Linear Subdivision Schemes for the Refinement of Geometric Objects

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Abstract. Subdivision schemes are efficient computational methods for the design, representation and approximation of surfaces of arbitrary topology in $\mathbb{R}^3$. Subdivision schemes generate curves/surfaces from discrete data by repeated refinements. This paper reviews some of the theory of linear stationary subdivision schemes and their applications in geometric modelling. The first part is concerned with "classical" schemes refining control points. The second part reviews linear subdivision schemes refining other objects, such as vectors of Hermite-type data, compact sets in $\mathbb{R}^n$ and nets of curves in $\mathbb{R}^3$. Examples of various schemes are presented.

Mathematics Subject Classification (2000). Primary 65D07, 65D17, 65D18; Secondary 41A15, 68U07.

Keywords. Subdivision schemes, geometric modelling, curves, surfaces, refinements, smoothness, arbitrary topology, nets of points, nets of curves, compact sets, approximation order.

1. Introduction

Subdivision schemes in geometric applications are efficient tools for the generation of curves/surfaces from discrete data, by repeated refinements.

The first subdivision schemes where devised by de Rahm [53] for the generation of functions with a first derivative everywhere and a second derivative nowhere.

In geometric modelling the first schemes were proposed for easy and quick rendering of B-spline curves. A B-spline curve has the form

$$C(t) = \sum_i P_i B_m(t-i) \tag{1}$$

with $\{P_i\}$ points in $\mathbb{R}^d$ ($d = 2$ or 3) termed control points, and B_m a B-spline of degree m with integer knots, namely $B_m|_{[i,i+1]}$ is a polynomial of degree m, $B_m \in C^{m-1}(\mathbb{R})$, $\operatorname{supp} B_m = [0, m+1]$. Equation (1) is a parametric representation of a B-spline curve. By using the refinement equation satisfied by a B-spline,

$$B_m(x) = \sum_{i=0}^{m+1} a_i^{[m]} B_m(2x-i)\,, \quad a_i^{[m]} = 2^{-m}\binom{m+1}{i}\,,\ i = 0,\dots,m+1\,. \tag{2}$$

$C(t)$ in (1) has the parametric representations

$$C(t) = \sum_i P_i^0 B_m(t-i) = \sum_i P_i^1 B_m(2t-i) = \dots \\ = \sum_i P_i^k B_m(2^k t - i) = \dots, \tag{3}$$

where

$$P_i^{\ell+1} = \sum_j a_{i-2j}^{[m]} P_j^\ell, \quad \ell = 0,1,2,\dots, \tag{4}$$

with the convention $a_i^{[m]} = 0$, $i \notin \{0,1,\dots,m+1\}$.

As is demonstrated in §2.3, the differences $\{P_i^k - P_{i-1}^k\}$ tend to zero as k increases, and since $B_m \geq 0$ and $\sum_i B_m(t-i) \equiv 1$ [6], the polygonal line through the control points $\{P_i^k\}$ is close to $C(t)$ for k large enough, and can be easily rendered.

The relation (4) encompasses the refinement rule for B-spline curves. The first scheme of this type was devised by Chaikin [10] for quadratic B-spline curves, and the schemes for general B-spline curves were investigated in [14]. All other subdivision schemes can be regarded as a generalization of the spline case.

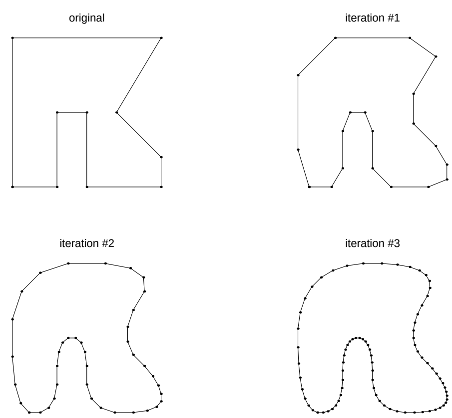

Figure 1. Refinements of a polygon with Chaikin scheme

In this paper we first review the "classical" subdivision schemes for the refinement of control points. The schemes for the generation of curves are direct generalizations of (4), in the sense that the coefficients, defined in (2), are replaced by other sets of coefficients. The "classical" schemes, and in particular those generating surfaces, are used extensively in Computer Graphics. In §2 we discuss the construction of such schemes, their approximation properties, tools for the analysis of their convergence and smoothness, and their application to the generation of surfaces from general nets of points in $\mathbb{R}^3$. Examples of important schemes are presented.

Subdivision schemes for the refinement of objects other than control points are reviewed in §3. These schemes include subdivision schemes refining vectors, in particular, vectors consisting of values of a function and its consecutive derivatives, schemes refining compact sets in $\mathbb{R}^n$ and a scheme refining nets of curves.

All the schemes reviewed in this paper are linear. Recently, various non-linear schemes were devised and analyzed (see, e.g., [26] and references therein). It seems that this is one of the future directions in the study of subdivision schemes. Applications of "classical" schemes to the numerical solution of special types of PDEs is another direction. (See, e.g., [11]).

New "classical" schemes are still being devised for particular applications. For example, adaptive refinements can be accomplished straightforwardly by refining according to topological rules different from the "classical" ones, therefore, corresponding linear schemes had to be devised (see, e.g. [47] and [58]).

2. Stationary linear schemes for the refinement of control points

A subdivision scheme $S_{\mathbf{a}}$ for the refinement of control points is defined by a finite set of coefficients called *mask* $\mathbf{a} = \{a_i \in \mathbb{R} : i \in \sigma(\mathbf{a}) \subset \mathbb{Z}^s\}$. Here $\sigma(\mathbf{a})$ denotes the finite support of the mask, $s = 1$ corresponds to curves and $s = 2$ to surfaces. The refinement rule is

$$P_\alpha^{k+1} = \sum_{\beta \in \mathbb{Z}^s} a_{\alpha-2\beta} P_\beta^k \,, \quad \alpha \in \mathbb{Z}^s \,. \tag{5}$$

Remark. In most of the paper we consider schemes defined on $\mathbb{Z}^s$, although, in geometric applications the schemes operate on finite sets of data. Due to the finite support of the mask, our considerations apply directly to closed curves/surfaces, and also to "open" ones, except in a finite zone near the boundary.

In the case $s = 1$, a subdivision scheme is termed *uniformly convergent* (or convergent for geometric applications) if the sequence $\{\mathcal{P}^k(t)\}$ of polygonal lines through the control points at refinement levels $k = 0, 1, 2, \ldots$ (with parametric representation as the piecewise linear interpolants to the data $\{(i2^{-k}, P_i^k) : i \in \mathbb{Z}\}$, $k = 0, 1, 2, \ldots$), converges uniformly in bounded intervals. In the case $s = 2$, we require the uniform convergence of the sequence of piecewise bi-linear interpolants to the data $\{(\alpha 2^{-k}, P_\alpha^k) : \alpha \in \mathbb{Z}^2\}$ on bounded squares [9], [33], [24].

The convergence of a scheme $S_{\mathbf{a}}$ implies the existence of a *basic-limit-function* $\phi_{\mathbf{a}}$, being the limit obtained from the initial data, $f_i^0 = 0$ everywhere on $\mathbb{Z}^s$ except $f_0^0 = 1$.

It follows from the linearity and uniformity of (5) that the limit obtained from any set of initial control points $\mathbf{P}^0 = \{P_\alpha^0 \in \mathbb{R}^d : \alpha \in \mathbb{Z}^s\}$, $S_{\mathbf{a}}^\infty \mathbf{P}^0$, can be written in terms of integer translates of $\phi_{\mathbf{a}}$, as

$$S_{\mathbf{a}}^\infty \mathbf{P}^0(x) = \sum_{\alpha \in \mathbb{Z}^s} P_\alpha^0 \phi_{\mathbf{a}}(x - \alpha) \,, \quad x \in \mathbb{R}^s. \tag{6}$$

For $s = 1$ and $d = 2$ or $d = 3$, (6) is a parametric representation of a curve in $\mathbb{R}^d$, while for $s = 2$ and $d = 3$, (6) is a parametric representation of a surface in $\mathbb{R}^3$. Also, by the linearity, uniformity and stationarity of the refinement (5), $\phi_{\mathbf{a}}$ satisfies the *refinement equation* (*two-scale relation*)

$$\phi_{\mathbf{a}}(x) = \sum_{\alpha \in \mathbb{Z}^s} a_\alpha \phi_{\mathbf{a}}(2x - \alpha)\,, \tag{7}$$

analogous to the refinement equation (2) for B-splines.

It follows from (5) or from (7) that $\mathrm{supp}(\phi_{\mathbf{a}})$ is contained in the convex hull of $\sigma(\mathbf{a})$ [9], as is the case for the B-spline schemes.

The choice of the mask in the design of good schemes is partly heuristic and partly aims at obtaining specific properties of the scheme as convergence, smoothness, locality, interpolation, shape preservation, and approximation order.

For the case $s = 1$, the topology of $\mathbb{Z}$ is sufficient to describe an ordered set of control points for curve design. For the case $s = 2$, the topology of $\mathbb{Z}^2$, where the point (i, j) is connected to the four points $(i \pm 1, j), (i, j \pm 1)$, is sufficient to describe a set of control points in $\mathbb{R}^3$, connected each to four neighboring points and constituting a *quad-mesh*. The above connectivity of $\mathbb{Z}^2$, with the additional connections of the point (i, j) to the points $(i+1, j+1), (i-1, j-1)$, forms the *three-direction mesh* which is sufficient to describe a regular triangulation (each vertex is connected to six neighboring vertices). These two types of topologies of $\mathbb{Z}^2$, are also relevant to general topologies of control points, since they are generated by most of the topological refinement rules. This is explained in §2.4.

2.1. The main construction methods of schemes. There are two main approaches to the construction of subdivision schemes. The first approach is by repeated averaging. In case $s = 1$, repeated averaging leads to B-spline schemes.

In this approach, the refinement rule (5) consists of several simple steps. The first is the trivial refinement

$$P_\alpha^{k+1,0} = P^k_{\left[\frac{\alpha}{2}\right]}\,, \quad \alpha \in \mathbb{Z}^s \tag{8}$$

with $\left[\frac{\alpha}{2}\right]$ the biggest integer smaller than or equal to $\frac{\alpha}{2}$ for $\alpha \in \mathbb{Z}$, and $\left(\left[\frac{\alpha_1}{2}\right], \left[\frac{\alpha_2}{2}\right]\right)$ for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$.

The trivial refinement is followed by a fixed number m of repeated averaging

$$P_\alpha^{k+1,j} = \frac{P_\alpha^{k+1,j-1} + P_{\alpha-e_j}^{k+1,j-1}}{2}\,, \quad \alpha \in \mathbb{Z}^s\,,\ j = 1, \dots, m\,,$$

where $\{e_1, \dots, e_m\}$ are non-zero vectors in $\mathbb{Z}^s$ with components in $\{0, 1\}$.

The case $s = 1$ corresponds to the B-spline scheme of degree m, while in case $s = 2$ one gets the tensor-product B-spline schemes for the choice $e_1 = \cdots = e_r = (1, 0)$, $e_{r+1} = \cdots = e_m = (0, 1)$, $1 \le r < m$, and the three-direction box-spline schemes [7] for the choice $e_1 = \cdots = e_r = (1, 0)$, $e_{r+1} = \cdots = e_\rho = (0, 1)$,

$e_{\rho+1} = \cdots = e_m = (1,1)$, $1 \le r < \rho < m$. One can get other box-spline schemes for more general choices of $e_1, \ldots, e_m$ [7].

The second construction of subdivision schemes is based on a local approximation operator A, approximating on $[0,1]^s$. A is defined in terms of samples of the approximated function in a set of points $\mathcal{A} \subset \mathbb{Z}^s$,

$$(Af)(x) = \sum_{\alpha\in\mathcal{A}} f(\alpha) w_\alpha(x)\,, \quad x \in [0,1]^s\,. \tag{9}$$

For geometrical applications, the set $\mathcal{A}$ contains the set E_s of extreme points of $[0,1]^s$, and is symmetric relative to $[0,1]^s$. The operator A has to be scale and shift invariant, so that (9) can be used in any refinement level and at any location. This leads to the choice of a polynomial approximation operator A.

The commonly derived refinement rule from (9) is

$$P^{k+1}_{2\alpha+\gamma} = \sum_{\beta\in\mathcal{A}} P^k_{\alpha+\beta} w_\beta\left(\frac{\gamma}{2}\right)\,, \quad \gamma \in E_s\,. \tag{10}$$

Another possibility is

$$P^{k+1}_{2\alpha+\gamma} = \sum_{\beta\in\mathcal{A}} P^k_{\alpha+\beta} w_\beta\left(g+\frac{\gamma}{2}\right)\,, \quad \gamma \in E_s\,,\ g = \{1/4\}^s\,. \tag{11}$$

In case $s = 1$, with Af the interpolation polynomial based on the symmetric set of points relative to $[0,1]$, $-N+1, \ldots, 0, 1, \ldots, N$, the resulting family of schemes obtained by (10) for $N = 1, 2, \ldots$ consists of the Dubuc–Deslaurier schemes [22]

$$P^{k+1}_{2i} = P^k_i\,, \quad P^{k+1}_{2i+1} = \sum_{\ell=-N+1}^{N} w_\ell\left(\frac{1}{2}\right) P^k_{i+\ell}\,, \quad w_i(x) = \prod_{\substack{j=-N+1\\ j\neq i}}^{N} \frac{x-j}{i-j} \tag{12}$$

The schemes in (12) are interpolatory, since the set of control points after refinement contains the control points before refinement. These schemes are convergent, and the limit curves interpolate the initial control points [22]. Interpolatory schemes in general are discussed in [34].

Recently this construction was extended to non-interpolatory schemes [30], by using (11) instead of (10) with $w_i(x)$ defined in (12).

The refinement rules are

$$P^{k+1}_{2i} = \sum_{\ell=-N+1}^{N} w_\ell\left(\frac{1}{4}\right) P^k_{i+\ell}\,, \quad P^{k+1}_{2i+1} = \sum_{\ell=-N+1}^{N} w_\ell\left(\frac{3}{4}\right) P^k_{i+\ell}\,.$$

It is checked in [30] that, for $N \le 10$, the schemes are convergent with limit curves of higher smoothness than the limit curves of the corresponding Dubuc–Deslaurier schemes. Yet, there is no proof that this holds in general.

In fact, (11) can be further extended to

$$P^{k+1}_{2\alpha+\gamma} = \sum_{\beta\in\mathcal{A}} P^k_{\alpha+\beta} w_\beta\big(g + (1-2\mu)\gamma\big)\,, \quad \gamma \in E_s\,,\ g = \{\mu\}^s$$

with $0 < \mu < \frac{1}{2}$.

This refinement was studied in [33], [39], [4], for $s = 1$ and A a linear interpolation operator at the points $x = 0$, $x = 1$. For $\mu = \frac{1}{4}$, this is Chaikin's scheme for generating quadratic B-spline curves [10]. For $\mu \neq \frac{1}{4}$ it is a general *corner cutting* scheme.

2.2. Approximation order of subdivision schemes. A convergent subdivision scheme S, constructed by the second approach of §2.1 with refinement rule (10), has the property of *reproduction of polynomials*.

Let the operator A map the set $f|_{\mathcal{A}}$ to a unique interpolation polynomial of total degree not exceeding m, interpolating the data $\{(x, f(x)) : x \in \mathcal{A}\}$. In the following, we denote by $\Pi_m(\mathbb{R}^s)$ the space of all s-variate polynomials of degree up to m. It is easy to verify that for $f \in \Pi_m(\mathbb{R}^s)$ and $\mathbf{f}^0 = \{f^0_\alpha = f(\alpha h) : \alpha \in \mathbb{Z}^s\}$, $h \in \mathbb{R}_+$, the refinement (10) generates data on f, namely $\mathbf{f}^k = S^k \mathbf{f}^0 = \{f^k_\alpha = f(2^{-k}\alpha h) : \alpha \in \mathbb{Z}^s\}$, and therefore $S^\infty \mathbf{f}^0 = f$, and the subdivision scheme reproduces polynomials in $\Pi_m(\mathbb{R}^s)$.

In case of the refinement rule (11), arguments as in [30] lead to $(S^\infty \mathbf{f}^0)(x) = f(x+2hg)$, with g as in (11). This property of the scheme S is termed reproduction *with a fixed shift* of polynomials in $\Pi_m(\mathbb{R}^s)$.

The reproduction of polynomials in $\Pi_m(\mathbb{R}^s)$ (with or without a shift), the representation of $S^\infty \mathbf{f}^0$ in terms of the compactly supported basic limit function ϕ of S,

$$S^\infty \mathbf{f}^0(x) = \sum_{\alpha \in \mathbb{Z}^s} f^0_\alpha \phi(x - \alpha)\,, \tag{13}$$

and classical quasi-interpolation arguments [5], lead to the error estimate

$$\sup_{x \in \Omega} \left| (S^\infty \mathbf{f}^0)(x) - f(x) \right| \leq C h^{m+1}. \tag{14}$$

In (14) $\mathbf{f}^0 = \{f^0_\alpha = f(\alpha h) : \alpha \in \mathbb{Z}^s\}$ for the refinement rule (10), while, for the refinement rule (11), $\mathbf{f}^0 = \{f^0_\alpha = f(\alpha h - 2gh) : \alpha \in \mathbb{Z}^s\}$, f is a smooth enough function, Ω is a bounded domain in $\mathbb{R}^s$, and the constant C may depend on S, f, Ω but not on h. A subdivision scheme satisfying (14) is said to have *approximation order* $m + 1$.

Subdivision schemes constructed by repeated averaging reproduce constant functions and hence have approximation order 1. If the repeated averaging is done in a symmetric way relative to $[0, 1]^s$, then the resulting scheme reproduces also linear polynomials, and the scheme has approximation order 2. For example, this property is shared by all the symmetric B-spline schemes of odd degrees. The mask of the scheme generating B-spline curves, based on the symmetric B-spline of degree $2\ell + 1$ is

$$\tilde{a}_i^{[2\ell+1]} = \frac{1}{2^{2\ell+1}} \binom{2\ell+2}{\ell+1+i}, \quad i = -\ell-1, \ldots, 0, \ldots, \ell+1\,.$$

The repeated averaging for such a symmetric mask takes the symmetric form

$$P_{2i}^{k+1,0} = P_i^k \,, \quad P_{2i+1}^{k+1,0} = \frac{1}{2}(P_i^k + P_{i+1}^k)\,,$$
$$P_i^{k+1,j} = \frac{1}{4}(P_{i-1}^{k+1,j-1} + 2P_i^{k+1,j-1} + P_{i+1}^{k+1,j-1}), \quad i \in \mathbb{Z}\,,\ j = 1,\ldots,\ell\,,$$
$$P_i^{k+1} = P_i^{k+1,\ell}\,, \quad i \in \mathbb{Z}\,.$$

2.3. Convergence and smoothness analysis. Given the coefficients of the mask of a scheme, one would like to be able to determine if the scheme is convergent, and what is the smoothness of the resulting basic limit function (which is the generic smoothness of the limits generated by the scheme in view of (13)). Such analysis tools are essential for the design of new schemes.

We present one method for convergence analysis of the two cases $s = 1, 2$. The method for smoothness analysis in case $s = 1$ is simpler and is given in full. Its extension to $s = 2$ is omitted, but some special cases are discussed. There are other methods for convergence and smoothness analysis, see, e.g., [18], [19], [20], [41], [44].

An important tool in the analysis of convergence, presented here, is the symbol of a scheme $S_{\mathbf{a}}$ with the mask $\mathbf{a} = \{a_\alpha : \alpha \in \sigma(\mathbf{a})\}$,

$$a(z) = \sum_{\alpha\in\sigma(\mathbf{a})} a_\alpha z^\alpha. \tag{15}$$

A first step towards the convergence analysis is the derivation of the necessary condition for uniform convergence,

$$\sum_{\beta\in\mathbb{Z}^s} a_{\alpha-2\beta} = 1\,, \quad \alpha \in E_s\,, \tag{16}$$

derived easily from the refinement rule

$$f_\alpha^{k+1} = \sum_{\beta\in\mathbb{Z}^s} a_{\alpha-2\beta} f_\beta^k\,, \quad \alpha \in \mathbb{Z}^s.$$

with $\mathbf{f}^k = \{f_\alpha^k \in \mathbb{R} : \alpha \in \mathbb{Z}^s\}$. The necessary condition (16) implies that we have to consider symbols satisfying

$$a(1) = 2\,, \quad a(-1) = 0 \quad \text{if } s = 1\,, \tag{17}$$

or

$$a(1,1) = 4\,, \quad a(-1,1) = a(1,-1) = a(-1,-1) = 0 \quad \text{if } s = 2\,. \tag{18}$$

In case $s = 1$, condition (17) is equivalent to

$$a(z) = (1+z)q(z) \quad \text{with} \quad q(1) = 1\,. \tag{19}$$

The scheme with symbol $q(z)$, $S_{\mathbf{q}}$, satisfies $S_{\mathbf{q}}\Delta = \Delta S_{\mathbf{a}}$ (see, e.g. [24]), where Δ is the difference operator

$$\Delta\mathbf{f} = \left\{(\Delta\mathbf{f})_i = f_i - f_{i-1} : i \in \mathbb{Z}\right\}. \tag{20}$$

A necessary and sufficient condition for the convergence of $S_{\mathbf{a}}$ is the contractivity of the scheme $S_{\mathbf{q}}$, namely $S_{\mathbf{a}}$ is convergent if and only if $S_{\mathbf{q}}^{\infty}\mathbf{f}^0 = 0$ for any $\mathbf{f}^0$ [33]. The contractivity of $S_{\mathbf{q}}$ is equivalent to the existence of a positive integer L, such that $\|S_{\mathbf{q}}^L\|_\infty < 1$. This condition can be checked for a given L by algebraic operations on the symbol $q(z)$ (see, e.g., [24], [25]).

For practical geometrical reasons, only small values of L have to be considered, since a small value of L guarantees "visual convergence" of $\{\mathcal{F}^k(t)\}$ to $S_{\mathbf{a}}^{\infty}\mathbf{P}^0$, already for small k, as the distances between consecutive control points contract to zero fast. A good scheme corresponds to $L = 1$ as the B-spline schemes, or to $L = 2$ as many of the schemes constructed by the second method in §2.1 (see the following examples).

For $s = 2$, the necessary condition (18) guarantees the existence of two decompositions of the form

$$(1 - z_i)a(z) = q_{i1}(z)(1 - z_1^2) + q_{i2}(z)(1 - z_2^2)\,, \quad i = 1, 2\,, \tag{21}$$

where $z = (z_1, z_2)$. The above two decompositions extend to $s = 2$ the factorization (19) written as $(1 - z)a(z) = (1 - z^2)q(z)$. The decompositions (21) guarantee the existence of a matrix subdivision scheme S_Q, with a matrix symbol $Q(z) = \{q_{ij}(z)\}_{i,j=1}^2$, satisfying $S_{\mathbf{Q}}(\Delta_1, \Delta_2)^T = (\Delta_1, \Delta_2)^T S_{\mathbf{a}}$. Here $(\Delta_1, \Delta_2)^T$ is the vector difference operator, extending (20) to $s = 2$,

$$(\Delta_1, \Delta_2)^T\mathbf{f} = \left\{((\Delta_1, \Delta_2)^T\mathbf{f})_\alpha = (f_\alpha - f_{\alpha-(1,0)}, f_\alpha - f_{\alpha-(0,1)})^T : \alpha \in \mathbb{Z}^2\right\}.$$

A sufficient condition for the convergence of $S_{\mathbf{a}}$ is the contractivity of $S_{\mathbf{Q}}$, which can be checked by algebraic operations on the symbol $Q(z)$ [9], [24], [43].

Since many of the schemes have symmetries relative to $\mathbb{Z}^2$, their symbols are factorizable and have the form $a(z) = (1 + z_1)(1 + z_2)q(z)$. As a simple extension of the case $s = 1$, we get that $S_{\mathbf{a}}$ is convergent if the two schemes with symbols $(1 + z_1)q(z), (1 + z_2)q(z)$ are contractive. If $a(z)$ is symmetric in the sense that $q(z_1, z_2) = q(z_2, z_1)$, then it is sufficient to check the contractivity of $(1 + z_1)q(z)$ (see, e.g., [25]).

The smoothness analysis in the case $s = 1$, relies on the result that if the symbol of a scheme has a factorization

$$a(z) = \left(\frac{1+z}{2}\right)^{\nu} b(z)\,, \tag{22}$$

such that the scheme $S_{\mathbf{b}}$ is convergent, then $S_{\mathbf{a}}$ is convergent and its limit functions are related to those $S_{\mathbf{b}}$ by

$$D^{\nu}(S_{\mathbf{a}}^{\infty}\mathbf{f}^0) = S_{\mathbf{b}}^{\infty}\Delta^{\nu}\mathbf{f}^0, \tag{23}$$

with D the differentiation operator [33], [24]. Thus, each factor $(1 + z)/2$ multiplying a symbol of a convergent scheme adds one order of smoothness. This factor is termed a *smoothing factor*.

The relation between (22) and (23) is a particular instance of the "algebra of symbols" [35]. If $a(z), b(z)$ are two symbols of converging schemes, then S_c with the symbol $c(z) = \frac{1}{2^s}a(z)b(z)$ is convergent, and

$$\phi_{\mathbf{c}} = \phi_{\mathbf{a}} * \phi_{\mathbf{b}}\,. \tag{24}$$

Example (B-spline schemes). The smoothness of the limit functions generated by the m-th degree B-spline scheme, having the symbol $a^{[m]}(z) = 2\left(\frac{1+z}{2}\right)^{m+1}$, can be concluded easily. The factor $b(z) = \frac{(1+z)}{2}^2$ corresponds to $S_{\mathbf{b}}$ generating a piecewise linear interpolant to the initial data $\{(i, f_i^0)\}$, which is continuous, and the factors $\left(\frac{1+z}{2}\right)^{m-1}$ add smoothness, so that $S_{\mathbf{a}^{[m]}}^\infty f^0 \in C^{m-1}$. Note that $a^{[m]}(z)$ consists of smoothing factors only. In fact the B-spline schemes are optimal, in the sense that for a given support size of the mask, the limit functions generated by the corresponding B-spline scheme is of maximal smoothness.

Example (the four-point scheme). Here we present the most general univariate interpolatory scheme which is based on four points [31], and describe briefly its convergence and smoothness analysis.

The refinement rule is

$$f_{2i}^{k+1} = f_i^k\,, \quad f_{2i+1}^{k+1} = -w(f_{i-1}^k + f_{i+2}^k) + \left(\tfrac{1}{2} + w\right)(f_i^k + f_{i+1}^k)\,,$$

with w a parameter controlling the shape of the limit curves. The symbol of the scheme is

$$a_w(z) = \tfrac{1}{2z}(z+1)^2\left[1 - 2wz^{-2}(1-z)^2(z^2+1)\right]. \tag{25}$$

Note that for $w = 0$, $a_0(z)$ is the symbol of the two-point scheme generating the polygonal line through the initial control points, and that for $w = 1/16$ it coincides with the symbol of the Dubuc–Deslauriers scheme based on four points (reproducing cubic polynomials).

The range of w for which $S_{\mathbf{a}_w}$ is convergent is the range for which $S_{\mathbf{b}_w}$ with symbol $b_w(z) = a_w(z)/(1+z)$ is contractive. The condition $\|S_{\mathbf{b}_w}\|_\infty < 1$ holds in the range $-3/8 < w < (-1+\sqrt{13})/8$, while the condition $\|S_{\mathbf{b}_w}^2\|_\infty < 1$ holds in the range $-1/4 < w < (-1+\sqrt{17})/8$. Thus $S_{\mathbf{a}_w}$ is convergent in the range $-3/8 < w < (-1+\sqrt{17})/8$. To find a range of w where $S_{\mathbf{a}_w}$ generates C^1 limits, the contractivity of $S_{\mathbf{c}_w}$ with $c_w(z) = 2a_w(z)/(1+z)^2$ has to be investigated. It is easy to check that $\|S_{\mathbf{c}_w}\|_\infty \geq 1$, but that $\|S_{\mathbf{c}_w}^2\|_\infty < 1$ for $0 < w < (\sqrt{5}-1)/8$.

The limit of $S_{\mathbf{a}_w}$ is not C^2 even for $w = 1/16$, although for $w = 1/16$ the symbol is divisible by $(1+z)^3$ (see, e.g., [31]). It is shown in [20] by other methods, that the basic limit function for $w = 1/16$, restricted to its support, has a second derivative only at the non-dyadic points.

For the case $s = 2$, the idea of smoothing factors generalizes straightforwardly. Two smoothing factors in two linearly independent directions in $\mathbb{Z}^2$ are sufficient for increasing the smoothness. A smoothing factor in direction $(u, v) \in \mathbb{Z}^2$ is $\frac{1}{2}(1 + z_1^u z_2^v)$. Specializing to the coordinate directions in $\mathbb{Z}^2$, $(1,0)$ and $(0,1)$, we get for a symbol $a(z) = (1+z_1)^m(1+z_2)^m b(z)$, such that $S_{\mathbf{b}}$ is convergent, that

$$\partial_{i,j} S_{\mathbf{a}}^\infty f^0 = S_{\mathbf{a_{i,j}}}^\infty \Delta_1^i \Delta_2^j f^0, \quad i,j = 0,\ldots,m\,, \tag{25}$$

with

$$a_{i,j}(z) = \frac{2^{i+j}a(z)}{(1+z_1)^i(1+z_2)^j}\,, \quad i,j = 0,\ldots,m\,, \tag{26}$$

and with ∂_{ij} the $(i+j)$-th partial derivative of orders i,j in directions $(1,0)$ and $(0,1)$ respectively.

For a symbol with the symmetry of the three direction mesh

$$a(z) = (1+z_1)^m(1+z_2)^m(1+z_1z_2)^m b(z)\,, \tag{27}$$

such that $S_{\mathbf{b}}$ is convergent, we get

$$\partial_{i,j,\ell} S_{\mathbf{a}}^\infty f^0 = S_{\mathbf{a_{i,j,\ell}}}^\infty \Delta_1^i \Delta_2^j (\Delta_1+\Delta_2)^\ell f^0, \quad i,j,\ell = 0,\dots,m\,, \tag{28}$$

with

$$a_{i,j,\ell}(z) = \frac{2^{i+j+\ell}a(z)}{(1+z_1)^i(1+z^2)^j(1+z_1z_2)^\ell}\,, \quad i,j,\ell = 0,\dots,m\,, \tag{29}$$

and with $\partial_{i,j,\ell}$ the $(i+j+\ell)$-th partial derivative of orders i,j,ℓ in directions $(1,0),(0,1),(1,1)$ respectively.

In particular $S_{\mathbf{a}}$ with the symbol $a(z) = (1+z_1)^2(1+z_2)^2 b(z)$ generates C^1 limit functions if the three schemes with the symbols

$$2(1+z_1)(1+z_2)b(z)\,, \quad 2(1+z_1)^2 b(z)\,, \quad 2(1+z_2)^2 b(z)\,,$$

are contractive. Similarly for $a(z) = (1+z_1)(1+z_2)(1+z_1,z_2)b(z)$, $\phi_{\mathbf{a}} \in C^1$ if two of the three schemes with the symbols $2(1+z_1)b(z), 2(1+z_2)b(z), 2(1+z_1z_2)b(z)$ are contractive.

The conditions for smoothness given above are only sufficient. Yet, in the case $s=1$, there is a large class of convergent schemes for which the factorization in (22) is necessary for generating C^ν limit functions. The schemes in this class are L_∞-stable, namely, satisfy

$$\|S_a^\infty \mathbf{f}^0\| \geq C\|\mathbf{f}^0\|_\infty\,, \quad \mathbf{f}^0 \in \ell_\infty(\mathbb{Z})\,, \tag{31}$$

with constant C dependent on $S_{\mathbf{a}}$ but not on $\mathbf{f}^0$. All relevant schemes for geometric applications are L_∞-stable, as the interpolatory schemes and the B-spline schemes.

This is not the case for $s=2$. The symbol of a convergent L_∞-stable scheme, generating smooth limit functions is not necessarily factorizable. Yet, many of the schemes in use have factorizable symbols.

2.4. Subdivision schemes generating surfaces. Schemes generating surfaces operate on control nets, and map a control net to a refined one.

A control net $N(V,E,F)$, consists of a set V of points in $\mathbb{R}^3$, termed *vertices*, with two sets of topological relations between them E and F, called *edges* and *faces* respectively (see, e.g., [48]). An edge denotes a pair of vertices. A face is a cyclic list of vertices where every pair of consecutive vertices constitutes an edge. The *valency* of a vertex is the number of edges that share it, the *valency* of a face is the number of vertices that belong to it. In Fig. 2 we present a schematic net. We consider here only closed nets, namely nets in which each edge is shared by two faces.

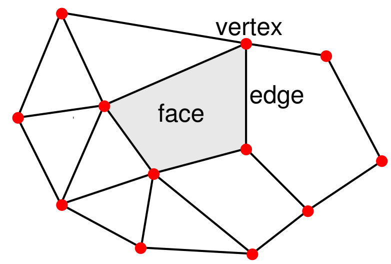

Figure 2. A schematic net

2.4.1. Topological refinement of nets. There are several topological rules for refining a net $N(V, E, F)$. The most common one defines the new set of vertices, as

$$V' = \{u(v) : v \in V\} \cup \{u(e) : e \in E\} \cup \{u(f) : f \in F\} = V'_V \cup V'_E \cup V'_F \,. \quad (32)$$

Here V'_V denotes all the new vertices, called *v-vertices*, corresponding to the vertices in V (in an interpolatory scheme $V'_V = V$); E'_V denotes all the new vertices, called *e-vertices*, corresponding to the edges in E, and V'_F denotes all the new vertices, called *f-vertices*, corresponding to the faces in F. The rule for determining the location in $\mathbb{R}^3$ of $u(v)$, $u(e)$ and $u(f)$ is the refinement rule of the subdivision scheme. For example, a new vertex $u(e)$ is a certain linear combination of the vertices in V, weighted according to the topological relation between each $v \in V$ and e.

The topological relations E', F' in the refined net $N'(V', E', F')$ are independent of the subdivision scheme, but depend only on E and F,

$$E' = \{(u(e), u(f)) : e \in f \in F\} \cup \{(u(e), u(v)) : v \in e \in E\} = E'_F \cup E'_E \quad (33)$$

and

$$F' = \{(u(v), u(e), u(f), u(\widetilde{e})) : v = e \cap \widetilde{e} \in f \in F\} \,. \quad (34)$$

Thus after one refinement step all faces have valency four and similarly all the vertices in the set V'_E. The valency of a vertex in V'_F is the same as that of the "parent" face, and the valency of a vertex in V'_V is the same as that of the "parent" vertex. From this observation we conclude that the nets obtained after two or more refinements have the topology of a quad-mesh (of $\mathbb{Z}^2$), except for a finite number of vertices with valency different from four (each equals the valency of an "ancestor" face or vertex in the initial net). The vertices with valency different from four are termed *irregular (extra-ordinary)* and a special local analysis of convergence and smoothness is required there [54]. Over the net, except in the vicinity of the irregular vertices, the analysis relative to $\mathbb{Z}^2$ is applicable.

For a net $N(V, E, F)$ with all faces of valency three, the topological refinement which is commonly used is such that the new vertices consist of v-vertices and

e-vertices only, with the topological refinement

$$E' = E'_E \cup E'_V \, , \quad F' = F'_V \cup F'_F \, . \tag{35}$$

In (35), E'_E is defined as in (33), and $E'_V = \{(u(e), u(\widetilde{e})) : e \cap \widetilde{e} \in V\}$. The new faces are of two types, $F'_V = \{(u(v), u(e), u(\widetilde{e})) : v \in e \cap \widetilde{e} \in V\}$, and $F'_F = \{(u(e_1), u(e_2), u(e_3)) : e_1, e_2, e_3 \in f \in F\}$. This refinement is presented in a schematic way in Fig. 3. As can be observed from Fig. 3, every face is replaced by

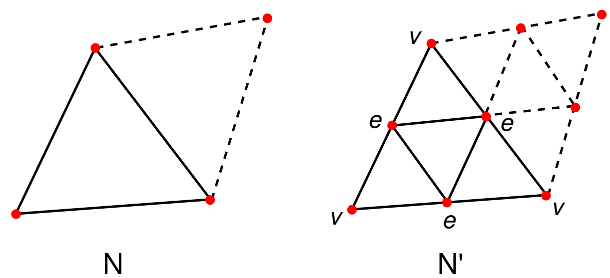

Figure 3. Schematic triangular topological refinement

four faces, one determined by the face itself, and three in F'_V, each consisting of three new vertices, one corresponding to one vertex of the face and two to the two edges of the face sharing that vertex.

Note that a face with valency three can be realized in $\mathbb{R}^3$ as a planar triangle, and therefore $N(V, E, F)$, with all faces of valency three, can be realized as a triangulation of the set V. According to the topological refinement (35), the e-vertices have valency six, while a v-vertex has the same valency as that of its "parent" vertex in V. Thus, after two or more topological refinements, most of the vertices in the triangulations have valency six. Only a finite set of *irregular (extra-ordinary)* vertices have valencies different from six, "inherited" from those in the initial triangulation. Also, each irregular vertex is connected by edges only to regular vertices (of valency six).

Thus for a triangulation refined as above, the analysis of convergence and smoothness relative to the three-direction mesh applies, except in the vicinity of a finite number of isolated points, where a special local analysis is required [54].

While the analysis on regular meshes can handle any order of smoothness, the analysis at irregular vertices is limited to C^1 smoothness (see, [51], [45], and references therein). This limitation is the main reason why subdivision schemes are used mainly in computer graphics. In many industrial applications the designed surfaces have to be C^2 everywhere.

2.4.2. Some popular schemes. The first schemes devised for general nets were the bivariate tensor-product B-spline schemes of low degree, with special rules near irregular vertices [8], [23]. A bivariate tensor-product scheme of a univariate scheme with symbol $a(z)$, is a scheme with the symbol $a(z_1, z_2) = a(z_1)a(z_2)$.

The most commonly used scheme of that type for the topological refinement (32)–(34) is the Catmull–Clark scheme, which is an extension of the tensor-product cubic B-spline scheme [8]. The weights, up to normalization, of this scheme are given in Fig. 4. The points designated by o are the new f-vertices, and the weight

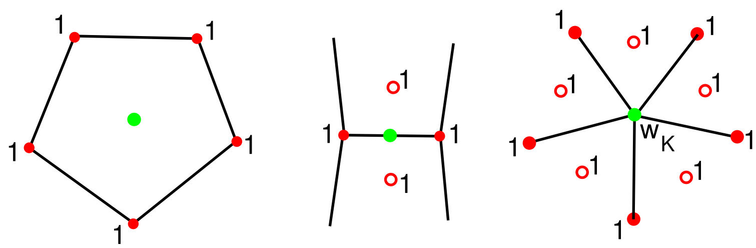

Figure 4. Weights for Catmull-Clark scheme: f-vertex (left), e-vertex (middle) and v-vertex (right)

of a vertex of valency k, in the rule for its "son", is $w_k = k(k-2)$, $k = 3, 4, \ldots$. Note that $w_4 = 8$, which is the weight in the tensor-produce cubic B-spline scheme. This scheme is easy to implement as can be inferred from Fig. 4. Different choices of w_k were considered in [2], [3] to improve the limit curvature at irregular vertices. Applications of the Catmull–Clark scheme are many. Here we refer to two important papers [21], [40].

In [46], the tensor-product four-point scheme is extended to an interpolatory scheme for general nets with the topological refinement (32)–(34).

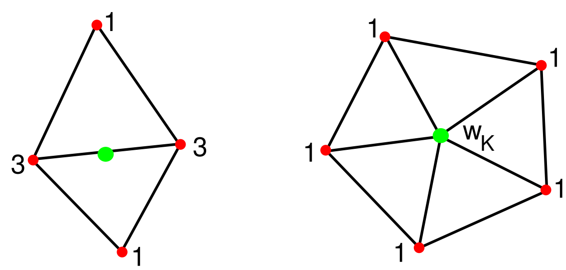

Figure 5. Weights for Loop scheme: e-vertex (left) and v-vertex (right)

For triangulations, the box-spline-based scheme of Loop [49] is very popular.

Loop scheme is an extension of a box-spline scheme with the symbol

$$a(z_1, z_2) = \frac{1}{16}(1+z_1)^2(1+z_2)^2(1+z_1z_2)^2\,,$$

generating C^2 piecewise quartic box-spline surfaces on the three-direction mesh. The support of the mask of Loop scheme is small, and the refinement rule involves only neighboring vertices. In Fig. 5 the weights for defining a new e-vertex and a new v-vertex are given up to normalization. The weight w_k of a vertex of valency k, involved in the rule for its "son", is

$$w_k = \frac{64k}{40 - \left(3 + 2\cos\frac{2\pi}{k}\right)}\,, \quad k = 3, 4, \ldots \tag{36}$$

Fig. 6 depicts an initial triangulation of a head, and the triangulations after two refinements with Catmull-Clark scheme and with Loop scheme.

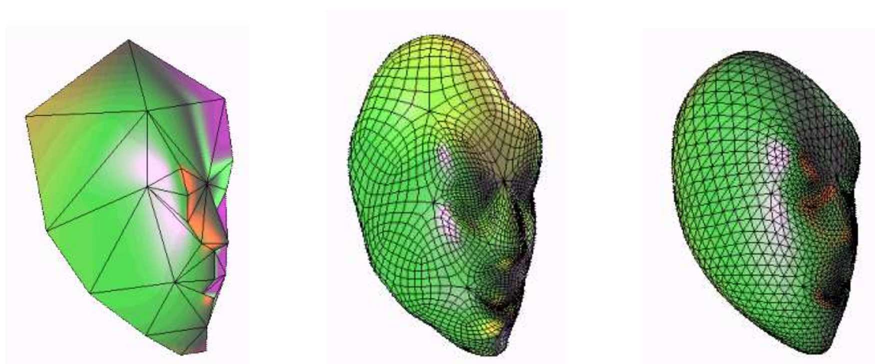

Figure 6. Head. Initial control net (left), after two refinements: with Catmull-Clark scheme (middle) and with Loop scheme (right)

An interpolatory scheme for general closed triangulations with a shape parameter is the butterfly scheme [32]. The weights defining a new e-vertex are depicted in the left figure of Fig. 7. Since the scheme is interpolatory, the new v-vertices coincide with the old vertices. The scheme generates C^1 surfaces if all vertices have valencies at least four and at most eight, depending on the value of w [43], [56]. Modified weights for e-vertices, corresponding to edges having an irregular vertex of any valency above three are derived in [60] for $w = 1/16$. These weights are depicted in the right figure of Fig. 7. The values $\{s_j\}$ are given by a formula depending on the valency k of the irregular vertex,

$$s_j = \frac{1}{k}\left(\frac{1}{4} + \cos\frac{2\pi j}{k} + \frac{1}{2}\cos\frac{4\pi j}{k}\right), \quad j = 0, 1, \ldots, k-1\,,\ k > 3\,. \tag{37}$$

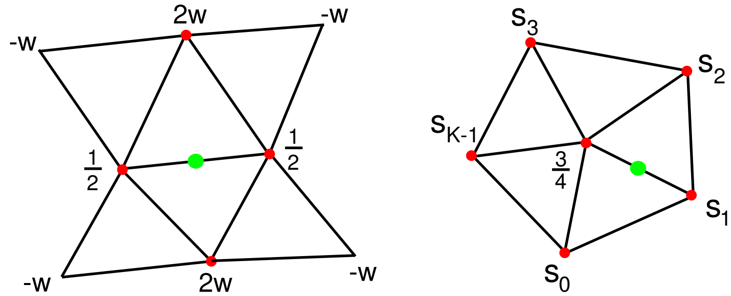

Figure 7. Weights for e-vertex: butterfly scheme (left), modified butterfly scheme (right)

With the modified weights, the generated surfaces are C^1 for any valency greater than three, and are better looking in the vicinity of irregular vertices of valency between four and eight.

3. Linear extensions

In this section, we review several extensions of stationary linear schemes for the refinement of points to stationary linear schemes which refine other objects.

3.1. Matrix subdivision schemes. Matrix schemes are defined by matrix masks and refine sequences of vectors. Although, in the geometric setting, the schemes of §2 refine sequences of control points in $\mathbb{R}^2$ or in $\mathbb{R}^3$, the schemes operate on each component of the vectors in the same way, such that the refinement of one component is independent of the other components. This property is very important in geometric applications, since the subdivision schemes commute with affine transformations (the schemes are *affine invariant*). The schemes presented here are not affine invariant, and their main application is in multiwavelets constructions [12], [57] and in the analysis of multivariate subdivision schemes for control points as indicated in §2.3 (see, e.g., [24]).

A finite set of matrices of order $d \times d$, $\mathbf{A} = \{A_\alpha : \alpha \in \sigma(\mathbf{A}) \subset \mathbb{Z}^s\}$, defines a matrix subdivision scheme $S_\mathbf{A}$ with a refinement rule

$$(S_\mathbf{A}\mathbf{v})_\alpha = \sum_{\beta \in \mathbb{Z}^s} A_{\alpha - 2\beta} v_\beta \,, \quad \mathbf{v} = \{v_\alpha \in \mathbb{R}^d : \alpha \in \mathbb{Z}^s\} \,. \tag{1}$$

Given initial "control vectors" $\mathbf{v}^0 = \{v^0_\alpha \in \mathbb{R}^d : \alpha \in \mathbb{Z}^s\}$, the matrix subdivision scheme $S_\mathbf{A}$ generates a sequence of control vectors by

$$\mathbf{v}^{k+1} = S_\mathbf{A}\mathbf{v}^k, \quad k = 1, 2, \ldots \tag{2}$$

The notion of uniform convergence from §2 can be extended to this case, by considering the convergence of each of the d components of the vectors. The convergence analysis has a linear algebra component to it, in addition to the analysis component. By considering the matrices

$$B_\gamma = \sum_{\beta \in \mathbb{Z}^s} A_{\gamma - 2\beta}\,, \quad \gamma \in E_s\,, \tag{3}$$

one can easily conclude a necessary condition for convergence. This condition is the analogue of condition (2.16), stating that for any initial control vectors $\mathbf{v}^0$, and any $x \in \mathbb{R}^s$,

$$(S_{\mathbf{A}}^\infty \mathbf{v}^0)(x) \in \text{span}\{u \in \mathbb{R}^d : B_\gamma u = u \text{ for all } \gamma \in E_s\}\,. \tag{4}$$

In the extreme case of schemes with $B_\gamma = I$, $\gamma \in E_s$, the space in (4) is $\mathbb{R}^d$, and no condition of linear-algebra type is imposed. Such are the schemes used in the analysis of convergence and smoothness of multivariate schemes for points. Schemes for which the space in (4) is $\mathbb{R}^d$, are very similar to schemes with a scalar mask [17]. In the other extreme case, the space in (4) is one dimensional with vectors of equal components, implying that the limit vector function $S_{\mathbf{A}}^\infty \mathbf{v}^0$, has equal components. An example of this type of schemes is provided by matrix subdivision schemes generating multiple knot B-spline curves (see, e.g., [52]). This latter extreme case is the most relevant to the construction of multiwavelets.

In [13] and in [38], univariate ($s = 1$) matrix schemes with the space (4) of dimension m, $1 \le m \le d$, are studied. An appropriate change of basis, depending on the structure of the space (4), facilitates the extension of the factorization of scalar symbols to a certain factorization of matrix symbols. This factorization is sufficient for convergence and smoothness analysis of matrix schemes, and is also necessary under an extension of the notion of L_∞-stability (see §2.3) to the matrix case. Multivariate matrix schemes with the space (4) of general dimension are considered in [55].

In the next section we discuss a special type of matrix subdivision schemes, which is relevant to curve design from locations and normals, and to the generation of functions from the point values of the functions and their derivatives. The use of analogous schemes for the generation of surfaces from locations and normals is not straightforward, and leads to non-linear schemes.

3.2. Hermite subdivision schemes. The first Hermite schemes to be studied were univariate and interpolatory [50]. Interpolatory Hermite subdivision schemes are matrix schemes, such that the components of the vectors are regarded as the value of a function and its consecutive derivatives up to a certain order at the points of $2^{-k}\mathbb{Z}^s$. Non-interpolatory Hermite subdivision schemes were introduced later [42].

3.2.1. Univariate interpolatory Hermite schemes. The most common construction of interpolatory Hermite subdivision schemes is similar to the second construction method presented in §2.1. The approximation operator A is an extension of the one in (2.9). For interpolatory schemes, it is a polynomial interpolation operator of the form

$$(Af)(x) = \sum_{\alpha\in\mathcal{A}} \sum_{i=0}^{d-1} w_{\alpha,i}(x) f^{(i)}(\alpha)\,, \tag{5}$$

satisfying $D^i(Af)(\alpha) = f^{(i)}(\alpha)$, $\alpha \in \mathcal{A}$, $i = 0, 1, \dots, d-1$.

The refinement is similar to (2.10), namely

$$v_{2\alpha}^{k+1} = v_\alpha^k\,, \quad \left(v_{2\alpha+1}^{k+1}\right)_j = \sum_{\beta\in\mathcal{A}} \sum_{i=0}^{d-1} D^j w_{\beta,i}(1/2)(v_{\alpha+\beta}^k)_i\,, \quad 0 \le j \le d-1\,. \tag{6}$$

In (6), $(v)_i$ denotes the i-th component of the vector v. The refinement (6) can be written in terms of a matrix mask as,

$$v_\alpha^{k+1} = \sum_{\beta\in\mathbb{Z}^s} A_{\alpha-2\beta}^{(k)} v_\beta^k\,, \quad \alpha \in \mathbb{Z}^s\,, \tag{7}$$

where the matrices with even indices are

$$A_{2\alpha}^{(k)} = \delta_{\alpha,0} I_{d\times d}\,, \quad \alpha \in \mathbb{Z}\,, \tag{8}$$

with $\delta_{\alpha,0} = 0$ for $\alpha \neq 0$, and $\delta_{0,0} = 1$. The matrices with odd indices depend on the refinement level k, and have the form

$$A_{2\alpha+1}^{(k)} = \Lambda_d(2^k) A_{2\alpha+1}^{(0)} \Lambda_d(2^{-k})\,, \quad \alpha \in \mathbb{Z}\,, \tag{9}$$

with $\Lambda_d(h) = \mathrm{diag}(1, h, h^2, \dots, h^{d-1})$ and

$$A_{1-2\alpha}^{(0)} = \left\{ D^i w_{\alpha,j}\left(\frac{1}{2}\right) \right\}_{i,j,=0}^{d-1}\,, \quad \alpha \in \mathcal{A}\,. \tag{10}$$

The powers of 2 in (9) are due to the fact that derivatives of polynomials are not scale invariant. More precisely if $q(x) = p(hx)$, with p a polynomial, then $(D^j q)(hx_0) = h^j (D^j p)(x_0)$.

An interpolatory Hermite scheme is termed *uniformly convergent* if there is a limit vector function F of the form $F = (D^j f,\, 0 \le j \le d-1)^T$, with $f \in C^{d-1}(\mathbb{R})$, satisfying for any closed interval $[a, b]$,

$$\lim_{k\to\infty} \sup_{\alpha\in 2^k[a,b]\cap\mathbb{Z}} \left\| F(2^{-k}\alpha) - v_\alpha^k \right\| = 0\,,$$

with $\|\cdot\|$ any norm in $\mathbb{R}^d$.

Example (a two-point Hermite interpolatory scheme). The scheme is given by the non-zero matrices of its mask:

$$A_0 = I_{2\times 2}\,, \quad A_1^{(k)} = \begin{pmatrix} \frac{1}{2} & \nu 2^{-k} \\ -\mu 2^k & \frac{1-\mu}{2} \end{pmatrix}, \quad A_{-1}^{(k)} = \begin{pmatrix} \frac{1}{2} & -\nu 2^{-k} \\ \mu 2^k & \frac{1-\mu}{2} \end{pmatrix}.$$

This scheme with $\nu = 1/8$ and $\mu = 3/2$ generates the C^1 piecewise Hermite cubic interpolant to the data $\{v_i^0 = (f(i), f'(i))^T : i \in \mathbb{Z}\}$, while for $\nu = 0$, $\mu = 1$, it generates the piecewise linear interpolant to the given function's values at the integers, which is only C^0. By the analysis to be reviewed, it can be shown that for $0 < \nu < 1/4$, $\mu = 4\nu + 1$, the limit functions generated by the scheme are C^1. (See, e.g. [36].)

One method for the convergence analysis of such schemes is based on deriving an equivalent stationary matrix scheme, refining vectors of $(d-1)$-th order divided differences, obtained from the original control vectors. The limit of such a scheme, if it exists, necessarily consists of equal components, which are the derivative of order $d-1$ of the smooth function f [37].

More precisely, the divided difference vector u_n^k at level k is defined for each $n \in \mathbb{Z}$ by

$$(u_n^k)_j = [\tau_{j+1}, \tau_{j+2}, \dots, \tau_{j+d}]f\,, \quad j = 0, \dots, d-1\,,$$

with $\tau_1 = \cdots = \tau_{d-1} = (n-1)2^{-k}$, $\tau_d = \tau_{d+1} = \cdots = \tau_{2d-1} = n2^{-k}$. Here we use the definition of divided differences, allowing repeated points for functions with enough derivatives (see, e.g., [6, Chapter 1]). In our setting all integer points have multiplicity d. The vector u_n^k can be derived from the vectors v_{n-1}^k and v_n^k.

The symbol $D(z)$ of the matrix scheme refining the control vectors $\mathbf{u}^k = \{u_n^k : n \in \mathbb{Z}\}$, can be obtained recursively from the symbol $D^{[0]}(z) = \sum_\alpha A_\alpha^{(0)} z^\alpha$, by algebraic manipulations, involving multiplication by certain matrix Laurent polynomials and their inverses.

It is proved in [37] that the matrix symbol $D(z)$ is a matrix Laurent polynomial if the scheme (7) reproduces polynomials of degree $\leq d-2$, and that necessarily a scheme of the form (7), which generates C^{d-1} functions, reproduces polynomials of degree $\leq d-1$. In (5), the degree of the interpolation polynomial is $d|\mathcal{A}|-1$, so the scheme (7), with the mask (8),(9),(10), reproduces polynomials of degree at least $2d-1$, as $\mathcal{A}$ contains at least the points $0, 1$. These arguments lead to the conclusion that the Hermite subdivision scheme $S_{\mathbf{A}}$, refining the control vectors $\mathbf{v}^k$ can be transformed into the matrix subdivision scheme $S_{\mathbf{D}}$ for the control vectors $\mathbf{u}^k$.

To determine the convergence of the scheme $S_{\mathbf{D}}$, which is equivalent to the convergence of the original Hermite subdivision scheme $S_{\mathbf{A}}$ to C^{d-1} functions, we observe that the component $(u_n^k)_j$, in case of convergence, approximates $f^{(d-1)}(2^{-k}n)$ for $j = 1, \dots, d$. Thus as in the case of control points, a necessary condition for convergence is the contractivity of the scheme which refines the differences $(u_n^k)_j - (u_n^k)_{j-1}$, $j = 2, \dots, d, (u_n^k)_1 - (u_{n-1}^k)_d$, $n \in \mathbb{Z}$. Indeed, such a scheme exists, and its symbol is a matrix Laurent polynomial when (7) reproduces polynomials of degree $\leq d-1$ [37], guaranteeing that the contractivity of this scheme can be checked by algebraic manipulations.

The analysis of higher order smoothness is along the same lines.

3.3. *B*-spline subdivision schemes for compact sets. In the last years, the univariate B-spline schemes were extended to operate on data consisting of compact sets [27], [28]. The motivation for the study of such schemes is the problem of approximating a 3D object from a discrete set of its 2D parallel cross-sections, and the problem of approximating a 2D shape from a discrete set of its 1D parallel cross-sections. In both problems, either the 3D object or the 2D shape is regarded as a univariate set-valued function, with its parallel cross-sections as images. The B-spline subdivision schemes are adapted to this setting, so that the limit set-valued function generated by the subdivision from samples of a continuous set-valued function, approximates it.

For initial data $\mathbf{F}^0 = \{F_i^0 \subset \mathbb{R}^n : i \in \mathbb{Z}\}$ consisting of convex compact sets, averages of numbers in the execution of a scheme, can be replaced by Minkowski averages of sets. A *Minkowski average* of sets $B_1, \ldots, B_\ell \subset \mathbb{R}^n$ with weights $\lambda_1, \ldots, \lambda_\ell \in \mathbb{R}$, $\sum_{i=1}^{\ell} \lambda_i = 1$, is the set

$$\sum_{i=1}^{\ell}{}_M \lambda_i B_i = \left\{ \sum_{i=1}^{\ell} \lambda_i b_i : b_i \in B_i \right\}.$$

Thus the m-th degree B-spline subdivision scheme (2.4) can be adapted to convex compact sets by the refinement rule

$$F_i^{k+1} = \sum_j{}_M a_{i-2j}^{[m]} F_j^k\,, \quad i \in \mathbb{Z}\,, \tag{11}$$

with $\mathbf{a}^{[m]} = \{a_i^{[m]}, i = 0, \ldots, m+1\}$ given in (2.2). Since the coefficients of the mask are positive, the sets $\mathbf{F}^k$, $k \geq 1$, generated by the subdivision scheme $S_{M,a^{[m]}}$ with the refinement rule (11) are compact and convex [27]. By the associativity and distributivity of the Minkowski average of convex sets with positive weights, it can be deduced straightforwardly that the limit generated by $S_{M,a^{[m]}}$ from $\mathbf{F}^0$, when $\mathbf{F}^0$ consists of convex compact sets, is

$$(S_{M,\mathbf{a}^{[m]}}^{\infty} \mathbf{F}^0)(t) = \sum_{i \in \mathbb{Z}}{}_M F_i^0 B_m(t-i)\,. \tag{12}$$

In (12) the convergence is in the Hausdorff metric, defined for two sets A, B in $\mathbb{R}^n$, by

$$\text{haus}(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a-b\|\,,\ \sup_{b \in B} \inf_{a \in A} \|a-b\| \right\}$$

with $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^n$.

The subdivision scheme $S_{M,a^{[m]}}$ has approximation properties. It is shown in [27] that for a set-valued function G with convex compact images, which is Lipschitz continuous, namely satisfies $\text{haus}(G(t), g(t+\Delta)) = O(\Delta t)$, and for initial data $\mathbf{F}_h^0 = \{G(ih) : i \in \mathbb{Z}\}$

$$\text{haus}\left((S_{M,\mathbf{a}^{[m]}}^{\infty} \mathbf{F}_h^0)(t), G(t)\right) = O(h)\,. \tag{13}$$

The subdivision $S_{M,a^{[m]}}$ fails to approximation set-valued functions with general compact images. As is shown in [29], for initial data $\mathbf{F}^0$, consisting of general compact sets

$$S^{\infty}_{M,\mathbf{a}^{[m]}}\mathbf{F}^0 = \sum_{i\in\mathbb{Z}} \langle F^0_i \rangle B_m(\cdot - i)$$

with $\langle F^0_i \rangle$ the convex hull of F^0_i. Thus $S^{\infty}_{M,\mathbf{a}^{[m]}}\mathbf{F}^0$ is convex even when the initial sets are non-convex, and it cannot approximate set-valued functions with general compact sets as images.

There is another adaptation of the B-spline subdivision schemes to compact sets [28], which yields approximation also in case of set-valued functions with general compact sets as images. This adaptation is obtained by using the first construction in §2.1 for $s = 1$, and by replacing the average of two numbers by the *metric average* of two compact sets, introduced in [1],

$$A \oplus_t B = \big\{ ta + (1-t)b : (a,b) \in \Pi(A,B) \big\}$$

with

$$\Pi(A_0, A_1) = \Big\{ (a_0, a_1) : a_i \in A_i,\ i = 0,1, \\ \|a_0 - a_1\| = \min_{a\in A_j} \|a_i - a\|,\ j = 1-i,\ \text{for } i = 0 \text{ or } 1 \Big\} .$$

The refinement rule of the resulting scheme $S_{MA,m}$ is achieved by the $m+1$ steps,

$$\begin{gathered} F^{k+1,0}_{2i} = F^k_i \quad F^{k+1,0}_{2i+1} = F^k_i\,, \quad i \in \mathbb{Z}\,, \\ F^{k+1,j}_i = F^{k+1,j-1}_i \oplus_{\frac{1}{2}} F^{k+1,j-1}_{i-1}\,, \quad i\in\mathbb{Z}\,,\ j = 1,\dots,m \\ F^{k+1}_i = F^{k+1,m}_i, \quad i \in \mathbb{Z} \end{gathered} \tag{14}$$

The refinement rule (14) is denoted formally by $\mathbf{F}^{k+1} = S_{MA,m}\mathbf{F}^k$.

Two important properties of the metric average, which are central to its application in B-spline subdivision schemes are

$$A \oplus_t A = A\,, \quad \text{haus}(A \oplus_t B, A \oplus_s B) = |s-t|\,\text{haus}(A,B)\,, \tag{15}$$

for $(s,t) \in [0,1]$.

Let the sequence $\{H^k\}$ consist of the "piecewise linear" set valued functions, interpolating $\{\mathbf{F}^k = S^k_{MA,m}\mathbf{F}^0\}$,

$$H^k(t) = F^k_i \oplus_{\lambda(t)} F^k_{i+1}\,, \quad 2^{-k}i \le t < 2^{-k}(i+1)\,,\ i\in\mathbb{Z}\,,\ k = 0,1,2,\dots, \tag{16}$$

with $\lambda(t) = i + 1 - 2^k t$. It is proved in [28], with the aid of the metric property of the metric average (the second equality in (15)) and the completeness of the metric space of compact sets with the Hausdorff metric, that the sequence $\{H^k(t)\}$ converges to a limit set-valued function denoted by $S^{\infty}_{MA,m}\mathbf{F}^0$.

Moreover, for G a Lipschitz continuous set valued function with general compact sets as images, the limit generated by the scheme $S_{MA,m}$ starting from $\mathbf{F}^0_h = \{G(ih) : i \in \mathbb{Z}\}$ approximates G with "error" given by

$$\text{haus}\big((S^{\infty}_{MA,m}\mathbf{F}^0_h)(t), G(t)\big) = O(h)\,, \quad t\in\mathbb{R}\,. \tag{17}$$

3.4. A blending-based subdivision scheme for nets of curves.

The quadratic B-spline scheme (Chaikin algorithm) was extended to the refinement of nets of curves in [15]. A net of curves with parameter $d > 0$ consists of two families of continuous curves

$$\left\{\phi_i(s) : 0 \le i \le n\,,\ s \in [0, md]\right\}, \quad \left\{\psi_j(t) : 0 \le j \le m\,,\ t \in [0, nd]\right\}$$

satisfying the compatibility condition

$$\phi_i(jd) = \psi_j(id)\,, \quad i = 0, \ldots, n\,,\ j = 0, \ldots, m\,.$$

Such a net is denoted by $\mathcal{N}(d, \{\phi_i\}_{i=0}^n, \{\psi_j\}_{j=0}^m)$. The *blending-based Chaikin-type* scheme refines a net of curves, $\mathcal{N}_0 = \mathcal{N}(d, \{\phi_i^0\}_{i=0}^n, \{\psi_j^0\}_{j=0}^m)$ into a net of curves $\mathcal{N}_1 = \mathcal{N}\big(\frac{d}{2}, \{\phi_i^1\}_{i=0}^{2n-1}, \{\psi_j^1\}_{j=0}^{2m-1}\big)$. A repeated application of such refinements generates a sequence of nets $\big\{\mathcal{N}_k = \mathcal{N}\big(\frac{d}{2^k}, \{\phi_i^k\}_{i=0}^{n_k}, \{\psi_j^k\}_{j=0}^{m_k}\big) : k \in \mathbb{Z}_+\big\}$, with $n_k = 2^k(n-1)+1$, $m_k = 2^k(m-1)+1$, which converges uniformly to a continuous surface [15].

The construction of the refinement rule is analogous to the second method in §2.1. The approximation operator A maps a net of curves $\mathcal{N}(d, \{\phi_i\}_{i=0}^n, \{\psi_j\}_{j=0}^m)$ into the piecewise Coons patch surface, interpolating the curves of the net,

$$\begin{gathered} \mathcal{C}(\mathcal{N})(s,t) = C(\phi_i, \phi_{i+1}, \psi_j, \psi_{j+1}; d)(s - jd, t - id)\,, \\ (s,t) \in [jd, jd+d] \times [id, id+d]\,, \quad i = 0, \ldots, n-1\,,\ j = 0, \ldots, m-1\,, \end{gathered} \tag{18}$$

with $C(\phi_i, \phi_{i+1}, \psi_j, \psi_{j+1}; d)$ a Coons patch [16].

Four continuous curves $\phi_0, \phi_1, \psi_0, \psi_1$ defined on $[0, h]$ and satisfying $\phi_i(jh) = \psi_j(ih)$, $i, j = 0, 1$, define a Coons patch on $[0, h]^2$. For $(s,t) \in [0,h]^2$ the Coons patch is given by

$$\begin{aligned} &C(\phi_0, \phi_1, \psi_0, \psi_1; h)(s,t) \\ &\qquad = \left[\left(1 - \frac{t}{h}\right)\phi_0(s) + \frac{t}{h}\phi_1(s)\right] + \left[\left(1 - \frac{s}{h}\right)\psi_0(t) + \frac{s}{h}\psi_1(t)\right] \\ &\quad - \left[\left(1-\frac{s}{h}\right)\left(\left(1-\frac{t}{h}\right)\phi_0(0)+\frac{t}{h}\phi_1(0)\right)+\frac{s}{h}\left(\left(1-\frac{t}{h}\right)\phi_0(h)+\frac{t}{h}\phi_1(h)\right)\right]. \end{aligned} \tag{19}$$

The Coons patch is blending between two surfaces. One is interpolating linearly between corresponding points of ϕ_0, ϕ_1 and the other between the corresponding points of ψ_0, ψ_1. (These two surfaces are the two first terms on the right-hand side of (19)). It is easy to verify that $C(\phi_0, \phi_1, \psi_0, \psi_1; h)$ coincides with the four curves on the boundary of $[0, h]^2$, namely that

$$\begin{aligned} C(\phi_0, \phi_1, \psi_0, \psi_1; h)(jh, t) &= \psi_j(t)\,, \quad j = 0, 1\,, \\ C(\phi_0, \phi_1, \psi_0, \psi_1; h)(s, ih) &= \phi_i(s)\,, \quad i = 0, 1\,. \end{aligned}$$

Regarding the Coons patch of four curves as the analogue of a linear segment between two points, the Chaikin scheme for the refinement of control points is "extended" to nets of curves, by sampling each of the Coons patches of $\mathcal{C}(\mathcal{N}_k)$ at 1/4 and 3/4 of the corresponding parameters values. Thus the refinement rule analogous to (2.11) is

$$\phi_{2i}^{k+1}(s) = \mathcal{C}(\mathcal{N}_k)\left(s, \left(i+\frac{1}{4}\right)\frac{d}{2^k}\right), \quad \phi_{2i+1}^{k+1} = \mathcal{C}(\mathcal{N}_k)\left(s, \left(i+\frac{3}{4}\right)\frac{d}{2^k}\right), \quad i = 0, \dots, n_k - 1, \tag{20}$$

$$\psi_{2j}^{k+1}(t) = \mathcal{C}(\mathcal{N}_k)\left(\left(j+\frac{1}{4}\right)\frac{d}{2^k}, t\right), \quad \psi_{2j+1}^{k+1} = \mathcal{C}(\mathcal{N}_k)\left(\left(j+\frac{3}{4}\right)\frac{d}{2^k}, t\right), \quad j = 0, \dots, m_k - 1. \tag{21}$$

This refinement rule generates a refined net of curves after a simple reparametrization. This is written formally as $\mathcal{N}_{k+1} = S_{BC}\mathcal{N}_k$.

The proof of convergence of the scheme S_{BC} is not an extension of the analysis of §2.3, but is based on the proximity of S_{BC} to a new subdivision scheme $S_{\mathbf{a}}$ for points, which is proved to be convergent by the analysis of §2.3.

Convergence proofs by proximity to linear stationary schemes for points are employed, e.g., in [35] for the analysis of linear non-stationary schemes, and in [59] for the analysis of a certain class of non-linear schemes.

Another important ingredient in the convergence proof is a property of a net of curves, which is preserved during the refinements with S_{BC}. A net of curves $\mathcal{N}(d, \{\phi_i\}_{i=0}^n, \{\psi_j\}_{j=0}^m)$ is said to have the *M-property* if the second divided differences of all curves of the net at three points restricted to intervals of the form $\left[\ell d, \left(\ell+\frac{1}{2}\right)d\right]$, $\ell \in (1/2)\mathbb{Z}$ in the domain of definition of the curves, are all bounded by a constant M.

The sequence $\{\mathcal{C}(\mathcal{N}_k) : k \in \mathbb{Z}_+\}$ of continuous surfaces is shown to be a Cauchy sequence for $\mathcal{N}_0$ with the M-property, by comparison of one refinement of S_{BC} with one refinement of $S_{\mathbf{a}}$. The scheme $S_{\mathbf{a}}$ is constructed to be in proximity to S_{BC} in the sense that

$$\left\|\mathcal{E}(S_{BC}\mathcal{N}_k) - S_{\mathbf{a}}(\mathcal{E}(\mathcal{N}_k))\right\| \leq \frac{3}{2}M\left(\frac{d}{2^{k+1}}\right)^2, \tag{22}$$

with $\mathcal{E}(\mathcal{N}_k) = \left\{C(\mathcal{N}_k)\left(i\frac{d}{2}, j\frac{d}{2}\right), 0 \leq i \leq 2m_k, 0 \leq j \leq 2n_k\right\}$, and with M the constant in the M-property satisfied by all the sets $\{\mathcal{N}_k : k \in \mathbb{Z}_+\}$ which are generated by S_{BC}.

Although the limit of the Cauchy sequence $\{\mathcal{C}(\mathcal{N}_k) : k \in \mathbb{Z}_+\}$ is only C^0, it is conjectured in [15] that S_{BC} generates C^1 surfaces from initial curves which are C^1. This conjecture is based on simulations.

Acknowledgement The author wishes to thank David Levin and Adi Levin for helping with the figures and the references.

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