure of reconstructing F, known as Chaikin algorithm when applied to scalar functions ([5], [6]). At level $k+1$, $k \geq 0$ of the procedure we calculate for $i = 0, ..., 2^k(N-1)$ the sets

$$F_{2i}^{k+1} = \frac{1}{4} F_{i+1}^k + \frac{3}{4} F_i^k, \tag{5}$$

$$F_{2i+1}^{k+1} = \frac{3}{4} F_{i+1}^k + \frac{1}{4} F_i^k. \tag{6}$$

Thanks to the positivity of the coefficients, we obtain convex compact sets at each stage of the process. Moreover, since the coefficients form a convex combination, the scheme has two noticeable properties:

1. It preserves monotonicity, i.e.

   $$\text{If for all } i \quad F_{i-1}^k \subseteq F_i^k \ , \qquad \text{then} \qquad F_{2i-1}^{k+1} \subseteq F_{2i}^{k+1} \subseteq F_{2i+1}^{k+1} \qquad \text{for all i.}$$

   This follows directly from the equalities

   $$F_{2i}^{k+1} - F_{2i-1}^{k+1} = \frac{1}{4}(F_{i+1}^k - F_{i-1}^k), \qquad F_{2i+1}^{k+1} - F_{2i}^{k+1} = \frac{1}{2}(F_{i+1}^k - F_i^k), \tag{7}$$

   which means that the first finite differences remain in $\mathcal{K}$ at each stage of the process if they are in $\mathcal{K}$ for $k = 0$.

2. It preserves convexity, namely the second differences remain in $\mathcal{K}$ at each iteration if they are in it for $k = 0$, since

   $$F_{2i}^{k+1} + F_{2i+2}^{k+1} - 2F_{2i+1}^{k+1} = \frac{1}{4}(F_i^k + F_{i+2}^k - 2F_{i+1}^k),$$

   $$F_{2i-1}^{k+1} + F_{2i+1}^{k+1} - 2F_{2i}^{k+1} = \frac{1}{4}(F_{i-1}^k + F_{i+1}^k - 2F_i^k).$$

At the k-th iteration ($k \geq 1$) we construct a piecewise linear multimap $F^k : [0, N-1+2^{-k}] \to \mathcal{C}(\mathbb{R}^n)$ satisfying

$$F^k(t) = \frac{t_i^k - t}{t_i^k - t_{i-1}^k} F_{i-1}^k + \frac{t - t_{i-1}^k}{t_i^k - t_{i-1}^k} F_i^k, \qquad t_{i-1}^k \leq t \leq t_i^k.$$

Lemma 3.1 *The sequence of set-valued functions $\{F^k(t)\}_{k=1}^{\infty}$ converges (uniformly in the interval $[0, N-1]$) to a Lipschitz continuous multifunction $F^{\infty}(t)$ with convex values.*

Proof: Denote the support functions $\delta_i^k(l) = \delta^*(F_i^k, l)$ and $\delta^k(l,t) = \delta^*(F^k(t), l)$. Then by (5), (6) and properties $5^o, 6^o$ of support functions it follows that for each fixed direction $l \in \mathbb{R}^n$

$$\delta_{2i}^{k+1}(l) = \frac{1}{4}\delta_{i+1}^k(l) + \frac{3}{4}\delta_i^k(l)$$

$$\delta_{2i+1}^{k+1}(l) = \frac{3}{4}\delta_{i+1}^k(l) + \frac{1}{4}\delta_i^k(l).$$

It means that for each given l the Chaikin subdivision procedure is realized on the scalar values $\delta_i^k(l)$. Hence by the well known convergence of the Chaikin algorithm for scalar subdivision (see e.g. [6]) for each fixed $l \in \mathbb{R}^n$ there exists a limit function

$$\delta^\infty(l,t) = \lim_{k\to\infty} \delta^k(l,t).$$

In the following we prove that $\delta^\infty(l,t)$ is Lipschitz continuous in l and t.
Let us fix a point $t \in [0, N-1]$. By properties $1^o, 2^o$ of the support functions $\delta^k(\cdot,t)$, it follows that the limit function $\delta^\infty(\cdot,t)$ satisfies $1^o, 2^o$. Moreover, since the initial sets $F_i^0,\ i = 0, \dots N$ are uniformly bounded, then by (5), (6) $F_i^k,\ i = 0, \dots 2^k(N-1)+1$ and $F^k(t)$ are uniformly bounded by the same constant. Hence by property 3^o the functions $\delta^k(\cdot,t)$ are Lipschitz continuous with an absolute constant, independent of t and k, and therefore the limit function $\delta^\infty(\cdot,t)$ is Lipschitz continuous with the same constant and the convergence is uniform with respect to l.

Moreover, properties $1^o, 2^o$ characterizing each support function hold for the limit function $\delta^\infty(\cdot,t)$ for fixed t. Therefore, by 4^o $\delta^\infty(\cdot,t)$ is the support function of a convex compact set which is denoted by $F^\infty(t)$. That $F^\infty(t) = \lim\limits_{k\to\infty} F^k(t)$ can be concluded from

$$\mathrm{haus}(F^\infty(t), F^k(t)) = \max_{l\in S_1} |\delta^\infty(l,t) - \delta^k(l,t)| \to 0 \quad \text{as} \quad k\to\infty.$$

To see that this convergence is uniform for $t \in [0, N-1]$, we observe by (7) and by property (iv) of J that

$$\mathrm{haus}(F_i^k, F_{i-1}^k) \le 2^{-k} \max_{0\le j\le N-1} \mathrm{haus}(F_{j+1}^0, F_j^0).$$

This means that the Lipschitz constants of the piecewise linear maps $F^k(t)$, and therefore of $F^\infty(t)$, (or of the functions $\delta^k(l,\cdot), \delta^\infty(l,\cdot)$) do not exceed the constant $\max\limits_{1\le i\le N} \mathrm{haus}(F_i^0, F_{i-1}^0)$. Hence $\delta^k(l,\cdot)$ and $F^k(\cdot)$ are uniformly Lipschitz continuous with the same constant in the interval $[0, N-1]$ and converge uniformly to $\delta^\infty(l,\cdot)$ and $F^\infty(\cdot)$) respectively, on this interval.

∎

The shape preserving properties of $F^\infty(t)$ and its smoothness follow from the analysis of more general schemes, done in the next section.

4 A Class of Shape Preserving Subdivision Schemes

Let the initial sequence $F_i^0,\quad i = 0, \dots, N$ of convex compact sets in $\mathbb{R}^n$ be given. For convenience we define $F_i^0 = \{0\}$ for $i \in \mathbb{Z} \setminus \{0, 1, ..., N\}$. Consider a finitely supported subdivision scheme given by

$$F_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{i-2j}^{[m]} F_j^k, \quad i \in \mathbb{Z}, \quad k = 0, 1, 2, ... \tag{8}$$

with the spline weights $a_i^{[m]} = \binom{m+1}{i}/2^m,\ i = 0, 1, ..., m+1$ and $a_i^{[m]} = 0$ for $i \in \mathbb{Z} \setminus \{0, 1, ..., m+1\}$. Note that Chaikin algorithm is the special case $m = 2$.

The scheme (8) when applied to scalar values $\{f_i^0\}_i$, is uniformly convergent and its limit function $f^\infty(\cdot)$ is of the form (see e.g. [6])

$$f^\infty(t) = \sum_{i\in\mathbb{Z}} f_i^0 B_m(t-i), \tag{9}$$

where the function $B_m(\cdot)$ is a B-spline of degree m, with integer knots and support $[0, m+1]$. In the following we obtain a set-valued analog of (9).

As in the previous section, at the k-th iteration ($k \geq 1$) we construct a piecewise linear multimap $F^k : \mathbb{R} \to \mathcal{C}(\mathbb{R}^n)$ satisfying $F^k(t_i^k) = F_i^k$ for $t_i^k = 2^{-k}i$.

Let the generating function of the sequence $\{a_i^{[m]}\}_i$ be

$$a^{[m]}(z) = \sum_{i=0}^{m+1} a_i^{[m]} z^i.$$

Then $a^{[m]}(\cdot)$ has the form

$$a^{[m]}(z) = \frac{(1+z)^{m+1}}{2^m}.$$

Clearly the coefficients $a_i^{[m]}$ are nonnegative numbers and satisfy

$$\sum_{i\in\mathbb{Z}} a_{2i}^{[m]} = 1, \qquad \sum_{i\in\mathbb{Z}} a_{2i+1}^{[m]} = 1. \tag{10}$$

Denote the differences $G_i^k = J(F_i^k) - J(F_{i-1}^k), \quad i \geq 1,\ k \geq 0$. Note that $G_i^k \in \mathcal{D}(\mathbb{R}^n)$.

It is easy to show as in the case of scalar functions (see e.g.[6]) that

Proposition 4.1 *The differences G_i^k satisfy*

$$G_i^{k+1} = \sum_j b_{i-2j}^{[m]} G_j^k, \quad i \in \mathbb{Z}, \quad k = 0, 1, ..., \tag{11}$$

where the generating function $b^{[m]}(\cdot)$ of the sequence $\{b_i^{[m]}\}_i$ is

$$b^{[m]}(z) = \sum_{i\in\mathbb{Z}} b_i^{[m]} z^i = \frac{a^{[m]}(z)}{1+z} = \frac{(1+z)^m}{2^m}. \tag{12}$$

It is clear from the last Proposition that the coefficients $b_i^{[m]}$ are nonnegative and

$$\sum_{i=0}^{[\frac{m}{2}]} b_{2i}^{[m]} = \sum_{i=0}^{[\frac{m}{2}]} b_{2i+1}^{[m]} = \frac{1}{2}. \tag{13}$$

This means that if the initial sets $\{F_i^0\}_{i=0}^N$ form a monotone increasing sequence (i.e. the differences $G_i^0,\ i = 1, ..., N$ are in the cone $\mathcal{K}$, then the differences $G_i^k,\ i = 1, ..., N$ remain in this cone at each stage of the subdivision process (11), i.e. for each k the sequence $\{F_i^k\}_i$ is monotone increasing.

By the same reasoning the second differences $H_i^k = G_i^k - G_{i-1}^k$ can be obtained by a subdivision scheme with a generating function

$$c^{[m]}(z) = \frac{b^{[m]}(z)}{1+z} = \frac{(1+z)^{m-1}}{2^m}. \qquad (14)$$

Since $c^{[m]}(z)$ has nonnegative coefficients, the second differences remain in $\mathcal{K}$ at each iteration k, provided they are in $\mathcal{K}$ for $k=0$. Therefore the scheme (8) is shape preserving.

By the above argumentation it is easy to conclude that the subdivision schemes (8) preserve the sign of the differences $\Delta^\nu F^k$ of order ν, $1 \le \nu \le m$, i.e. $(\Delta^\nu F^k)_i$ belongs to $\mathcal{K}$, provided $(\Delta^\nu F^0)_i \in \mathcal{K}$ for all i.

Thus we have proved the following

Proposition 4.2 *The subdivision scheme (8) is monotonicity and convexity preserving, i.e.*

$$F_j^k \subseteq F_{j+1}^k \text{ for all } j \implies F_i^{k+1} \subseteq F_{i+1}^{k+1} \text{ for all } i, \qquad (15)$$

$$F_{j-1}^k + F_{j+1}^k \subseteq 2F_j^k \text{ for all } j \implies F_{i-1}^{k+1} + F_{i+1}^{k+1} \subseteq 2F_i^{k+1} + F_{i+1}^{k+1} \text{ for all } i. \qquad (16)$$

Theorem 4.1 *The set-valued mappings $\{F^k(\cdot)\}_{k=1}^\infty$ converge uniformly on $\mathbb{R}$ to a spline multimap $F^\infty(\cdot)$ with convex images of the form*

$$F^\infty(t) = \sum_{i\in\mathbb{Z}} F_i^0 B_m(t-i) \quad \text{for each } t \in \mathbb{R}. \qquad (17)$$

Proof: Fix a direction $l \in \mathbb{R}^n$ and denote $\delta_i^k(l) = \delta^*(F_i^k, l)$, $\delta^k(l,t) = \delta^*(F^k(t), l)$.
Then (8) implies

$$\delta_i^{k+1}(l) = \sum_{j\in\mathbb{Z}} a_{i-2j}^{[m]} \delta_j^k(l),$$

hence by (9) for each l

$$\delta^\infty(l,t) = \sum_{i\in\mathbb{Z}} \delta_i^0(l) B_m(t-i).$$

Since the last sum is finite for any t, and since the coefficients $B_m(t-i)$ are nonnegative, it follows that for fixed t the scalar function $\delta^\infty(\cdot,t)$ is a support function of the set

$$F^\infty(t) = \sum_{i\in\mathbb{Z}} F_i^0 B_m(t-i). \qquad (18)$$

The fact that $F^\infty(t) = \lim_{k\to\infty} F^k(t)$ can be proved by the same argumentation as in the proof of Lemma 3.1 with (7) replaced by (11) and (13). ∎

The next corollary follows from the fact that $B_m \in C^{m-1}$.

Corollary 4.1 $F^\infty \in \mathcal{S}$ *and* $F^\infty \in C^{m-1}$.

For $m=1$, F^∞ is piecewise linear multifunction interpolating $\{F_i^0\}_i$ i.e. satisfying $F^\infty(i) = F_i^0$, $F^\infty(t) = (t-i)F_{i+1}^0 + (i+1-t)F_i^0$ for $i \le t \le i+1$, $i \in \mathbb{Z}$.
For $m=2$ (the Chaikin scheme), $F^\infty \in C^1$ and it is piecewise quadratic in the sense that for each fixed l the support function $\delta(l,\cdot)$ is a linear combination of quadratic B-splines.

The following shape preserving properties of F^∞ follow from Proposition 4.2 and the discussion above it.

Corollary 4.2 *Let $\{F_i^0\}_{i=1}^N$ be an initial sequence of compact convex sets.*

(a) If the initial sequence $\{F_i^0\}_{i=1}^N$ is monotone increasing, i.e. $F_i^0 \subseteq F_{i+1}^0$, $i = 1, \dots, N-1$, then the map $F^\infty(t)$ is monotone increasing in the sense of Remark 2.3.

(b) If the initial sequence is convex, i.e. $F_i^0 + F_{i+2}^0 \subseteq F_{i+1}^0$, $i = 0, \dots N-2$, then the map $F^\infty(t)$ has a convex graph.

Proof:

(a) With the notations of the previous theorem, it follows from (15) that for every $l \in \mathbb{R}^n$

$$\delta_i^k(l) \leq \delta_{i+1}^k(l).$$

Since a piecewise linear interpolating scalar function of monotone data is itself monotone, it follows that for each $l \in \mathbb{R}^n$

$$\delta^k(l, t+h) \geq \delta^k(l, t)$$

for every t and $h > 0$. The last inequality implies

$$F^k(t+h) \supseteq F^k(t)$$

for all t and $h > 0$. Therefore the set-valued map $F^k(\cdot)$ is monotone increasing. The last conclusion holds also for the limit $F^\infty(t) = \lim_{k\to\infty} F^k(t)$.

The proof of (b) is similar, based on (16) and the fact that a scalar piecewise linear interpolant of convex (concave) data is convex (concave). Note that in this case the set-valued functions are convex, but their support functions are concave for each fixed direction l (see Remark 2.3). ■

The limit multimaps generated by the above subdivision processes approximate smooth multimaps in $\mathcal{S}$ with the same approximation order as the limit functions generated by the corresponding scalar schemes approximate smooth scalar functions.

Proposition 4.3 *Let $F \in \mathcal{S} \cap C^r$ with $r > 0$. Then*

$$haus(F(t), \sum_{j\in\mathbb{Z}} F(jh) B_m(\frac{t}{h} - j)) \leq ch^s,$$

where $s = \min\{r, 2\}$ and the constant c depends only on F.

Proof: Let $F(t) = \sum_{i\in I} A_i f_i(t)$, where I is a finite subset of $\mathbb{Z}$, and $f_i \in C^r$. Denote $\tilde{f}_i(t) = \sum_{j\in\mathbb{Z}} f_i(jh) B_m(\frac{t}{h} - j)$. Then by a well-known approximation result for scalar functions (see e.g. [3]), we obtain

$$\|f_i - \tilde{f}_i\|_\infty \leq c_i h^s.$$

with the constant c_i depending only on f_i. Let $\tilde{F}(t) = \sum_{i\in I} A_i \tilde{f}_i(t)$, then

$$\tilde{F}(t) = \sum_{j\in\mathbb{Z}} B_m(\frac{t}{h} - j) F(jh).$$

Hence for each t

$$\text{haus}(F(t),\tilde{F}(t)) = \|J(\sum_{i\in I} A_i f_i(t)) - J(\sum_{i\in I} A_i \tilde{f}_i(t))\| \le \sum_{i\in I} \|A_i\| \, \|f_i - \tilde{f}_i\|_\infty \le ch^s, \qquad (19)$$

where we used property (iv) of the embedding J. ■

For general multimaps (not from $\mathcal{S}$) which are Hausdorff continuous, we get a weaker approximation result.

Proposition 4.4 *Let F be a general multimap with values in $\mathcal{C}(\mathbb{R}^n)$ which is Hausdorff continuous, then*

$$haus(F(t), \sum_{j\in\mathbb{Z}} F(jh) B_m(\frac{t}{h} - j)) = o(1).$$

Proof: For a fixed t there is only a finite number of terms in $\sum_{j\in\mathbb{Z}} F(jh)B_m(\frac{t}{h}-j)$. Now since $\sum_{j\in\mathbb{Z}} B_m(\frac{t}{h}-j) = 1$, we can write $F(t) = \sum_{j\in\mathbb{Z}} F(t)B_m(\frac{t}{h}-j)$. Using an argument similar to (19) and by the fact that F is continuous, we obtain

$$\text{haus}(F(t), \sum_{j\in\mathbb{Z}} F(jh) B_m(\frac{t}{h} - j)) = \| \sum_{j\in(\frac{t}{h}-m,\frac{t}{h})\cap\mathbb{Z}} \big(J(F(t)) - J(F(jh))\big) B_m(\frac{t}{h} - j)\| = o(1).$$

■

If we assume enough smoothness on the support functions of the sets $F(t)$ for all t , we get a similar result to that of Proposition 4.3.

Proposition 4.5 *Let F be a multimap defined on $\mathbb{R}$ with values in $\mathcal{C}(\mathbb{R}^n)$ such that the t-dependent support function $\delta^*(F(\cdot),l)$ has a second derivative in t uniformly bounded in t and in $l \in S_{n-1}$. Then*

$$haus(F(t), \sum_{j\in\mathbb{Z}} F(jh) B_m(\frac{t}{h} - j)) = O(h^2).$$

Proof: Let us denote the spline multifunction

$$S_m^h F(t) = \sum_{j\in\mathbb{Z}} F(jh) B_m(\frac{t}{h} - j) = \sum_{j\in(\frac{t}{h}-m,\frac{t}{h})\cap\mathbb{Z}} F(jh) B_m(\frac{t}{h} - j).$$

Then

$$\text{haus}(F(t), (S_m^h F)(t)) = \|\delta^*(F(t),\cdot) - \delta^*((S_m^h F)(t),\cdot)\|_{\infty, S_{n-1}}.$$

Denote for any $l \in \mathbb{R}^n$ $\quad \delta_l(t) = \delta(t,l) = \delta^*(F(t),l)$. For scalar functions it is known that

$$|f(t) - (S_m^h f)(t)| \le \frac{1}{2} \sup_t |\frac{d^2 f(t)}{dt^2}| h^2. \qquad (20)$$

Set $f(t) = \delta_l(t)$. Since $S_m^h \delta^*(F(\cdot),l) = \delta^*(S_m^h F(\cdot),l)$, we get

$$\delta^*(F(t),l) - \delta^*((S_m^h F)(t),l)| \le \frac{1}{2} \|\frac{d^2}{dt^2}\delta_l(\cdot)\|_\infty h^2.$$

Now, by our assumptions $\sup_{l\in S_{n-1}} |\frac{d^2}{dt^2}\delta_l(\cdot)| \le L$, and we obtain for every t

$$\|\delta^*(F(t),\cdot) - \delta^*((S_m^h F)(t),\cdot)\|_{\infty,S_{n-1}} \le \frac{1}{2}Lh^2.$$

■

Note that if F is defined in the finite interval $[0,N]$, then the estimate near the boundary of the interval is $O(h)$. This follows from the corresponding result in the scalar case .

Before concluding the paper, we state a conjecture that was inspired by the famous theorem of R. Aumann [1] stating that the integral of a compact-valued multimap is a convex set. Since the Riemann integral is the limit of Riemann sums, which are in essence averages with positive weights, we expect that the repeated application of (8) with the spline weights generates in the limit a convex- valued multifunction, even when the initial sets are compact but not convex.

Conjecture: Any of the subdivision methods (8) with $m > 1$ applied to initial compact sets, $\{F_j^0\}$, generates as a limit a convex-valued multifunction, given by

$$F^\infty(t) = \sum_{j\in\mathbb{Z}} \mathrm{co}F_j^0 B_m(t-j),$$

where $\mathrm{co}A$ denotes the convex hull of A.

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