Serdica Math. J. **28** (2002), 349-360

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

SPLINE SUBDIVISION SCHEMES FOR COMPACT SETS. A SURVEY

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Communicated by P. P. Petrushev

Dedicated to the memory of our colleague Vasil Popov
January 14, 1942 – May 31, 1990

Abstract. Attempts at extending spline subdivision schemes to operate on compact sets are reviewed. The aim is to develop a procedure for approximating a set-valued function with compact images from a finite set of its samples. This is motivated by the problem of reconstructing a 3D object from a finite set of its parallel cross sections. The first attempt is limited to the case of convex sets, where the Minkowski sum of sets is successfully applied to replace addition of scalars. Since for nonconvex sets the Minkowski sum is too big and there is no approximation result as in the case of convex sets, a binary operation, called *metric average*, is used instead. With the metric average, spline subdivision schemes constitute approximating operators for set-valued functions which are Lipschitz continuous in the Hausdorff metric. Yet this result is not completely satisfactory, since 3D objects are not continuous in the Hausdorff metric near points of change of topology, and a special treatment near such points has yet to be designed.

2000 *Mathematics Subject Classification:* 26E25, 28B20, 41A15, 41A36.
Key words: compact sets, spline subdivision schemes, metric average, Minkowski sum.
*Partially supported by ISF-Center of Excellence, and by The Hermann Minkowski Center for Geometry at Tel Aviv University, Israel

1. Introduction. The interest in developing subdivision schemes for compact sets is motivated by the problem of the reconstruction of 3D objects from a set of their 2D parallel cross sections, or the reconstruction of a 2D shape from a set of its 1D parallel cross sections. For a review on methods for the reconstruction of 3D objects from a finite set of parallel cross sections see [10].

In our approach every n-dimensional body is regarded as a univariate set-valued function with compact sets of dimension $n-1$ as images, determined by parallel cross sections [9]. The set-valued function is then approximated from the given samples (cross sections). The approximating procedure we use is an extension to compact sets of spline subdivision schemes.

A spline subdivision scheme generates from data consisting of real values attached to the integer points, a smooth function. In case of data sampled from a smooth function, the limit function, generated by such a scheme, approximates the sampled function, and has shape preserving properties [4, 5, 11].

Here we consider spline subdivision schemes operating on data consisting of compact sets. A spline subdivision scheme generates from such initial data a sequence of set-valued functions, with compact sets as images. This sequence converges in the Hausdorff metric to a limit set-valued function. In the case of 2D sets, the limit set valued function, with 2D sets as images, describes a 3D object.

For the case of initial data consisting of convex compact sets, we introduced in [6] spline subdivision schemes, where the usual addition of numbers is replaced by Minkowski sums of sets. Then the spline subdivision schemes generate limit set-valued functions with convex compact images which can be expressed as linear combinations of integer shifts of a B-spline, with the initial sets as coefficients. The subdivision techniques are used to conclude that these limit "set-valued spline functions" have shape preserving properties similar to those of scalar spline functions, but with shape properties relevant to sequences of sets and to set-valued functions.

In the case of nonconvex initial sets it is shown in [7] that the limit set-valued function, generated by a spline subdivision scheme, using the Minkowski sums, coincides with the limit set-valued function, generated by the same subdivision scheme from the convex hulls of the initial sets. Therefore, a set-valued function generated in such a way, has too big images to be a good approximation to the set-valued function from which the initial nonconvex sets are sampled.

To define spline subdivision schemes for general compact sets, which do not convexify the initial data, i.e. preserve the non-convexity, the usual Minkowski average is replaced by a binary operation between two compact sets, the *metric average*, introduced in [1] and applied within subdivision schemes in

[8]. As is shown in [8], spline subdivision schemes, based on the metric average, converge in the Hausdorff metric. The limit set-valued function generated by such a scheme, from initial data sampled at distance h from a Lipschitz continuous set-valued function with compact images, approximates to order $O(h)$ the sampled function.

2. Preliminaries. First we introduce some notations. The collection of all nonempty compact subsets of $\mathbb{R}^n$ is denoted by $\mathcal{K}_n$, $\mathcal{C}_n$ denotes the collection of convex sets in $\mathcal{K}_n$, $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^n$, $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$, S^{n-1} is the unit sphere in $\mathbb{R}^n$, co$A$ denotes the convex hull of the set A.

The Hausdorff distance between two sets A and B in $\mathbb{R}^n$ is defined by

$$\text{haus}(A,B) = \max\{ \sup_{x\in A} \text{dist}(x,B) \ , \ \sup_{y\in B} \text{dist}(y,A) \ \},$$

where the Euclidean distance from a point x to a set $A \in \mathcal{K}_n$ is

$$\text{dist}(x,A) = \min\{|x-y| \ : \ y \in A \ \}.$$

The support function $\delta^*(A,\cdot) : \mathbb{R}^n \to \mathbb{R}$ is defined for $A \in \mathcal{K}_n$ as

$$\delta^*(A,l) = \max_{a\in A}\langle l,a\rangle, \quad l \in \mathbb{R}^n.$$

The set of all projections of x on the set A is

$$\Pi_A(x) = \{ \ a \in A \ : \ |a-x| = \text{dist}(x,A) \ \}.$$

The set difference of $A, B \in \mathcal{K}_n$ is

$$A \setminus B = \{ \ a \ : \ a \in A, \ a \notin B \ \}.$$

A linear Minkowski combination of two sets A and B is

$$\lambda A + \mu B = \{ \ \lambda a + \mu b \ : \ a \in A, \ b \in B \ \},$$

for $A, B \in \mathcal{K}_n$ and $\lambda, \mu \in \mathbb{R}$. The Minkowski sum $A + B$ corresponds to a linear Minkowski combination with $\lambda = \mu = 1$. A Minkowski average (a Minkowski convex combination) of two sets is a linear Minkowski combination with λ, μ non-negative, summing up to 1.

We denote by $\mathcal{S}$ the class of multifunctions (set-valued functions) of the form,

$$F(t) = \sum_{i=1}^{N} A_i f_i(t), \tag{1}$$

where N is finite and $A_i \in \mathcal{C}_n$. We say that $F \in \mathcal{S}$ is C^k if in (1) $f_i \in C^k$ for $i = 1, \ldots, N$.

The notions convergence, continuity, Lipschitz continuity for set-valued functions or for sets, are to be understood with respect to the Hausdorff metric (distance). Let us recall that $\mathcal{K}_n$ is a complete metric space with respect to this metric.

3. Spline subdivision schemes for points in $\boldsymbol{R^d}$. A spline curve in R^d of degree m is defined by

$$C(t) = \sum_{i\in\mathbb{Z}} P_i^0 B_m(t-i) \quad \text{for each } t \in \mathbb{R}, \tag{2}$$

where $P^0 = \{P_i^0 \in \mathbb{R}^d, i \in \mathbb{Z}\}$ are the control points and $B_m(\cdot)$ is a B-spline of degree m. Due to the compact support of B_m, the treatment of the case in (2) applies also to curves defined by a finite set of control points.

The curve in (2) is the limit of a sequence of piecewise linear curves, each interpolating the points generated by the spline subdivision scheme S_m at a certain refinement level according to the refinement step,

$$P_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{i-2j}^{[m]} P_j^k, \quad i \in \mathbb{Z}, \quad k = 0,1,2,\dots \tag{3}$$

with the spline weights $a_i^{[m]} = \binom{m+1}{i}/2^m$, $i = 0,1,\dots,m+1$ and $a_i^{[m]} = 0$ for $i \in \mathbb{Z} \setminus \{0,1,\dots,m+1\}$.

For $m = 1$, the above scheme has coefficients $a_0 = \frac{1}{2}$, $a_1 = 1$, $a_2 = \frac{1}{2}$, and the refinement step is:

$$P_{2i}^{k+1} = \frac{1}{2}P_i^k + \frac{1}{2}P_{i-1}^k, \tag{4}$$

$$P_{2i+1}^{k+1} = P_i^k. \tag{5}$$

In this case the limit is a linear spline curve, interpolating the initial points $\{P_i^0 \ : \ i \in \mathbb{Z}\}$.

A quadratic spline curve is obtained as a limit in case $m = 2$, with the well-known scheme of Chaikin. The coefficients of this scheme are: $a_0 = \frac{1}{4}$, $a_1 = \frac{3}{4}$, $a_2 = \frac{3}{4}$, $a_3 = \frac{1}{4}$, and the refinement step is

$$P_{2i}^{k+1} = \frac{1}{4}P_i^k + \frac{3}{4}P_{i-1}^k, \tag{6}$$

$$P_{2i+1}^{k+1} = \frac{3}{4}P_i^k + \frac{1}{4}P_{i-1}^k. \tag{7}$$

An important result (see e.g. [4], [5]) is that the scheme (3), starting from $\{P_i^0 \ : \ i \in \mathbb{Z}\} \in l_\infty^d$, converges to a function $f(\cdot) \in C(\mathbb{R})^d$, i.e. $\lim\limits_{k\to\infty} \sup\limits_{i\in\mathbb{Z}} |f(2^{-k}i) - P_i^k| = 0$ if and only if $\lim\limits_{k\to\infty} \sup\limits_{t\in\mathbb{R}} |f(t) - \sum\limits_{i\in\mathbb{Z}} P_i^k h(2^k t - i)| = 0$, where $h(\cdot)$ is the "hat function"

$$h(t) = \begin{cases} 1 - |t| & \text{for } |t| \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

The limit function $f(t)$ is denoted by $S_m^\infty P^0$.

4. Extension to convex compact sets. The case of convex compact sets is investigated in [6].

We assume that the initial data $\{F_i^0,\ i \in \mathbb{Z}\}$ are convex compact sets. Then the addition operation in (3) is replaced by the Minkowski sum of sets, and the multiplication of a set by a scalar is defined as

$$\mu A = \{ \ \mu a \ : \ a \in A \ \}, \quad \mu \in \mathbb{R}. \tag{9}$$

The refinement step becomes

$$F_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{i-2j}^{[m]} F_j^k, \quad i \in \mathbb{Z}, \quad k = 0, 1, 2, \ldots \tag{10}$$

We note that convex compact sets are generated at each step of (10), if F_i^0, $i \in \mathbb{Z}$ are compact and convex.

It is shown in [6] that the set-valued spline function

$$F_m^\infty(t) = \sum_{i\in\mathbb{Z}} F_i^0 B_m(t - i) \quad \text{for each } t \in \mathbb{R}, \tag{11}$$

is the uniform limit in the Hausdorff metric of the subdivision scheme,

$$\lim_{k\to\infty} \sup_{i\in\mathbb{Z}} \operatorname{haus}(F_m^\infty(2^{-k}i), F_i^k) = 0,$$

or equivalently, that $\lim\limits_{k\to\infty} \sup\limits_{t\in\mathbb{R}} \operatorname{haus}\left(F_m^\infty(t), \sum\limits_{i\in\mathbb{Z}} F_i^k h(2^k t - i)\right) = 0$, where $h(\cdot)$ is the hat function defined in (8). The proofs in [6] are based on the support functions parametrization of convex compact sets. The linear (and ordering) properties of the support functions, reflecting the corresponding properties of the Minkowski operations on convex sets, allow to reduce the subdivision process on convex compact sets to subdivision on the support functions, and to apply known results on subdivision of scalar functions.

An easy way (suggested by David Levin) to see that F_m^∞ in (11) is the limit of the spline subdivision scheme, is based on the associativity and distributivity of the Minkowski sum and the positive-scalar multiplication of sets. Writing $F^0 = \sum_{i\in\mathbb{Z}} F_i^0 \delta^{[i]}$ with

$$\delta_j^{[i]} = \begin{cases} 1 & \text{for } j = i \\ 0 & j \neq i. \end{cases}$$

we get $F_m^\infty = \sum_i F_i^0 S_m^\infty \delta^{[i]}$. Since $S_m^\infty \delta^{[i]} = B_m(\cdot - i)$ (see e.g. [4, 5]), (11) follows.

The spline subdivision schemes have the following shape preserving properties:

1. Monotonicity preservation: If $F_i^0 \subset F_{i+1}^0$ for all i, then $F_i^k \subset F_{i+1}^k$ for all k, i, and F_m^∞ is monotone in the sense that $F_m^\infty(t) \subset F_m^\infty(t+h)$ for any $t \in \mathbb{R}$ and $h > 0$.

2. Convexity preservation: If $2\ F_{i+1}^0 \supset F_i^0 + F_{i+2}^0$ for all i, then $2F_{i+1}^k \supset F_i^k + F_{i+2}^k$ for all k, i, and F_m^∞ is convex in the sense that its graph is convex, i.e., $2F_m^\infty(t+h) \supset F_m^\infty(t) + F_m^\infty(t+2h)$ for all $h, t \in \mathbb{R}$.

As already mentioned, subdivision schemes for compact sets constitute a method for the approximate reconstruction of 3D objects from their 2D parallel crossections, or, respectively, of 2D shapes from their 1D parallel crossections. Thus the rate of approximation of these schemes is of importance. Indeed, for continuous set-valued functions we have an approximation result.

If the set-valued function $G(\cdot)$ has convex compact images (it is not necessary that its graph is convex), and is Lipschitz continuous, that is, $\text{haus}(G(t+\Delta t), G(t)) = O(\Delta t)$, and the initial data for the spline subdivision scheme consist of samples of G, of the form $F_i^0 = G(i\Delta t)$, $i \in \mathbb{Z}$, then

$$\text{haus}(G(t), F_m^\infty(t)) = O(\Delta t).$$

One can easily use the method of proof in [6] to show that if $G(\cdot)$ is only continuous, then the right-hand side of the last estimate is $O(\omega(G, t, \Delta t))$, where $\omega(G, t, \Delta t)$ is the modulus of continuity of G defined in terms of the Hausdorff distance.

The estimate $\text{haus}(G(t), F_m^\infty(t)) = O((\Delta t)^2)$ is obtained for a multifunction $G(t)$ which has a support function $\delta^*(G(t), l)$ with second derivative with respect to t, uniformly bounded in $l \in S^{n-1}$. Clearly, every multifunction G from $\mathcal{S} \cap C^2$ satisfies this condition.

It is well known that Minkowski averages with equal weights of a large number of nonconvex sets, tend as the number of sets grows, to the limit of the averages with equal wights of the convex hulls of the sets. It turns out that for every spline subdivision scheme, since a fixed Minkowski convex combination of a small number of sets is repeated an infinite number of times, the limit set-valued function equals to the limit multifunction obtained by the same scheme from the convex hulls of the initial sets [7]:

$$F_m^\infty(t) = \sum_{i\in\mathbb{Z}} (\mathrm{co}F_i^0)\ B_m(t-i) \quad \text{for each } t \in \mathbb{R}, \tag{12}$$

The proof of (12) is based on the use of a measure of nonconvexity of a set, the so-called *inner radius*, which is an upper bound for the Hausdorff distance between the set and its convex hull. Two important ingredients are used in the proof. A Pythagorean type upper estimate for the inner radius of a Minkowski sum of compact sets by the inner radii of the summands, proved by Cassels [3], and the fact that the coefficients of averaging in the refinement step of the spline subdivision schemes (10) are non-negative and sum up to 1. With these two ingredients it can be shown that the Hausdorff distance between the set F_i^k and its convex hull vanishes as $k \to \infty$, uniformly in i, as a geometric progression with a ratio less than 1.

Therefore, with the Minkowski averages, no approximation result can be expected for set-valued functions with nonconvex images. This failure of the Minkowski sum for nonconvex sets is in accordance with the observation that the Minkowski average of convex sets has properties, which do not hold for nonconvex sets. Let $A, B, C \in \mathcal{C}_n$, $0 \le \lambda \le 1$. Then

1. $\lambda A + (1-\lambda)A = A, \quad \lambda \in (0,1);$
2. $\lambda A + (1-\lambda)B = \lambda A + (1-\lambda)C \Longrightarrow B = C.$

These two properties do not hold for nonconvex sets. Indeed, for a nonconvex set $A \in \mathcal{K}_n$, $\lambda A + (1-\lambda)A \supset A$.

Here is a simple example, showing that Minkowski averages for nonconvex sets are too big. $A = \{0,1\}$, $A_n = \frac{1}{n}\sum_{i=1}^{n} A = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\right\}$. Moreover, in the Hausdorff metric, $\lim_{n\to\infty} A_n = \mathrm{co}A$, demonstrating the convexification nature of Minkowski averaging processes.

5. The metric average. A binary operation introduced in [1], and called in [8] "metric average", has several properties which make it appropriate for our purposes.

Definition 5.1. *Let $A, B \in \mathcal{K}_n$ and $0 \le t \le 1$. The* **t-weighted metric average of A and B is**

$$A \oplus_t B = \{t\{a\} + (1-t)\Pi_B(a) : a \in A\} \cup \{t\Pi_A(b) + (1-t)\{b\} : b \in B\} \tag{13}$$

where the linear combinations above are in the Minkowski sense.

The following properties of the metric average are easy to observe [8].
Let $A, B, C \in \mathcal{K}_n$ and $0 \leq t \leq 1$, $0 \leq s \leq 1$. Then

1. $A \oplus_0 B = B, \qquad A \oplus_1 B = A, \qquad A \oplus_t B = B \oplus_{1-t} A.$
2. $A \oplus_t A = A.$
3. $A \cap B \subseteq A \oplus_t B \subseteq tA + (1-t)B \subseteq co(A \cup B).$

The metric property of this average, which is essential for our applications and which gave it its name, is proved in [1]:

4. $\mathrm{haus}(A \oplus_t B, A \oplus_s B) = |t-s|\mathrm{haus}(A, B).$

The metric average of sets in $\mathbb{R}$ has several more properties [2].
Let $A, B, C \in \mathcal{K}_1$, $D, E \in \mathcal{C}_1$, $t \in [0,1]$ and let $\mu(A)$ denote the Lebesgue measure of the set A. Then

- $D \oplus_t E = tD + (1-t)E,$
- $\mu(A \oplus_t B) = t\mu(A) + (1-t)\mu(B).$
- $\mu(\mathrm{co}(A \oplus_t B) \setminus (A \oplus_t B)) = t\mu(\mathrm{co}A \setminus A) + (1-t)\mu(\mathrm{co}B \setminus B).$
- $A \oplus_t B = A \oplus_t C \Longrightarrow B = C.$

In the next example we have plotted the one-dimensional sets A, B and the set $C_t = A \oplus_t B$ in one picture, giving B at the y-coordinate 0, A at y=1, and C_t at y= t for $t = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ (see Figure 1). The two sets are

$$A = [0,1] \cup [5,6] \cup [7.5,8] \cup [9,10] \cup [11.5,14], \quad B = [1,4] \cup [5,6.5] \cup [14,16].$$

It follows from the definition of the metric average that the metric average of two sets produces a subset of the Minkowski average and also that the metric average of a set with itself is the set. Indeed, this binary operation, being smaller than the Minkowski average, does not convexify repeated averaging processes. Since it is defined as a binary operation between two sets, in order to use it in spline subdivision schemes we need another representation of these schemes in terms of repeated binary averaging.

6. Spline subdivision schemes with metric averages. First, we represent the spline subdivision schemes in terms of repeated binary averages.

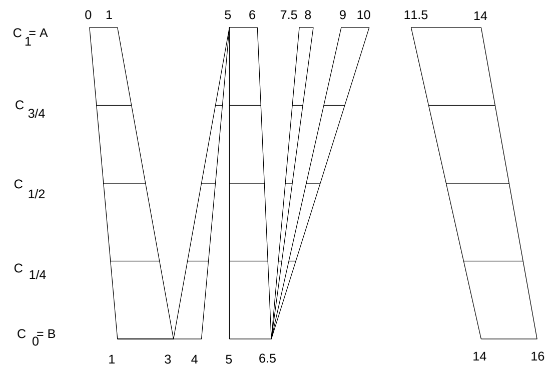

Figure 1. The sets A, B and C_t

The refinement step (3) can be obtained by one step of refinement of the linear spline subdivision, followed by a sequence of binary averages. The sequence of steps which replaces (3) consists of first defining

$$P_{2i}^{k+1,0} = P_i^k, \qquad P_{2i+1}^{k+1,0} = \frac{1}{2}(P_i^k + P_{i+1}^k), \qquad i \in \mathbb{Z}, \tag{14}$$

and then defining for $1 \leq j \leq m-1$ the intermediate averages

$$P_{i+\frac{1}{2}}^{k+1,j} = \frac{1}{2}(P_i^{k+1,j-1} + P_{i+1}^{k+1,j-1}), \qquad i \in I_j, \tag{15}$$

where

$$I_j = \begin{cases} \mathbb{Z}, & j \text{ odd} \\ \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}, & j \text{ even}. \end{cases} \tag{16}$$

The final values at level $k+1$ are

$$P_i^{k+1} = P_i^{k+1,m-1} \qquad \text{for m odd, } i \in \mathbb{Z},$$

$$P_i^{k+1} = P_{i-\frac{1}{2}}^{k+1,m-1} \qquad \text{for m even, } i \in \mathbb{Z}.$$

For example, in case $m = 2$, one step of (14) followed by one step of (15) is equivalent to the refinement step of the Chaikin scheme.

The above procedure is carried over to compact sets, with the metric average replacing the averaging operations in (14) and (15) [8]. First refining with metric averages

$$F_{2i}^{k+1,0} = F_i^k, \qquad F_{2i+1}^{k+1,0} = F_i^k \oplus_{\frac{1}{2}} F_{i+1}^k, \qquad i \in \mathbb{Z}, \tag{17}$$

and then, for $1 \le j \le m-1$, replacing the sequence $\{F_i^{k+1,j-1} \ : \ i \in \mathbb{Z}\}$ by metric averages of pairs of consecutive sets

$$F_{i+\frac{1}{2}}^{k+1,j} = F_i^{k+1,j-1} \oplus_{\frac{1}{2}} F_{i+1}^{k+1,j-1}, \qquad i \in I_j. \tag{18}$$

The final refined sets are

$$F_i^{k+1} = F_i^{k+1,m-1} \quad \text{for m odd, } i \in \mathbb{Z},$$

$$F_i^{k+1} = F_{i-\frac{1}{2}}^{k+1,m-1} \quad \text{for m even, } i \in \mathbb{Z}.$$

The convergence of this scheme follows from the metric property of the metric average. Denote $d^k = \sup_i \text{haus}(F_i^k, F_{i+1}^k), k = 0, 1, \dots$. Then $d^{k+1} \le \frac{1}{2} d^k$. At the k-th stage of the subdivision, the set-valued function $F^k(t)$ is constructed as follows:

$$F^k(t) = F_i^k \oplus_{\lambda(t)} F_{i+1}^k, \qquad i2^{-k} \le t \le (i+1)2^{-k},$$

where $\lambda(t) = (i+1) - t2^k$. It follows from the metric property of the metric average that

$$\sup_t \text{haus}(F^{k+1}(t), F^k(t)) = O(2^{-k}),$$

therefore $\{F^k(t)\}_{k \in \mathbb{Z}_+}$ is a Cauchy sequence in the complete metric space $\mathcal{K}_n$. Thus the limit of this sequence exists and is denoted by $S_m^\infty F^0(t)$.

The approximation property below justifies the reconstruction of objects from their parallel cross sections by a spline subdivision scheme which uses metric averages instead of Minkowski averages.

Let the univariate multifunction $G(t)$ have compact images and let it be Lipschitz continuous. If $F_i^0 = G(i\Delta t), \ i \in \mathbb{Z}$, then

$$\sup_t \text{haus}(S_m^\infty F^0(t), G(t)) = O(\Delta t).$$

An example [8] of a shell included between two quarters of spheres is represented in Figure 2.

This body can be represented by the following set-valued function $F(x)$, defined for $0 \le x \le 1$:

$$F(x) = \{ \ (y,z) \in \mathbb{R}^2 \ | \ z \le 0, \ r(x) \le y^2 + z^2 \le 0.2 + r(x) \ \},$$

where $r(x) = 1 - x^2$. Given the initial crossections $F(0), F(h), F(2h), \dots, F(1)$, we reconstruct this shell by a metric subdivision scheme of Chaikin type, and obtain a sequence of piecewise linear (in a metric sense) set-valued functions $\{F^k(t)\}_{k=0}^\infty$, with $F^k(t)$ interpolating the sets generated at level k. The crossections $F^3\left(\frac{h}{2} + 0.25i\right)$, $i = 0, 1, 2, 3$ of F^3, obtained after three subdivision iterations from the initial sets as above with $h = 0.125$, are presented in Figure 3.

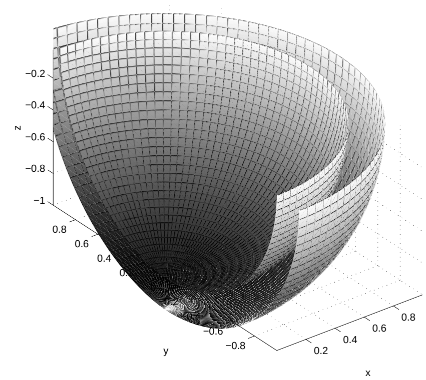

Figure 2. A shell included between two quarters of spheres

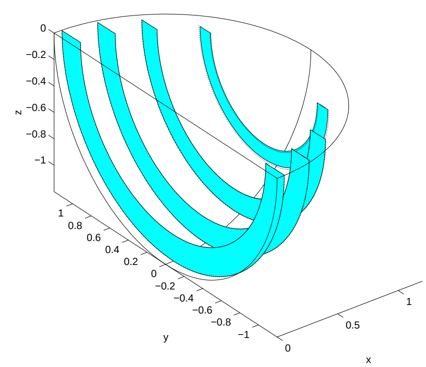

Fig. 3. Four cross-sections of the final body

The maximal Hausdorff distance between these crossections at the third iteration and the corresponding crossections of the initial object is 0.0122.

Since 2D shapes and 3D objects, when regarded as univariate multifunctions, are usually discontinuous in the Hausdorff metric at points of change of topology, the above approximation result does not hold near such points. This observation calls for a special treatment near points of change of topology, a subject which is still under investigation.

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Received September 30, 2002