A C^2 Four-Point Subdivision Scheme with Fourth Order Accuracy and its Extensions

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Abstract. We present a new four-point subdivision scheme that generates C^2 curves. It reproduces cubic polynomials, has a basic limit function with support $[-4, 3]$, and is close to being interpolatory. The refinement rule is based on local cubic interpolation, followed by evaluation at 1/4 and 3/4 of the refined interval. We investigate the approximation properties of this four-point scheme and extend it to a new family of $2n$-point schemes. The performance of the new schemes is demonstrated by several examples.

§1. Introduction

Subdivision schemes are efficient methods for generating curves and surfaces from discrete sets of control points. The important schemes for applications are schemes for surfaces, yet schemes generating curves constitute a basic tool for the design, study, and understanding of schemes generating surfaces. In this paper we present a new idea for the design of good subdivision schemes for curves.

In Sections 2 and 3 we construct and analyse a new four-point scheme, which generates C^2 curves, reproduces cubic polynomials and has a basic limit function supported on $[-4, 3]$. The size of the support measures the locality of the scheme and the degree of polynomials reproduced determines the approximation order of the subdivision operator. For our new four-point scheme the approximation order is 4. The refinement rule of this scheme is obtained by constructing, for each interval (or edge) in the coarser level, a cubic polynomial that interpolates the four points closest to the interval, and then evaluating this polynomial at 1/4 and 3/4 of the interval. The collection of the pairs of points, corresponding to each edge in the coarser level, constitute the refined set of points.

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In comparison, the well known interpolatory four-point scheme of [4] and [6] (with tension parameter $w = 1/16$), has smoothness C^1, the same approximation order 4, and $[-3, 3]$ as support of its basic limit function. The refinement rule for this "classical" scheme is obtained by adding to the points at the coarser level the points of the above local cubic interpolating polynomials, evaluated at the mid points of the corresponding intervals.

In the spirit of [6], we construct a family of four-point schemes with a tension parameter in Section 4. Each scheme in this family is a convex combination of the new four-point scheme and the Chaikin scheme [2], which can be obtained by our construction, if the local cubic interpolating polynomials are replaced by local linear polynomials that interpolate the two end points of the intervals. Among these four-point schemes with tension parameter, we choose the one that is closest to being interpolatory and demonstrate its superior performance by several examples given in Section 6.

As in [3], we extend the new four-point scheme to $2n$-point schemes, for any integer n, by replacing the local cubic polynomials in our construction by local interpolating polynomials of degree $2n-1$, each based on the $2n$ points that are closest to the corresponding interval. Our analysis of few of these schemes for small values of n indicates that the schemes converge and that their smoothness increases with n. This observation is yet to be proved for general n.

Section 6 concludes the paper with some numerical examples. The graphs of the basic limit functions corresponding to several of the new schemes introduced in the paper, as well as their derivatives are displayed. We also give two examples of curves generated by these schemes and compare them with the curves generated by the Chaikin scheme and the "classical" four-point scheme from the same set of control points.

§2. The Scheme

Suppose we are given the data f_i for $i \in \mathbb{Z}$. We set $f_i^0 = f_i$, for $i \in \mathbb{Z}$, and define for each $k = 0, 1, 2, \ldots$, and $i \in \mathbb{Z}$,

$$\begin{aligned} f_{2i}^{k+1} &= -\frac{7}{128} f_{i-1}^k + \frac{105}{128} f_i^k + \frac{35}{128} f_{i+1}^k - \frac{5}{128} f_{i+2}^k, \\ f_{2i+1}^{k+1} &= -\frac{5}{128} f_{i-1}^k + \frac{35}{128} f_i^k + \frac{105}{128} f_{i+1}^k - \frac{7}{128} f_{i+2}^k. \end{aligned} \tag{1}$$

This scheme comes from interpolating the data $(2^{-k}(i+j), f_{i+j}^k)$, $j = -1, 0, 1, 2$, by a cubic polynomial and evaluating it at $2^{-k}(i + 1/4)$ and $2^{-k}(i + 3/4)$ for the values f_{2i}^k and f_{2i+1}^k respectively. It is sufficient to consider p_3, the cubic polynomial such that $p_3(j) = f_j$, for $j = -1, 0, 1, 2$.

Since

$$p_3(t) = \sum_{j=-1}^{2} L_j(t) f_j, \qquad L_j(t) = \prod_{\substack{k=-1\\k\neq j}}^{2} \frac{t-k}{j-k},$$

we find

$$\begin{aligned} p_3(1/4) &= -\frac{7}{128}f_{-1} + \frac{105}{128}f_0 + \frac{35}{128}f_1 - \frac{5}{128}f_2,\\ p_3(3/4) &= -\frac{5}{128}f_{-1} + \frac{35}{128}f_0 + \frac{105}{128}f_1 - \frac{7}{128}f_2. \end{aligned} \tag{2}$$

§3. Analysis

Following the framework of [7], the scheme *converges* if there is a function $f : \mathbb{R} \to \mathbb{R}$ with the property that for any compact set $K \subset \mathbb{R}$,

$$\lim_{k\to\infty} \max_{i\in\mathbb{Z}\cap 2^k K} |f_i^k - f(2^{-k}i)| = 0. \tag{3}$$

Theorem 1. *The scheme (1) converges and has smoothness C^2.*

Proof: Following the notation of [7], we can write the scheme (1) as

$$f_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{i-2j} f_j^k,$$

or simply as

$$\mathbf{f}^{k+1} = S_a \mathbf{f}^k.$$

The symbol of S_a is the Laurent polynomial

$$a(z) = \sum_i a_i z^i = \frac{1}{128}(-5z^4 - 7z^3 + 35z^2 + 105z + 105 + 35z^{-1} - 7z^{-2} - 5z^{-3}).$$

This can be written as

$$a(z) = \frac{(1+z)^3}{4} b(z),$$

where

$$b(z) = \frac{1}{32}(-5z + 8 + 26z^{-1} + 8z^{-2} - 5z^{-3}).$$

By Corollary 4.11 of [7], if S_b is contractive then S_a is C^2. Defining

$$\|S_b^{[\ell]}\|_\infty := \max\left\{ \sum_{j\in\mathbb{Z}} |b^{[\ell]}_{i-2^\ell j}| : 0 \le i < 2^\ell \right\},$$

where

$$b^{[\ell]}(z) := b(z)b(z^2)\cdots b(z^{2^{\ell-1}}),$$

we find that

$$\|S_b^{[1]}\|_\infty = \frac{1}{32}\max\{5+26+5, 8+8\} = \frac{9}{8} \geq 1,$$

which does not show that S_b is contractive. However, since

$$\begin{aligned} b^{[2]}(z) &= b(z)b(z^2) \\ &= \frac{1}{1024}(-5z+8+26z^{-1}+8z^{-2}-5z^{-3}) \\ &\quad \times(-5z^2+8+26z^{-2}+8z^{-4}-5z^{-6}) \\ &= \frac{1}{1024}(25z^3-40z^2-170z+24+103z^{-1}+272z^{-2}-596z^{-3} \\ &\quad +272z^{-4}+103z^{-5}+24z^{-6}-170z^{-7}-40z^{-8}+25z^{-9}), \end{aligned}$$

we find that

$$\begin{aligned} \|S_b^{[2]}\|_\infty = \frac{1}{1024}\max\{&25+103+103+25, 40+272+24, \\ &170+596+170, 24+272+40\} = \frac{117}{128} < 1, \end{aligned}$$

which shows that S_b is contractive. □

Using the method described in [5] we further find the Hölder regularity of the scheme to be at least 2.67.

Consider next the support of the scheme (1). This is simply the support of the limit function ϕ generated by the data $f_0 = 1$ and $f_i = 0$ for $i \neq 0$. It is a simple matter to show that ϕ has support $[-4, 3]$. This is only slightly larger than the support $[-3, 3]$ of the "classical" four-point scheme of Dubuc [4].

As regards the approximation order of the scheme, we begin by showing that if the data is sampled from a cubic polynomial then the limit function is simply a shifted version of that polynomial. We denote by π_3 the space of all cubic polynomials.

Lemma 1. *If $f_i^0 = g(i-1/2)$, for some $g \in \pi_3$, then the limit of the subdivision scheme (1) is g.*

Proof: Let $g^{[k]}(x) = g(x-2^{-k-1})$, for $k = 0, 1, 2, \ldots$. By (1) and (2),

$$\begin{aligned} f_{2i}^1 &= g(i-1/4) = g(2^{-1}2i-1/4), \\ f_{2i+1}^1 &= g(i+1/4) = g(2^{-1}(2i+1)-1/4), \end{aligned}$$

and thus for all $i \in \mathbb{Z}$,

$$f_i^1 = g(2^{-1}i - 1/4) = g^{[1]}(2^{-1}i).$$

By a similar argument, if $f_i^{k-1} = g^{[k-1]}(2^{-k+1}i)$ then $f_i^k = g^{[k]}(2^{-k}i)$. Therefore we get

$$|f_i^k - g(2^{-k}i)| = |g(2^{-k}i - 2^{-k-1}) - g(2^{-k}i)|,$$

and so by (3) and the continuity of g, the limit function is $S^\infty \mathbf{f}^0 = g$. $\square$

Similarly, if the data for the scheme is $f_i^0 = g((i-1/2)h)$, for some $g \in \pi_3$, then since $g(h\,\cdot)$ is also in π_3, the limit is clearly $g(h\,\cdot)$. Due to this "shifted" cubic precision, we can show that the scheme is fourth order accurate.

Theorem 2. *Suppose $g : \mathbb{R} \to \mathbb{R}$ has four continuous derivatives on $\mathbb{R}$. Then the limit $f = S^\infty \mathbf{f}^0$ of the scheme (1) with initial data $f_i^0 = g\big((i - 1/2)h\big)$ satisfies*

$$\|f - g\|_{\infty,K_1} \le Ch^4 \|g^{(4)}\|_{\infty,K_2},$$

for any closed interval $K_1 = [a,b]$ and with $K_2 = [a - 7h/2, b + 7h/2]$.

Proof: Let $R_{h\mathbb{Z}}$ denote the restriction operator to $h\mathbb{Z}$ and let σ_τ denote the translation operator $\sigma_\tau g = g(\cdot + \tau)$. The operator $Q_h = S^\infty R_{h\mathbb{Z}}\, \sigma_{-h/2}$ is a quasi-interpolating operator [1] with the properties

1. Q_h is exact for π_3 (by Lemma 1), namely $Q_h p = p$ for all $p \in \pi_3$.
2. Q_h is local (in the terminology of [1]). This property follows from the explicit form of Q_h,

$$Q_h g = \sum_{i \in \mathbb{Z}} g\big((i - 1/2)h\big)\phi(h^{-1} \cdot - i), \tag{4}$$

and from the fact that for a fixed x, the values of $g((i-1/2)h)$ involved in $(Q_h g)(x)$ correspond to i satisfying

$$(i - 1/2)h \in (x - 7h/2, x + 7h/2). \tag{5}$$

Thus

$$g - Q_h g = (1 - Q_h)(g - p)$$

for any $p \in \pi_3$, and by (4) and (5),

$$|(g - Q_h g)(x)| \le (1 + \|Q_h\|_\infty)\|g - p\|_{\infty, I_{x,h}},$$

where $I_{x,h} = [x - 7h/2, x + 7h/2]$. To bound $\|Q_h\|_\infty$ we refer again to (4) and obtain

$$\|Q_h g\|_\infty \le 7\|g\|_\infty \|\phi\|_\infty,$$

using the observation that at each point x at most seven terms in the sum (4) do not vanish. Thus

$$\|Q_h\|_\infty = \sup_{\|g\|_\infty=1} \|Q_h g\|_\infty \le 7\|\phi\|_\infty.$$

Finally, taking p to be, for example, the Taylor expansion of g about x, we get

$$|(g - Q_h g)(x)| \le (1 + 7\|\phi\|_\infty)\|g^{(4)}\|_{\infty,I_{x,h}}(7/2)^4 h^4/4!.$$

Thus for any interval $K_1 = [a, b]$,

$$\|g - Q_h g\|_{\infty,K_1} \le (1 + 7\|\phi\|_\infty)\|g^{(4)}\|_{\infty,K_2}(7/2)^4 h^4/4!,$$

where $K_2 = \cup_{x\in K_1} I_{x,h} = [a - 7h/2, b + 7h/2]$, which proves the claim of the theorem with $C = (1 + 7\|\phi\|_\infty)7^4/(2^4 4!)$. □

§4. Tension Parameter

By viewing the scheme as a perturbation of Chaikin's scheme [2], we can easily introduce a *tension parameter* w, giving the extended scheme

$$\begin{aligned} f_{2i}^{k+1} &= -7wf_{i-1}^k + \left(\frac{3}{4} + 9w\right) f_i^k + \left(\frac{1}{4} + 3w\right) f_{i+1}^k - 5wf_{i+2}^k, \\ f_{2i+1}^{k+1} &= -5wf_{i-1}^k + \left(\frac{1}{4} + 3w\right) f_i^k + \left(\frac{3}{4} + 9w\right) f_{i+1}^k - 7wf_{i+2}^k, \end{aligned} \tag{6}$$

with $w = 0$ corresponding to the Chaikin scheme and $w = 1/128$ corresponding to the new four-point scheme (1).

As regards the smoothness of this scheme, we know that it is C^1 when $w = 0$ because Chaikin's scheme generates a C^1 quadratic spline. On the other hand it is reasonable to expect there to be a range of values of w around $1/128$ for which the scheme has C^2 continuity. In fact we would expect the range to include $(0, 1/128]$ because Chaikin's scheme is "close" to C^2. For the general scheme, we have (as before),

$$a_w(z) = \frac{(1+z)^3}{4} b_w(z),$$

but now with

$$b_w(z) = z^{-1} + 4w(-5z + 8 - 6z^{-1} + 8z^{-2} - 5z^{-3}).$$

Using the same approach as in the proof of Theorem 1 we find that

$$||S_{b_w}^{[1]}||_\infty = \max\{|40w| + |1 - 24w|, |64w|\} \ge 1$$

for all w, which does not show that S_{b_w} is a contraction. But we further find

$$||S_{b_w}^{[2]}||_\infty = \max\{|40w + 320w^2| + |1 - 48w - 704w^2|,\\ 1024w^2 + |32w + 256w^2|, 800w^2 + |24w + 224w^2|\} < 1$$

for $0 < w < \frac{\sqrt{6}-1}{80}$, which shows the scheme to be C^2 for an even larger range than the expected one. In fact, by analysing $S_{b_w}^{[\ell]}$ with a computer algebra system like *Maple* for ℓ up to 12, one can show that the scheme has C^2 continuity for w in the range of $(0, 1/48]$. For these schemes, the basic limit function ϕ_w has support $[-4, 3]$ for all positive w and support $[-2, 1]$ for $w = 0$.

Among these C^2 schemes we can search for the "tightest" one, i.e. for the scheme which is closest to being interpolatory. In view of Theorem 2, we measure the *tightness* by how close the values $\phi_w(i - 1/2)$ are to the Kronecker delta,

$$T(w) := \sum_{i\in\mathbb{Z}} (\phi_w(i - 1/2) - \delta_{i,0})^2 = (\phi_w(-1/2) - 1)^2 + 2\sum_{i=1}^{3} \phi_w(i - 1/2)^2.$$

In order to find the values $\phi_w(i-1/2)$ we need to analyse the 6×6 subblock

$$M_w = \begin{pmatrix} -7w & 3/4+9w & 1/4+3w & -5w & 0 & 0 \\ -5w & 1/4+3w & 3/4+9w & -7w & 0 & 0 \\ 0 & -7w & 3/4+9w & 1/4+3w & -5w & 0 \\ 0 & -5w & 1/4+3w & 3/4+9w & -7w & 0 \\ 0 & 0 & -7w & 3/4+9w & 1/4+3w & -5w \\ 0 & 0 & -5w & 1/4+3w & 3/4+9w & -7w \end{pmatrix}$$

of the subdivision matrix. The size 6 comes from the fact that the scheme has an invariant neighbourhood of size 6. The right and left eigenvectors of M_w corresponding to the eigenvalue 1 are

$$x_w^{[1]} = (1,1,1,1,1,1)^t \qquad \text{and} \qquad \tilde{x}_w^{[1]} = (\alpha, \beta, \gamma, \gamma, \beta, \alpha)^t$$

with

$$\alpha = \frac{40w^2}{1-8w}, \qquad \beta = -\frac{8w(7w+1)}{1-8w}, \qquad \gamma = \frac{32w^2 + 8w + 1}{2(1-8w)}.$$

The values $\phi_w(i - 1/2)$ can now be obtained by taking the inner product between $\tilde{x}_w^{[1]}$ and the vector of the six values f_k^1 with $2i - 3 \le k \le 2i + 2$, generated by one subdivision step from the initial data $f_k^0 = \delta_{k,0}$. Thus,

$$\phi_w(-1/2) = \frac{2432w^3 - 80w^2 - 44w - 3}{4(8w-1)},$$
$$\phi_w(1/2) = \frac{448w^3 - 576w^2 + 56w - 1}{8(8w-1)},$$
$$\phi_w(3/2) = \frac{-864w^3 - 92w^2 + 5w}{2(8w-1)},$$
$$\phi_w(5/2) = \frac{200w^3}{8w-1}.$$

The global minimum of $T(w)$ is obtained for $w_* \approx 0.013723$, giving a C^2 scheme that we call the *tight four-point scheme*. Like the scheme (1), it has support size 7, but its accuracy is only $O(h^2)$ as for Chaikin's scheme.

The following table lists the values of the basic limit function at the half-integers and the tightness for some values of w:

w	$\phi_w(-1/2)$	$\phi_w(1/2)$	$\phi_w(3/2)$	$\phi_w(5/2)$	$T(w)$
0	0.750000	0.125000	0	0	0.093750
1/128	0.892660	0.071391	-0.017619	-0.000102	0.022336
w_*	1.014525	0.020871	-0.027553	-0.000581	0.002601

§5. Generalization to Arbitrary Degree

Another way to extend our construction is to locally fit a Lagrange interpolation polynomial of degree $2n-1$ to the $2n$ points that are closest to the interval to be refined, and to evaluate it at 1/4 and 3/4 of the interval. Here n can be any fixed integer. For example, $n = 1$ gives Chaikin's scheme and $n = 2$ gives our new four-point scheme. Analogously to the analysis in Section 3, it can be shown that these new $2n$-point schemes reproduce polynomials up to degree $2n-1$ and have approximation order $O(h^{2n})$. The support of the basic limit function is $[-2n, 2n-1]$, i.e. the support size is $4n-1$. We computed lower bounds for the Hölder regularity of the functions generated by the new $2n$-point schemes for few small values of n. These values are presented in the following table:

n	1	2	3	4	5	6	7	8	9	10
R_H	1	2.67	3.51	4.11	4.56	5.27	5.62	6.17	6.49	7.15

It is yet to be shown that these schemes converge for any n and that they have increasing smoothness with increasing n.

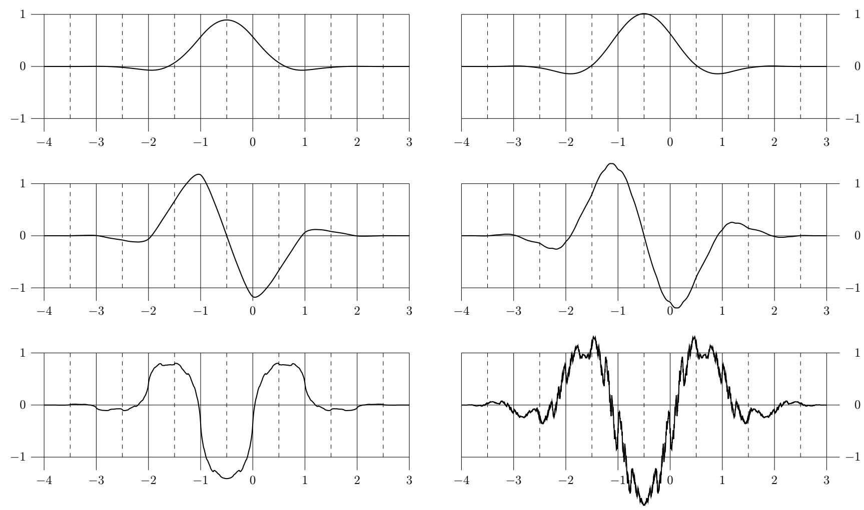

Fig. 1. The basic limit function and its first and second derivative of the new four-point (left) and the tight four-point scheme (right).

§6. Numerical Examples

Figures 1–3 show the basic limit functions for several of the new schemes as well as their first and second derivatives, and Figures 4–7 show the results of the schemes in comparison with the Chaikin and the classical four-point scheme.

The examples confirm our theoretical results and show that the tight four-point scheme generates curves that almost interpolate the control points. Thus, if one is interested in a curve that closely follows the control polygon, then the tight four-point scheme offers an alternative to the classical four-point scheme with the advantage of having C^2 smoothness. On the other hand, if an optimal approximation order is required, then the new family of $2n$-point schemes is an alternative to the classical family of interpolating $2n$-point schemes [3]. Their support size is only slightly bigger ($4n-1$ as compared to $4n-2$) and they have a higher degree of smoothness, at least for small values of n. It further seems that the new schemes become more interpolatory for higher n, but proving this observation as well as carefully analysing the smoothness is left to future work.

§7. References

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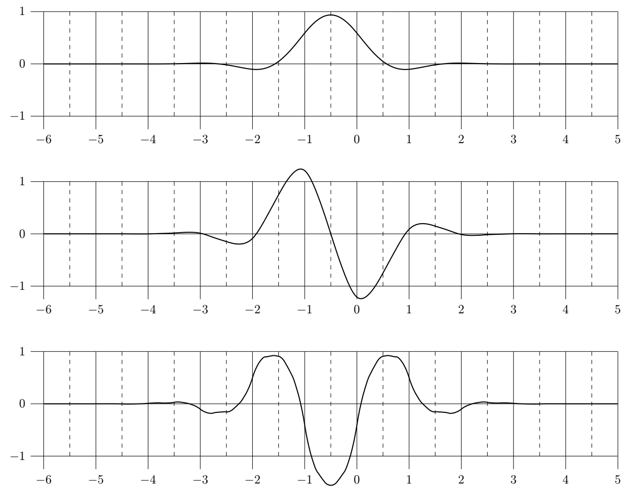

Fig. 2. Basic limit function of the new six-point scheme, and its first and second derivative.

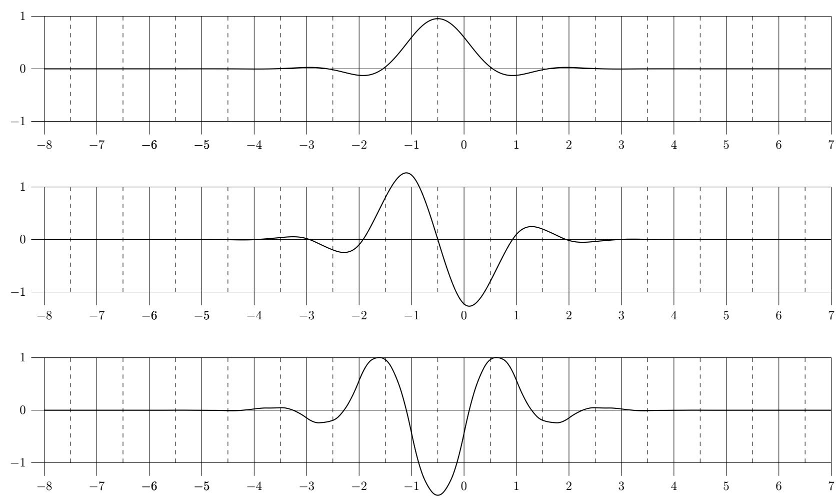

Fig. 3. Basic limit function of the new eight-point scheme, and its first and second derivative.

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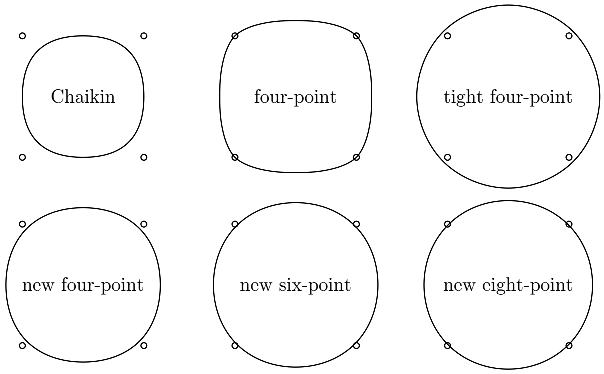

Fig. 4. Comparison of subdivision results.

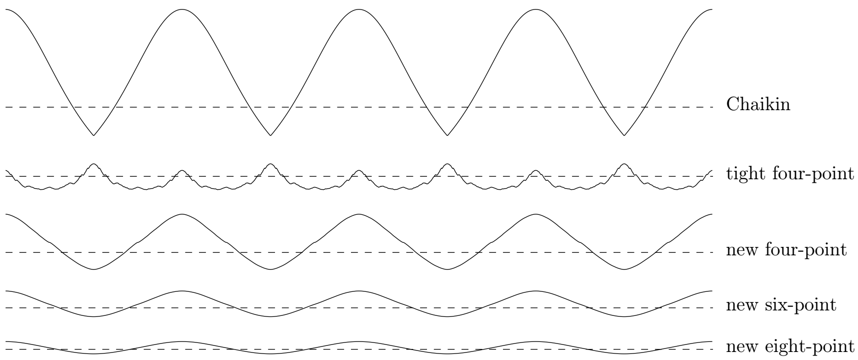

Fig. 5. Curvature plot of the curves in Figure 4. The dashed line indicates the curvature of the interpolating circle.

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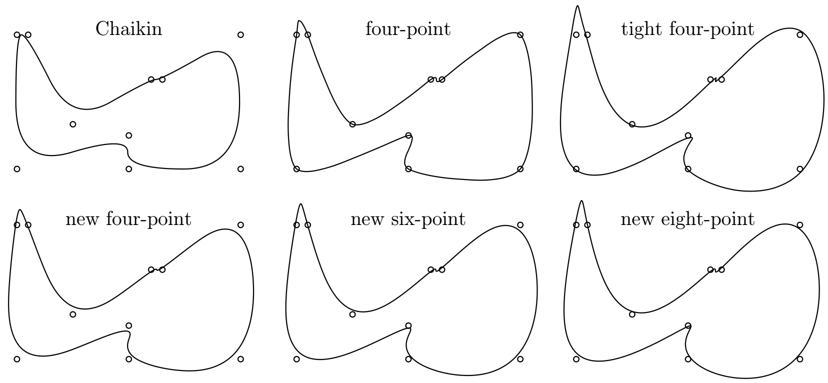

Fig. 6. Comparison of subdivision results.

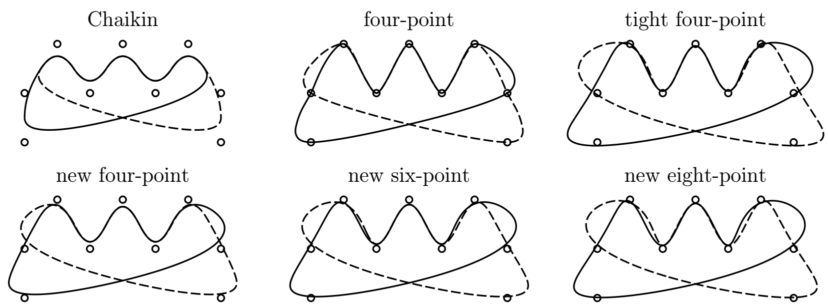

Fig. 7. Locality of the basis functions, visualized by moving the bottom left control point (full curve) to the right (dashed curve).

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