Spline Subdivision Schemes for Compact Sets with Metric Averages

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Abstract. To define spline subdivision schemes for general compact sets, we use the representation of spline subdivision schemes in terms of repeated averages, and replace the usual average (convex combination) by a binary averaging operation between two compact sets, introduced in [1] and termed here the "metric average". These schemes are shown to converge in the Hausdorff metric, and to provide $O(h)$ approximation.

§1. Introduction

In this paper, we introduce spline subdivision schemes for general compact sets.

Motivated by the problem of the reconstruction of 3D objects from their 2D cross-sections, we consider spline subdivision schemes operating on data consisting of compact sets. A spline subdivision scheme generates from such initial data a sequence of set-valued functions, with compact sets as images, which converges in the Hausdorff metric to a limit set-valued function. In the case of 2D sets, the limit set valued function, with 2D sets as images, describes a 3D object.

For the case of initial data consisting of convex compact sets, we introduced in [3] spline subdivision schemes, where the usual addition of numbers is replaced by Minkowski sums of sets. Then, the spline subdivision schemes generate limit set-valued functions with convex compact images which can be expressed as linear combinations of integer shifts of a B-spline, with the initial sets as coefficients. The subdivision techniques are used to conclude that these limit "set-valued spline functions" have shape preserving properties similar to those of scalar spline functions, for shape properties defined on sequences of sets and on set-valued functions.

For the case of non-convex initial sets, it is shown in [4] that the limit set-valued function, generated by a spline subdivision scheme, using the Minkowski sums, coincides with the limit set-valued function, generated by the same

Trends in Approximation Theory
Kirill Kopotun, Tom Lyche, and Marian Neamtu (eds.), pp. 95–104.

ISBN 0-8265-1379-4.

subdivision scheme from the convex hulls of the initial sets. Therefore, this generated set-valued function has too large images to be a good approximation to the set-valued function from which the initial non-convex sets were sampled.

To define spline subdivision schemes for general compact sets which do not convexify the initial data, *i.e.*, preserve the non-convexity, we use the representation of spline subdivision schemes in terms of repeated averages, as presented in Section 2. The usual Minkowski average is replaced by a binary operation between two compact sets, introduced in [1]. This binary operation between sets, termed here the "metric average", is discussed in Section 3. As is shown in Section 4, spline subdivision schemes, based on the metric average, converge, in the Hausdorff metric, to set-valued functions which are Lipschitz continuous. Also, for initial data sampled from a Lipschitz continuous set-valued function with compact images, the limit function of a spline subdivision scheme approximates the sampled set-valued function to order $O(h)$.

§2. Spline Subdivision Schemes via Repeated Binary Averages

An m-th degree spline subdivision scheme, in the scalar setting, refines the values

$$f^{k-1} = \{f^{k-1}_\alpha | \alpha \in \mathbb{Z}\} \subset \mathbb{R},$$

where f^k is defined by

$$f^k_\alpha = \sum_{\beta \in \mathbb{Z}} a^{[m]}_{\alpha - 2\beta} f^{k-1}_\beta, \quad \alpha \in \mathbb{Z}, \quad k = 1, 2, 3, ..., \tag{1}$$

with the mask $a^{[m]}_{\alpha - [\frac{m+1}{2}]} = \binom{m+1}{\alpha}/2^m$, $\alpha = 0, 1, ..., m+1$, and $a^{[m]}_{\alpha - [\frac{m+1}{2}]} = 0$, for $\alpha \in \mathbb{Z} \setminus \{0, 1, ..., m+1\}$. It is clear from (1) and the mask formulae that f^k_α is an average of two or more values from f^{k-1}. It is well known that the values f^k can be obtained by one step of first degree spline subdivision followed by a sequence of binary averaging. Thus, first define

$$f^{k,0}_{2\alpha} = f^{k-1}_\alpha, \qquad f^{k,0}_{2\alpha+1} = \tfrac{1}{2}(f^{k-1}_\alpha + f^{k-1}_{\alpha+1}), \qquad \alpha \in \mathbb{Z}. \tag{2}$$

Then, for $1 \le j \le m-1$, define the intermediate averages

$$f^{k,j}_{\alpha + \frac{1}{2}} = \tfrac{1}{2}(f^{k,j-1}_\alpha + f^{k,j-1}_{\alpha+1}), \qquad \alpha \in I_j,$$

where

$$I_j = \begin{cases} \mathbb{Z}, & j \text{ odd}, \\ \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}, & j \text{ even}. \end{cases} \tag{3}$$

The final values at level k are

$$f^k_\alpha = f^{k,m-1}_\alpha, \quad \text{for } m \text{ odd}, \ \alpha \in \mathbb{Z},$$
$$f^k_\alpha = f^{k,m-1}_{\alpha - \frac{1}{2}}, \quad \text{for } m \text{ even}, \ \alpha \in \mathbb{Z}.$$

For example, in the case $m = 2$, one step of averaging yields the Chaikin algorithm in the form

$$f^k_{2i} = \tfrac{1}{4} f^{k-1}_{i-1} + \tfrac{3}{4} f^{k-1}_{i},$$
$$f^k_{2i-1} = \tfrac{3}{4} f^{k-1}_{i-1} + \tfrac{1}{4} f^{k-1}_{i}.$$

At each level k, the piecewise linear function, interpolating the data $(2^{-k}\alpha, f^k_\alpha)$, $\alpha \in \mathbb{Z}$, is defined on $\mathbb{R}$ by

$$f^k(t) = \left(\frac{2^{-k}(\alpha+1) - t}{2^{-k}}\right) f^k_\alpha + \left(\frac{t - 2^{-k}\alpha}{2^{-k}}\right) f^k_{\alpha+1}. \tag{4}$$

for $\alpha 2^{-k} \le t \le (\alpha+1)2^{-k}$. Note that, by (4), every value of $f^k(t)$ is a weighted average of two consecutive elements in f^k. If the constructed sequence $\{f^k(t)\}_{k\in\mathbb{Z}_+}$ converges uniformly to a continuous function $f^\infty(t)$, then $f^\infty(\cdot)$ is defined as the limit function of the subdivision scheme [2]. Thus, the limit function of spline subdivision schemes can be described in terms of binary averages only.

In this paper, we study spline subdivision schemes for compact sets instead of scalars. Since we want to avoid the convexification caused by the use of Minkowski averages [4], we propose, in the next section, to use instead the binary metric average introduced in [1].

§3. The Metric Average of Two Sets

In many applications, averages of sets are defined as Minkowski averages. Here, we propose to use a different kind of a weighted average of two sets, which is a subset of the Minkowski convex combination and which possesses several important properties.

First, we introduce some notations. The collection of all compact subsets of $\mathbb{R}^n$ is denoted by $\mathcal{K}_n$, $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$, $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$, the Hausdorff distance between the sets $A, B \in \mathcal{K}_n$ is $\mathrm{haus}(A,B)$, $\mathrm{co}\, A$ denotes the convex hull of A, the Euclidean distance from a point x to a set $A \in \mathcal{K}_n$ is $\mathrm{dist}(x, A)$. The set of all projections of x on the set A is

$$\Pi_A(x) = \big\{ a \in A \ : \ |a - x| = \mathrm{dist}(x, A) \big\}.$$

The set difference of $A, B \in \mathcal{K}_n$ is

$$A \setminus B = \{a \ : \ a \in A,\ a \notin B\}.$$

A linear Minkowski combination of two sets A and B is

$$\lambda A + \mu B = \{\lambda a + \mu b \ : \ a \in A,\ b \in B\},$$

for $A, B \in \mathcal{K}_n$ and $\lambda, \mu \in \mathbb{R}$. The Minkowski sum $A + B$ corresponds to a linear Minkowski combination with $\lambda = \mu = 1$.

A segment is denoted by

$$[c, d] = \{\lambda c + (1-\lambda)d \ : \ 0 \le \lambda \le 1\}, \quad c, d \in \mathbb{R}^n$$

Definition 3.1. *Let $A, B \in \mathcal{K}_n$ and $0 \le t \le 1$. The t-weighted metric average of A and B is:*

$$A \oplus_t B = M(A, t, B) \bigcup M(B, 1-t, A) \tag{5}$$

with

$$M(A, t, B) = \bigcup_{a \in A} \big(t\{a\} + (1-t)\Pi_B(a)\big), \tag{6}$$

where the linear combinations in the last equality are in the Minkowski sense.

The metric average and its two components in (5) have several noticeable properties.

Theorem 3.2. *Let $A, B, C \in \mathcal{K}_n$ and $0 \le t \le 1$, $0 \le s \le 1$. Then, the following properties hold:*

1. $M(M(A, t, B), s, B) = M(A, ts, B)$.
2. $M(A \cap B, t, B) = A \cap B \subseteq M(B, s, A)$.
3. $A \oplus_t B = (A \cap B) \bigcup M(A \setminus B, t, B) \bigcup M(B \setminus A, 1-t, A)$.
4. $A \oplus_0 B = B$, $A \oplus_1 B = A$, $A \oplus_t B = B \oplus_{1-t} A$.
5. $A \oplus_t A = A$.
6. $A \oplus_t B \subseteq tA + (1-t)B \subseteq co(A \cup B)$.
7. *If B is a convex superset of A, then, for $0 \le t < s \le 1$,*

$$A \subseteq A \oplus_s B \subseteq A \oplus_t B \subseteq B \ .$$

8. *The Hausdorff distance between $A \oplus_t B$ and each of the given sets A and B is a linear function of t, or, more generally, for $s, t \in [0, 1]$,*

$$\mathrm{haus}(A \oplus_t B, A \oplus_s B) = |t-s| \, \mathrm{haus}(A, B).$$

Proof: The proof of Property 1 is easily obtained by the observation that, for every $b \in \Pi_B(a)$ and $0 \le t < 1$,

$$\Pi_B(ta + (1-t)b) = \{b\}.$$

Otherwise, there is a closer point to a from B. Therefore,

$$\begin{aligned} M\big(M(A, t, B), s, B\big) &= \big\{s(ta + (1-t)b) + (1-s)b \ | a \in A, \ b \in \Pi_B(a)\big\} \\ &= \big\{(ts)a + (1-ts)b \ | a \in A, \ b \in \Pi_B(a)\big\} \\ &= M(A, ts, B). \end{aligned}$$

Properties 2–6 follow from the definition. Property 8 is proved in [1].

To prove Property 7, we note that since $A \subseteq B$, $A = A \cap B \subseteq B$. By Property 2, $M(A,t,B) = M(A \cap B,t,B) \subseteq M(B,1-t,A)$, therefore $A \oplus_t B = M(B,1-t,A)$. Hence, by Property 6 and the convexity of B,

$$A = A \cap B \subseteq M(B,1-t,A) = A \oplus_t B \subseteq \mathrm{co}(A \cup B) = B.$$

Thus, it remains to prove that $M(B,1-s,A) \subseteq M(B,1-t,A)$. This is easily established by the convexity of B, which yields that, for each $b \in B$ and $a \in \Pi_A(b)$, the whole segment $[a, ta+(1-t)b]$ is a subset of $M(B,1-t,A)$. Since, for $s \geq t$, $[a, sa+(1-s)b] \subseteq [a, ta+(1-t)b]$, the desired inclusion follows. □

Remark 3.3. In contrast to Property 5, the equality $tA + (1-t)A = A$ with the Minkowski average, is true only if A is convex. Generally, only an inclusion holds: $A \subseteq tA + (1-t)A \subseteq \mathrm{co}\, A$. It is well-known that the sequence of increasing Minkowski averages of a set A tends to $\mathrm{co}\, A$. (See *e.g.,* [4, Example 3.11]).

Note that $A \oplus_t B$ may be non-convex even for convex sets $A,\ B \subset \mathbb{R}^n$, for $n \geq 2$ ([1]).

It follows from Property 2 that $A \subseteq B \Longrightarrow A \subseteq A \oplus_t B$. In general, in the nonconvex case, it is not true that $A \subseteq B \Longrightarrow A \oplus_t B \subseteq B$. This is true only if B is convex and is proved in Property 7.

Example 3.4. Let the sets A and B in $\mathbb{R}^2$ be a ring and its center, respectively:

$$A = \big\{(x,y) \in \mathbb{R}^2 \ : \ r_1 \leq x^2 + y^2 \leq r_2\big\}, \qquad B = \big\{(0,0)\big\}.$$

Then it is not hard to see that the metric average of A and B with a weight $t \in [0,1]$ is the t times contracted ring A:

$$A \oplus_t B = \big\{(x,y) \in \mathbb{R}^2 \ : \ t^2 r_1 \leq x^2 + y^2 \leq t^2 r_2\big\}.$$

§4. Metric Spline Subdivision Schemes for Compact Sets

Given $\{F^0_\alpha\}_{\alpha\in\mathbb{Z}}$, a sequence of compact sets in $\mathbb{R}^n$, we define recursively a sequence of sequences $\big\{\{F^k_\alpha\}_{\alpha\in\mathbb{Z}}\big\}_{k\in\mathbb{Z}_+}$ of compact sets.

First, we define the initial sets at level k, from the sets at level $k-1$, by

$$F^{k,0}_{2\alpha} = F^{k-1}_\alpha, \qquad F^{k,0}_{2\alpha+1} = F^{k-1}_\alpha \oplus_{\frac{1}{2}} F^{k-1}_{\alpha+1}, \qquad \alpha \in \mathbb{Z}. \tag{7}$$

Then, for $1 \leq j \leq m-1$, we define the intermediate metric averages

$$F^{k,j}_{\alpha+\frac{1}{2}} = F^{k,j-1}_\alpha \oplus_{\frac{1}{2}} F^{k,j-1}_{\alpha+1}, \qquad \alpha \in I_j, \tag{8}$$

where I_j is defined in (3). The final sets at level k are defined as

$$F_\alpha^k = F_\alpha^{k,m-1} \quad \text{for } m \text{ odd}, \tag{9}$$
$$F_\alpha^k = F_{\alpha-\frac{1}{2}}^{k,m-1} \quad \text{for } m \text{ even}, \tag{10}$$

for $\alpha \in \mathbb{Z}$, with the corresponding piecewise-linear interpolating set-valued function $F^k(\cdot)$,

$$F^k(t) = F_{\alpha+1}^k \oplus_{2^k t-\alpha} F_\alpha^k, \qquad \alpha 2^{-k} \le t \le (\alpha+1)2^{-k}, \quad \alpha \in \mathbb{Z}. \tag{11}$$

First, we prove two basic metric results which are used in the proof of the convergence theorem and in the proof of the approximation result.

Lemma 4.1. *Let $F^k = \{F_\alpha^k \mid \alpha \in \mathbb{Z}\}$ be defined as above and let*

$$d^k = \sup_{\alpha \in \mathbb{Z}} \text{haus}(F_\alpha^k, F_{\alpha+1}^k).$$

Then

$$d^k \le 2^{-k} d^0. \tag{12}$$

Proof: If we denote

$$d^{k,j} = \sup_{\alpha \in I_{j-1}} \text{haus}(F_\alpha^{k,j}, F_{\alpha+1}^{k,j}), \tag{13}$$

then it follows from (7) and Property 8 that

$$\text{haus}(F_\beta^{k,0}, F_{\beta\pm1}^{k,0}) \le \tfrac{1}{2} d^{k-1}, \qquad \beta \in \mathbb{Z},$$

and, therefore

$$d^{k,0} \le \tfrac{1}{2} d^{k-1}. \tag{14}$$

Also, for $\alpha \in I_j$, by (8), the triangle inequality and Property 8,

$$\begin{aligned}\text{haus}(F_{\alpha-\frac{1}{2}}^{k,j}, F_{\alpha+\frac{1}{2}}^{k,j}) &\le \text{haus}(F_{\alpha-\frac{1}{2}}^{k,j}, F_\alpha^{k,j-1}) + \text{haus}(F_\alpha^{k,j-1}, F_{\alpha+\frac{1}{2}}^{k,j}) \\ &\le \tfrac{1}{2}\text{haus}(F_{\alpha-1}^{k,j-1}, F_\alpha^{k,j-1}) + \tfrac{1}{2}\text{haus}(F_\alpha^{k,j-1}, F_{\alpha+1}^{k,j-1}) \\ &\le d^{k,j-1},\end{aligned}$$

which implies $d^{k,j} \le d^{k,j-1}$, and therefore

$$d^{k,j} \le d^{k,0}. \tag{15}$$

This, together with (12) yields

$$d^k = d^{k,m-1} \le \tfrac{1}{2} d^{k-1}, \tag{16}$$

hence (12) holds. □

Lemma 4.2. *Let the sequence* $\{F^k(\cdot)\}_{k\in\mathbb{Z}_+}$ *be defined as above. Then,*

$$\text{haus}\big(F^{k+1}(t), F^k(t)\big) \le C d^k, \quad k \in \mathbb{Z}_+, \tag{17}$$

where $C = \frac{3}{4} + \frac{m}{4}$, *with* m *the degree of the spline subdivision scheme.*

Proof: First, we prove the inequality

$$\text{haus}(F_{2\alpha}^{k+1}, F_\alpha^k) \le \frac{m-1}{4} d^k, \quad k \in \mathbb{Z}_+. \tag{18}$$

We prove (18) for m odd. The case of m even is similar. By (7), (9) and the triangle inequality we get

$$\begin{aligned}\text{haus}(F_\alpha^k, F_{2\alpha}^{k+1}) &= \text{haus}(F_{2\alpha}^{k+1,0}, F_{2\alpha}^{k+1,m-1}) \\ &\le \text{haus}(F_{2\alpha}^{k+1,0}, F_{2\alpha+\frac{1}{2}}^{k+1,1}) + \text{haus}(F_{2\alpha+\frac{1}{2}}^{k+1,1}, F_{2\alpha}^{k+1,2}) + \dots \\ &\quad + \text{haus}(F_{2\alpha+\frac{1}{2}}^{k+1,m-2}, F_{2\alpha}^{k+1,m-1}).\end{aligned}$$

It follows by (8), (13) and Property 8, that

$$\text{haus}(F_\alpha^k, F_{2\alpha}^{k+1}) \le \sum_{j=0}^{m-2} \tfrac{1}{2} d^{k+1,j}.$$

Using (15) and (14), one gets

$$\text{haus}(F_\alpha^k, F_{2\alpha}^{k+1}) \le \frac{(m-1)d^{k+1,0}}{2} \le \frac{(m-1)d^k}{4}.$$

Now, we prove (17). Let $\alpha 2^{-k} \le t \le (\alpha + \frac{1}{2})2^{-k}$. It follows from (11) and the metric Property 8, that

$$\text{haus}\big(F_\alpha^k, F^k(t)\big) \le \tfrac{1}{2} d^k, \quad \text{haus}\big(F^{k+1}(t), F_{2\alpha}^{k+1}\big) \le d^{k+1},$$

hence, by the triangle inequality, (18), Property 8 and (16), we obtain

$$\begin{aligned}\text{haus}\big(F^{k+1}(t), F^k(t)\big) &\le \text{haus}\big(F^{k+1}(t), F_{2\alpha}^{k+1}\big) + \text{haus}\big(F_{2\alpha}^{k+1}, F_\alpha^k\big) \\ &\quad + \text{haus}\big(F_\alpha^k, F^k(t)\big) \\ &\le d^{k+1} + \frac{m-1}{4} d^k + \tfrac{1}{2} d^k \le C d^k.\end{aligned}$$

For $(\alpha + \frac{1}{2})2^{-k} \le t \le (\alpha + 1)2^{-k}$, we have a similar bound, using $F_{2(\alpha+1)}^{k+1}$ instead of $F_{2\alpha}^{k+1}$. □

Theorem 4.3. *The sequence* $\{F^k(\cdot)\}_{k\in\mathbb{Z}_+}$ *converges uniformly to a set-valued function* $F^\infty(\cdot)$, *which is Hausdorff Lipschitz continuous with a Lipschitz constant* $d^0 = \sup_\alpha \text{haus}(F_\alpha^0, F_{\alpha+1}^0)$.

Proof: By Lemma 4.1, (11), and Property 8, it follows that, for every δ

$$\text{haus}\big(F^k(t+\delta), F^k(t)\big) \le \delta 2^k d^k \le \delta d^0.$$

Hence, the set-valued functions $F^k(\cdot)$ are uniformly Lipschitz continuous with the constant d^0. By the triangle inequality,

$$\operatorname{haus}\big(F^{k+M}(t), F^k(t)\big) \le \sum_{i=k}^{k+M-1} \operatorname{haus}\big(F^{i+1}(t), F^i(t)\big),$$

and, by Lemmas 4.1 and 4.2, for any positive integer M,

$$\operatorname{haus}\big(F^{k+M}(t), F^k(t)\big) \le C \sum_{i=k}^{k+M-1} d^i \le C \frac{d^0}{2^{k-1}}, \tag{19}$$

where C is defined in Lemma 4.2. This implies that, for every t, the sequence $\{F^k(t)\}_k$ is a Cauchy sequence in $\mathcal{K}_n$, and, since $\mathcal{K}_n$ is a closed metric space under the Hausdorff metric, the sequence $\{F^k(t)\}_k$ tends, for each t, to a compact set $F^\infty(t)$. The convergence is uniform in t by (19). The uniform Lipschitz continuity of $\{F^k(\cdot)\}_k$ yields that $F^\infty(t)$ is Lipschitz continuous with the same constant d^0. □

Theorem 4.4. *Let the set-valued function* $G(\cdot) : \mathbb{R} \to \mathcal{K}_n$ *be Lipschitz continuous with a Lipschitz constant* L, *and let the initial sets be given by* $F^0_\alpha = G(a+\alpha h)$, $\alpha \in \mathbb{Z}$, *for arbitrary* $a \in [0,h)$. *Then,*

$$\operatorname{haus}\big(F^k(t), G(t)\big) \le C_k h, \quad \text{for each } k = 0, 1, ..., \tag{20}$$

where $F^k(\cdot)$ *is defined in (11), and*

$$C_k = \left(\frac{7+m}{2}\right) L, \quad k \ge 1, \quad C_0 = 2L.$$

Proof: By (19), the triangle inequality and the metric Property 8, we get, for $k \ge 1$ and t satisfying $\alpha h \le t \le (\alpha+1)h$,

$$\begin{aligned}\operatorname{haus}\big(F^k(t), G(t)\big) &\le \operatorname{haus}\big(F^k(t), F^0(t)\big) + \operatorname{haus}\big(F^0(t), F^0_\alpha\big) + \operatorname{haus}\big(F^0_\alpha, G(t)\big)\\ &\le 2Cd^0 + d^0 + Lh \le 2(C+1)Lh,\end{aligned}$$

where C is defined in Lemma 4.2. Here, we used the Lipschitz condition on G which yields that $d^0 \le Lh$. This proves the claim of the theorem, since, for $k=0$, the term with C is missing. □

Corollary 4.5. *Under the assumptions of Theorem 4.4, the distance between the original set-valued function* $G(\cdot)$ *and the limit set-valued function* $F^\infty(\cdot)$ *is bounded by*

$$\max_t \operatorname{haus}(F^\infty(t), G(t)) \le \frac{7+m}{2} Lh. \tag{21}$$

Remark 4.6. The last corollary indicates that the metric spline subdivision scheme produces good approximations of $G(\cdot)$ even for non-convex-valued set-valued function $G(\cdot)$ with non-convex images. An analogous result for spline

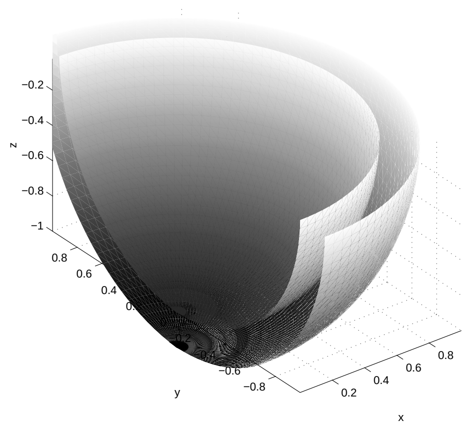

Fig. 1. A shell included between two quarters of spheres.

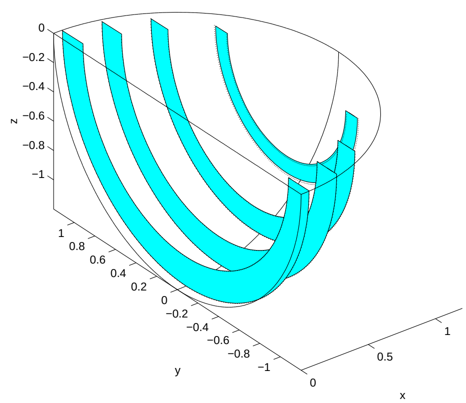

Fig. 2. Four cross-sections of the final body.

subdivision schemes, based on Minkowski sums of sets, is true only for convex-valued $G(\cdot)$, since, as is shown in [4], the limit set-valued functions generated by such schemes are convex-valued. The inequality (21) implies that, if the initial set-valued function G is not convex-valued, then, for small h, the limit set-valued function F^∞ has non-convex images, which are close to those of G in the Hausdorff metric.

This approximation property is also attained by $F^0(\cdot)$, according to Theorem 4.4. Yet, it is expected that F^∞ is "smoother" than F^0, as is indicated by our numerical tests. In the future work, we intend to develop smoothness measures for such set-valued functions in order to quantify this statement.

Example 4.7. A shell included between two quarters of spheres is represented in Figure 1. This body can be represented by the following set-valued function $F(x)$, defined, for $0 \le x \le 1$, by

$$F(x) = \left\{(y,z) \in \mathbb{R}^2 \mid z \le 0,\ r(x) \le y^2 + z^2 \le R(x)\right\},$$

where $r(x) = 1 - x^2$, $R(x) = (1.2)^2 - x^2$. Given the initial cross-sections $F(0), F(h), F(2h), ..., F(1)$, we reconstruct this shell by a metric subdivision scheme of Chaikin type, and obtain a sequence of piecewise linear (in a metric sense) set-valued functions $F^k : [a_k, b_k] \to \mathcal{K}_n$, where $a_k = \frac{1}{2}(1-2^{-k})h$, $b_k = 1 - a_k$. The cross-sections $F^3(\frac{h}{2} + 0.25i), i = 0,1,2,3$, of F^3, obtained after three subdivision iterations from the initial sets as above with $h = 0.125$, are presented in Figure 2. The maximal error between these cross-sections at the third iteration and the corresponding cross-sections of the initial object is 0.0122. The calculations and pictures are obtained using MATLAB 6.

Remark 4.8. The application of Corollary 4.5 to the reconstruction of 3D objects from their 2D cross-sections, as in Example 4.7, is not as wide as might be expected. It is very easy to observe that, for a nonconvex compact 3D object, even if its boundaries (outer and inner boundaries) are smooth, the univariate set-valued function, with images the 2D parallel cross-sections of the object, might be discontinuous at certain points of the boundary. Discontinuity points are boundary points with a tangent plane parallel to the cross-sections planes, and the last planes have empty intersection with the object in the neighborhood of the contact point. Our conclusion is that a 3D object can be approximated by a spline subdivision scheme, for any direction of parallel cross-sections, if it is a smooth convex object with smooth convex holes.

In our approach, we approximate the object "inside" its domain, namely, for those values of the 1D variable corresponding to non-empty cross-sections. Our method, as it is represented here, does not approximate well the object near the boundary of its domain.

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