1

Linear and Nonlinear Subdivision Schemes in Geometric Modeling

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Abstract

Subdivision schemes are efficient computational methods for the design, representation and approximation of 2D and 3D curves, and of surfaces of arbitrary topology in 3D. Subdivision schemes generate curves/surfaces from discrete data by repeated refinements. While these methods are simple to implement, their analysis is rather complicated.

The first part of the paper presents the "classical" case of linear, stationary subdivision schemes refining control points. It reviews univariate schemes generating curves and their analysis. Several well known schemes are discussed.

The second part of the paper presents three types of nonlinear subdivision schemes, which depend on the geometry of the data, and which are extensions of univariate linear schemes. The first two types are schemes refining control points and generating curves. The last is a scheme refining curves in a geometry-dependent way, and generating surfaces.

1.1 Introduction

Subdivision schemes are efficient computational tools for the generation of functions/curves/surfaces from discrete data by repeated refinements. They are used in geometric modeling for the design, representation and approximation of curves and of surfaces of arbitrary topology. A linear stationary scheme uses the same linear refinement rules at each location and at each refinement level. The refinement rules depend on a finite number of *mask coefficients*. Therefore, such schemes are easy to implement, but their analysis is rather complicated.

The first subdivision schemes where devised by G. de Rahm (1956) for the generation of functions with a first derivative everywhere and a second derivative nowhere. Infact, in most cases, the limits generated by subdivision schemes are convolutions of a smooth function with a fractal one.

This paper consists of two parts. The first is a review of "classical" subdivision schemes, namely linear, stationary schemes refining control points. The second brings several constructions of nonlinear schemes, taking into account the geometry of the refined objects. The nonlinear schemes are all extensions of "good" linear univariate schemes.

In the first part we review only linear univariate subdivision schemes (schemes generating curves from initial points). Although the main application of subdivision schemes is in generating surfaces, we limit the presentation here to these schemes. The main reasons for this choice are two. The theory of univariate schemes is much more complete and easier to present, yet it consists of most of the aspects of the theory of schemes generating surfaces. The understanding of this theory is a first essential step towards the understanding of the theory of schemes generating surfaces. Also, the schemes that are extended in the second part to nonlinear schemes, are univariate.

Section 1.2 is devoted to the presentation of stationary, linear, univariate subdivision schemes. Important examples such as the B-spline schemes and the interpolatory 4-point scheme are discussed in details. Among these important examples are the schemes that are extended in the second part to nonlinear schemes. A sketch of tools for analysis of convergence and smoothness of linear, univariate schemes is given in §1.2.2. The relation between subdivision schemes and the construction of wavelets is briefly discussed in §1.2.3.

The second part of the paper consists of three sections. In Section 1.3 linear schemes are extended to refine manifold-valued data. This is done in two steps. First, the refinement rules of any convergent linear scheme are presented (in several possible ways) in terms of repeated binary averages. This is demonstrated by several examples in §1.3.1. Then in §1.3.2 the manifold-valued subdivision schemes are constructed, either by replacing every linear binary average in the linear refinement rules by a geodesic average, yilding a geodesic analoguous scheme, or by replacing every linear binary average in the linear refinement rules by its projection to the manifold, yielding a projection analogue scheme. The analysis of the so constructed manifold-valued subdivision schemes, is done in §1.3.3, via their *proximity* to the linear schemes from which they are derived.

In Section 1.4 two data-dependent extensions of the linear interpolatory 4-point scheme are discussed. In both extensions the refinement is adapted to the local geometry of the four points used. These two geometric (data-dependent) 4-point schemes are effective in case of an initial control polygon with edges of significantly different length. For such initial control polygons the limits generated by the linear 4-point scheme have unwanted features (artifacts), while the geometric 4-point schemes tend to generate artifact free limits.

The last section deals with repeated refinements of curves for the generation of a surface. Here the scheme that is extended is the quadratic B-spline scheme (called Chaikin algorithm). It is used to refine curves by first constructing a *geometric correspondence* between pairs of curves, and then applying the refinment to corresponding points. Since this is a work in progress only partial results are presented.

1.2 Linear subdivision schemes for the generation of curves

In this section we discuss stationary, linear schemes, generating curves by repeated refinements of points. Such a subdivision scheme is defined by a finite set of real coefficients called *mask*

$$\mathbf{a} = \{a_i \in \mathbb{R},\ i \in \sigma(\mathbf{a}) \subset \mathbb{Z}\},$$

where $\sigma(\mathbf{a})$ denotes the finite support of the mask. The scheme with the mask $\mathbf{a}$ is denoted by $S_{\mathbf{a}}$.

The refinement rules of $S_{\mathbf{a}}$ have the form

$$(S_{\mathbf{a}}\mathcal{P})_\alpha = \sum_{\beta\in\mathbb{Z}} a_{\alpha-2\beta} P_\beta\,, \quad \alpha \in \mathbb{Z}, \tag{1.1}$$

where $\mathcal{P}$ denotes the polygonal line through the points $\{P_i\}_{i\in\mathbb{Z}} \subset \mathbb{R}^d$, with $d \geq 1$.

Note that there are two different refinement rules in (1.1), one corresponding to odd α and one to even α, involving the odd respectively even coefficients of the mask.

Remark 1.1 *In this section we consider schemes operating on points defined on $\mathbb{Z}$, although, in geometric applications the schemes operate on finite sets of points. Due to the finite support of the mask, our considerations apply directly to closed curves and also to 'open' ones, except in a finite zone near the boundary.*

The initial points to be refined are called *control points*, and the corresponding polygonal line through them is called *control polygon*. These terms are used also for the points and the corresponding polygonal line at each refinement level. The subdivision scheme is a sequence of repeated refinements of control polygons,

$$\{\mathcal{P}^{k+1} = S_{\mathbf{a}}\mathcal{P}^k,\ k \geq 0\}, \tag{1.2}$$

with $\mathcal{P}^0$ the initial control polygon. The refinements in (1.2) are stationary, since at each refinement level the same scheme $S_{\mathbf{a}}$ is used. Material on nonstationary subdivision schemes can be found in the review paper Dyn & Levin (2002).

A subdivision scheme is termed *uniformly convergent* if the sequence of control polygons $\{\mathcal{P}^k\}$ converges uniformly on compact sets of $\mathbb{R}^d$. This is the notion of convergence relevant to geometric applications.

Each control polygons in (1.2) has a parametric representation as the vector in $\mathbb{R}^d$ of the piecewise linear interpolant to the data $\{(i2^{-k}, P_i^k)\,,\ i \in \mathbb{Z}\}$, and the uniform convergence is analyzed in the setting of continuous functions. (see e.g. Dyn (1992)).

For $\mathcal{P}^0 \subset \mathbb{R}^d$ with $d = 1$, the scheme converges to a univariate function, while in case $d \geq 2$ the scheme converges to a curve in $\mathbb{R}^d$.

The convergence of a scheme $S_{\mathbf{a}}$ implies the existence of a *basic-limit-function* $\phi_{\mathbf{a}}$, which is the limit obtained from the initial data, $\delta_i^0 = 0$ everywhere on $\mathbb{Z}$ except $\delta_0^0 = 1$.

It follows from the linearity and uniformity of (1.1) that the limit $S_A^\infty \mathcal{P}^0$, obtained from any initial control polygon $\mathcal{P}^0$ passing through the initial control points $\{P_\alpha^0 \in \mathbb{R}^d,\ \alpha \in \mathbb{Z}\}$, can be written in terms of integer translates of $\phi_{\mathbf{a}}$, as

$$(S_{\mathbf{a}}^\infty \mathcal{P}^0)(t) = \sum_{\alpha \in \mathbb{Z}} P_\alpha^0 \phi_{\mathbf{a}}(t - \alpha)\,, \quad t \in \mathbb{R}. \tag{1.3}$$

Equation (1.3) is a parametric representation of a curve in $\mathbb{R}^d$ for $d \geq 2$.

Also, by the linearity, uniformity and stationarity (see (1.2)) of the subdivision scheme, and since $(S_{\mathbf{a}}\delta^0)_\alpha = a_\alpha\,,\ \alpha \in \mathbb{Z}\,,\ \ \phi_{\mathbf{a}}$ satisfies the *refinement equation* (*two-scale relation*)

$$\phi_{\mathbf{a}}(t) = \sum_{\alpha \in \mathbb{Z}} a_\alpha \phi_{\mathbf{a}}(2t - \alpha)\,. \tag{1.4}$$

It is easy to obtain from (1.1) or from (1.4) that the support of $\phi_{\mathbf{a}}$ equals the convex hull of $\sigma(\mathbf{a})$ (see e.g. Cavaretta, Dahmen & Micchelli (1991)).

Remark 1.2 *The discussion above leads to the conclusion that a convergent subdivision scheme gives rise to a unique continuous basic-limit-function, satisfying a refinement equation with the mask of the scheme. The converse is not generally true. Yet a continuous function satisfing a refinement equation is a basic-limit-function of a convergent subdivision scheme, if its integer shifts are linearly independent (see Cavaretta, Dahmen & Micchelli (1991)).*

1.2.1 The main types of schemes

The first subdivision schemes in geometric modelling were proposed for easy and quick rendering of B-spline curves. A B-spline curve has the form

$$C(t) = \sum_i P_i B_m(t-i) \tag{1.5}$$

with $\{P_i\} \subset \mathbb{R}^d$ the control points, and B_m a B-spline of degree m with integer knots, namely $B_m|_{[i,i+1]}$ is a polynomial of degree m for each $i \in \mathbb{Z}$, $B_m \in C^{m-1}(\mathbb{R})$, and B_m has a compact support $[0, m+1]$.

Equation (1.5) is a parametric representation of a B-spline curve. The B-spline curves (1.5) are a powerful design tool, since their shape is similar to the shape of the control polygon $\mathcal{P}$ corresponding to the control points in (1.5) (see e.g. Prautzsch, Boehm & Paluszny (2002)).

Such curves can be well approximated by the control polygons generated by the repeated refinements (1.2), using the mask $\mathbf{a}^{[m]}$ with coefficients

$$a_i^{[m]} = 2^{-m}\binom{m+1}{i}, \; i = 0, \ldots, m+1\,. \tag{1.6}$$

The repeated refinements of a B-spline scheme of degree m are thus

$$P_i^{\ell+1} = \sum_j a_{i-2j}^{[m]} P_j^{\ell}\,, \quad i \in \mathbb{Z}\,, \quad \ell = 0, 1, 2, \ldots\,. \tag{1.7}$$

In (1.7) we use the convention $a_i^{[m]} = 0$, $i \notin \{0, 1, \ldots, m+1\}$.

By the convergence analysis, presented in §1.2.2, it is easy to check that the subdivision scheme with the mask (1.6) is convergent. It converges to B-spline curves of degree m, since B_m is the basic-limit-function of $S_{\mathbf{a}^{[m]}}$.

This can be easily concluded in view of Remark 1.2 from the definition of B-splines, by observing that B_m satisfies a refinement equation (1.4) with the mask $\mathbf{a}^{[m]}$ (see e.g. Dyn (1992)), and that the integer translates of B_m are linearly independent.

Thus the control polygon $\mathcal{P}^k = S_{\mathbf{a}}^k \mathcal{P}$ approximates $C(t)$ for k large enough, and $C(t)$ can be easily rendered by rendering its approximation $\mathcal{P}^k$. In practice a large enough k is about 4 as can be seen in Figure 1.1.

The first scheme of this type was devised in Chaikin (1974) for a fast geometric rendering of quadratic B-spline curves. It has the refinement rules

$$P_{2i}^{k+1} = \frac{3}{4} P_{i-1}^k + \frac{1}{4} P_i^k \,, \quad P_{2i+1}^{k+1} = \frac{1}{4} P_{i-1}^k + \frac{3}{4} P_i^k \,. \tag{1.8}$$

Figure 1.1 illustrates three refinement steps with this scheme, applied to a closed initial control polygon. Chaikin scheme is extended to nonlinear schemes in §1.3 and in §1.5.

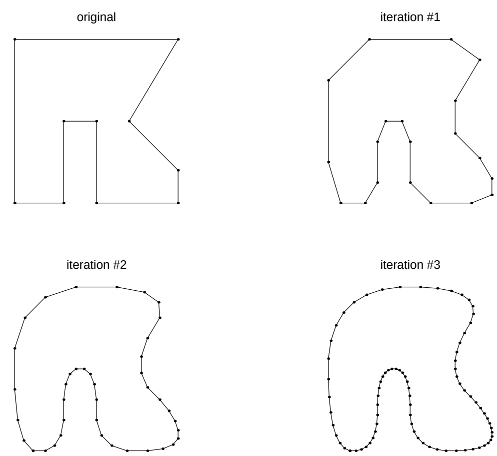

Fig. 1.1. Refinements of a polygon with Chaikin scheme

The schemes for general B-spline curves were introduced and investigated in Cohen, Lyche & Riesenfeld (1980). All other subdivision schemes can be regarded as a generalization of the B-spline schemes.

The B-spline schemes generate curves with a similar shape to the shape of the initial control polygons, but do not pass through the initial control points. Schemes with limit curves that interpolate the initial control points, were introduced in the late 1980's. These schemes are called *interpolatory*, and have the even refinement rule $P_{2i}^{k+1} = P_i^k$. Thus the control points at refinement level k are contained in those at refinement level $k+1$. The odd refinement rule, called in case of interpolatory schems *insertion rule*, is designed by a local approximation based on nearby control points.

The first schemes of this type were introduced by Deslauriers & Dubuc (1988) and by Dyn, Gregory & levin (1987). In the first paper the insertion rule for the point P_{2i+1}^{k+1} is obtained by first interpolating the data $\{((i+j), P_{i+j}^k),\ j = -N+1, \dots, N\}$ by a vector polynomial of degree $2N-1$, and then sampling it at the point $i + \frac{1}{2}$. This is done for a fix positive N. Rgarding N as a parameter, this construction yields a one-parameter family of convergent interpolatory subdivision schemes, with masks of increasing support, and with basic-limit-functions of increasing smoothness (see also the book Daubechies (1992)). We denote the scheme in this family corresponding to N by DD_N.

In the second paper above, a one-parameter family of 4-point interpolatory schemes is introduced, with the insertion rule

$$P_{2i+1}^{k+1} = -w(P_{i-1}^k + P_{i+2}^k) + (\frac{1}{2} + w)(P_i^k + P_{i+1}^k)\,. \tag{1.9}$$

Here w is a shape parameter. For $w = 0$ the limit is the initial control polygon, while as w increases to $w = 1/8$ the limit is a C^1 curve which becomes looser relative to the initial control polygon, as demonstrated in Figure 1.3. Thus w acts as a *tension parameter*. For $w = \frac{1}{16}$ this scheme coincides with DD_2. The local approximation which is used in the construction of the insertion rule (1.9) is a convex combination of the cubic interpolant used in DD_2 and the linear interpolant used in DD_1. In the next subsection the dependence of the convergence and smoothness of the 4-point scheme on the parameter w is discussed.

More about general interpolatory subdivision schemes, including multivariate schemes, can be found in Dyn & Levin (1992).

While in case of the B-spline schemes the limit was known and convergence was guaranteed, in case of the interpolastory schemes, analysis tools had to be developed. The analysis in Deslauriers & Dubuc (1988) and Daubecheis (1992) and in references therein is mainly in the Fourier domain, while in Dyn, Gregory & levin (1987) it is done in the geometric domain, based on the symbol of the scheme. Hints on this method of analysis, which was further developed in Dyn, Gregory & Levin (1991), are given in the next subsection.

1.2.2 Analysis of convergence and smoothness

Given the coefficients of the mask of a scheme, one would like to be able to determine if the scheme is convergent, and what is the smoothness of the resulting basic limit function (which is the generic smoothness of the limits generated by the scheme in view of (1.3)). Such analysis tools are also essential for the design of new schemes. We sketch here the method of convergence and smoothness analysis in Dyn, Gregory & Levin (1991) (see also Dyn (1992)).

An important tool in the analysis is the symbol of a scheme $S_{\mathbf{a}}$ with the mask $\mathbf{a} = \{a_\alpha : \alpha \in \sigma(\mathbf{a})\}$,

$$a(z) = \sum_{\alpha \in \sigma(\mathbf{a})} a_\alpha z^\alpha. \tag{1.10}$$

In the following we use also the notation $S_{a(z)}$ for $S_{\mathbf{a}}$.

A first step towards the convergence analysis is the derivation of the necessary condition for uniform convergence,

$$\sum_{\beta \in \mathbb{Z}} a_{\alpha - 2\beta} = 1\,, \quad \alpha = 0 \text{ or } 1 \,(\text{mod } 2), \tag{1.11}$$

The condition in (1.11) is derived easily from the refinement step

$$(S_{\mathbf{a}}^{k+1}\mathcal{P})_\alpha = \sum_{\beta \in \mathbb{Z}} a_{\alpha - 2\beta} (S_{\mathbf{a}}^k \mathcal{P})_\beta\,, \quad \alpha \in \mathbb{Z},$$

for k large enough so that for all $\ell \geq k$, $\|S_{\mathbf{a}}^\ell \mathcal{P} - S_{\mathbf{a}}^\infty \mathcal{P}\|_\infty$ is small enough.

The necessary condition (1.11) implies that we have to consider symbols satisfying

$$a(1) = 2\,, \quad a(-1) = 0\,. \tag{1.12}$$

Condition (1.12) is equivalent to

$$a(z) = (1+z)q(z) \quad \text{with} \quad q(1) = 1\,. \tag{1.13}$$

The scheme with symbol $q(z)$, $S_{\mathbf{q}}$, satisfies $S_{\mathbf{q}}\Delta = \Delta S_{\mathbf{a}}$, where Δ is the difference operator

$$\Delta \mathcal{P} = \big\{ (\Delta \mathcal{P})_i = P_i - P_{i-1}\,, \quad i \in \mathbb{Z} \big\}\,. \tag{1.14}$$

A necessary and sufficient condition for the convergence of $S_{\mathbf{a}}$ is the contractivity of the scheme $S_{\mathbf{q}}$, namely $S_{\mathbf{a}}$ is convergent if and only if $S_{\mathbf{q}}^\infty \mathcal{P} \equiv 0$ for any $\mathcal{P}$. The contractivity of $S_{\mathbf{q}}$ is equivalent to the existence of a positive integer L, such that $\|S_{\mathbf{q}}^L\|_\infty < 1$. This condition can be checked for a given L by algebraic operations on the symbol $q(z)$.

For practical geometrical reasons, only small values of L have to be considered, since a small value of L guarantees "visual convergence" of $\{\mathcal{P}^k\}$ to $S_{\mathbf{a}}^{\infty}\mathcal{P}^0$, already for small k, as the distances between consecutive control points contract to zero fast. A good scheme corresponds to $L = 1$ as the B-spline schemes, or to $L = 2$ as the 4-point scheme.

The smoothness analysis relies on the result that if the symbol of a scheme has a factorization

$$a(z) = \left(\frac{1+z}{2}\right)^{\nu} b(z)\,, \tag{1.15}$$

such that the scheme $S_{\mathbf{b}}$ is convergent, then $S_{\mathbf{a}}$ is convergent and its limit functions are related to those of $S_{\mathbf{b}}$ by

$$D^{\nu}(S_{\mathbf{a}}^{\infty}\mathcal{P}) = S_{\mathbf{b}}^{\infty}\Delta^{\nu}\mathcal{P}, \tag{1.16}$$

with D the differentiation operator.

Thus, each factor $(1 + z)/2$ multiplying a symbol of a convergent scheme adds one order of smoothness. This factor is termed *smoothing factor*.

The relation between (1.15) and (1.16) is a particular instance of the "algebra of symbols" (see e.g. Dyn & Levin (1995)). If $a(z), b(z)$ are two symbols of converging schemes, then S_c with the symbol $c(z) = \frac{1}{2}a(z)b(z)$ is convergent, and

$$\phi_{\mathbf{c}} = \phi_{\mathbf{a}} * \phi_{\mathbf{b}}\,. \tag{1.17}$$

Example (B-spline schemes). In view of (1.6), the symbol of the scheme generating B-spline curves of degree m is

$$a(z) = (1+z)^{m+1}/2^m. \tag{1.18}$$

The known smoothness of the limit functions generated by the m-th degree B-spline scheme, can be concluded easily, using the tools of analysis presented in this subsection. The factor $b(z) = \frac{(1+z)^2}{2}$ corresponds to the scheme $S_{\mathbf{b}}$ generating the initial control polygon as the limit curve, which is continuous, while the factors $\left(\frac{1+z}{2}\right)^{m-1}$ add smoothness, so that $S_{\mathbf{a}^{[m]}}^{\infty}\mathcal{P}^0 \in C^{m-1}$.

Example (the 4-point scheme). The analysis sketched above , is the tool by which the following results were obtained for the interpolatory 4-point scheme with the insertion rule (1.9). The symbol of the scheme can be written as

$$a_w(z) = \frac{1}{2z}(z+1)^2\left[1 - 2wz^{-2}(1-z)^2(z^2+1)\right]. \tag{1.19}$$

The range of w for which $S_{a_w(z)}$ is convergent is the range for which $S_{q_w(z)}$ with symbol $q_w(z) = a_w(z)/(1+z)$ is contractive. The condition $\|S_{q_w(z)}\|_\infty < 1$ holds in the range $-3/8 < w < (-1+\sqrt{13})/8$, while the condition $\|S^2_{q_w(z)}\|_\infty < 1$ holds in the range $-1/4 < w < (-1+\sqrt{17})/8$. Thus $S_{a_w(z)}$ is convergent in the range $-3/8 < w < (-1+\sqrt{17})/8$. In fact it was shown by M.D. Powell that the convergence range is $|w| \leq \frac{1}{2}$.

To find a range of w where $S_{a_w(z)}$ generates C^1 limits, the contractivity of $S_{c_w(z)}$ with $c_w(z) = 2a_w(z)/(1+z)^2$ has to be investigated. It is easy to check that $\|S_{c_w(z)}\|_\infty \geq 1$, but that $\|S^2_{c_w(z)}\|_\infty < 1$ for $0 < w < (\sqrt{5}-1)/8$. Only a year ago the maximal positive w for which the limit is C^1 was obtained in Hechler, Mößner & Reif (2008).

The limit of $S_{a_w(z)}$ is not C^2 even for $w = 1/16$, although for $w = 1/16$ the symbol is divisible by $(1+z)^3$. It is shown in Daubechies & Lagarias (1992), by other methods, that the basic limit function for $w = 1/16$, restricted to its support, has a second derivative only at the non-dyadic points there.

The conditions for smoothness given above are only sufficient. Yet, there is a large class of convergent schemes for which the factorization in (1.15) is necessary for generating C^ν limit functions. This class contains the B-spline schemes and the interpolatory schemes. See e.g. Dyn & Levin (2002).

1.2.3 Subdivision schemes and the construction of wavelets

Any convergent subdivision scheme $S_\mathbf{a}$ defines a sequence of nested spaces in terms of its basic-limit-function $\phi_\mathbf{a}$. For every $k \in \mathbb{Z}$ define the space

$$\begin{aligned} V_k &= \operatorname{span}\{\phi_\mathbf{a}(2^k(\cdot - i))\,, \quad i \in 2^{-k}\mathbb{Z}\} \\ &= \operatorname{span}\{\phi_\mathbf{a}(2^k \cdot - i)\,, \quad i \in \mathbb{Z}\}\,. \end{aligned} \tag{1.20}$$

Then in view of the refinement equation (1.4) satisfied by $\phi_\mathbf{a}$, these spaces are nested, namely

$$\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots \tag{1.21}$$

Such a bi-infinite sequence of spaces is the framework in which wavelets are constructed. It is called a multiresolution analysis when the integer translates of $\phi_\mathbf{a}$ constitute a Riesz basis of V_0 (see Daubechies (1992)). In the wavelets literature one starts from a solution of (1.4), termed *scaling* *function*, which is not necessarily the limit of a converging subdivision scheme, as indicated by Remark 1.2.

The choise of the mask coefficients for the construction of wavelets depends on the properties required from the wavelets. For example the orthonormal wavelets of Daubechies are generated by masks (*filters*) that are related to the masks of the DD_N schemes. The symbols of these masks, denoted by $\tilde{a}_N(z)$, satisfy $|\tilde{a}_N(z)|^2 = a_N(z)$, with $a_N(z)$ the symbol of the scheme DD_N. This relation between the symbols can be expressed in terms of the corresponding scaling functions as

$$\phi_{\tilde{\mathbf{a}}_\mathbf{N}} * \phi_{\tilde{\mathbf{a}}_\mathbf{N}}(-\cdot) = \phi_{\mathbf{a}_\mathbf{N}} ,$$

showing that the integer translates of $\phi_{\tilde{\mathbf{a}}_\mathbf{N}}$ constitute an orthonormal system due to the interpolatory nature of $\phi_{\mathbf{a}_\mathbf{N}}$, which vanishes at all integers except at zero where it is 1.

The next sections bring several constructions of non-linear schemes, based on linear schemes. In §1.3.1 more information on linear schemes, needed for the construction of schemes on manifolds, is presented.

1.3 Curve subdivision schemes on manifolds

To design subdivision schemes for curves on a manifold, we require that the control points generated at each refinement level are on the manifold, and that the limit of the sequence of corresponding control polygons is on the manifold. Such schemes are nonlinear.

The first approach to this problem is presented in Rahman, Drori, Stodden, Donoho & Schröder (2005). It is based on adapting a linear univariate subdivision scheme $S_\mathbf{a}$. Given control points $\{P_i\}$ on the manifold, let $\mathcal{P} = \{P_i\}$ denote the corresponding control polygon, and let T denote the adaptation of $S_\mathbf{a}$ to the manifold. Then the point $(T\mathcal{P})_i$ is defined by first executing the linear refinement step of $S_\mathbf{a}$ on the projections of the points $\{P_j\ ,\ a_{i-2j} \neq 0\}$ to a tangent plane at a chosen point P_i^*, and then projecting the obtained point to the manifold. This can be written as

$$(T\mathcal{P})_i = \psi_{P_i^*}^{-1}\Big(\sum_{j\in\mathbb{Z}} a_{i-2j}\psi_{P_i^*}(P_j)\Big), \tag{1.22}$$

where P_i^* is some chosen "center" of the points $\{P_j\ ,\ a_{i-2j} \neq 0\}$, and $\psi_{P_i^*}$ is the projection from the manifold to the tangent plan at P_i^*. Recently it was shown in Xie & Yu (2008) and Xie & Yu (2008a), that with a proper choise of the "center" point many properties of the linear scheme, such as convergence, smoothness and approximation order, are shared by the nonlinear scheme derived from it.

Here we discuss two other constructions of subdivision schemes on manifolds from converging linear schemes. These constructions are based on the observation that the refinement rules of any convergent linear scheme can be calculated by repeated binary averages (see Wallner & Dyn (2005)).

1.3.1 Linear schemes in terms of repeated binary averages

A linear scheme for curves, S, is defined by two refinement rules of the form,

$$(S\mathcal{P})_j = \sum_i a_{j-2i} P_i \ , \quad j = 0 \text{ or } 1(\text{mod } 2) \ , \tag{1.23}$$

where $\mathcal{P} = \{P_i\}$. As discussed in §1.2.2, any convergent linear scheme is affine invariant, namely $\sum_i a_{j-2i} = 1$. It is shown in Wallner & Dyn (2005), that for a convergent linear scheme, each of the refinement rules in (1.23) is expressible, in a non-unique way, by repeated binary averages. A reasonable choice is a symmetric representation relative to the topological relations in the control polygon.

For example, with the notation $Av_\alpha(P,Q) = (1-\alpha)P + \alpha Q\,,\ \alpha \in \mathbb{R},\ P,Q \in \mathbb{R}^d$, the insertion rule of the interpolatory 4-point scheme (1.9) can be rewritten as

$$P^k_{2j+1} = Av_{\frac{1}{2}}\Big(Av_{(-2w)}(P_j, P_{j-1})\ ,\ Av_{(-2w)}(P_{j+1}, P_{j+2})\Big)\ ,$$

or as

$$P^k_{2j+1} = Av_{(-2w)}\Big(Av_{\frac{1}{2}}(P_j, P_{j+1})\ ,\ Av_{\frac{1}{2}}(P_{j-1}, P_{j+2})\Big)\ .$$

Refinement rules represented in this way are termed hereafter *refinement rules in terms of repeated binary averages*.

Among the linear schemes there is a class of *factorizable schemes* for which the symbol $a(z) = \sum_i a_i z^i$, can be written as a product of linear real factors. For such a scheme, the control polygon obtained by one refinement step of the form (1.1), can be computed by several simple global steps, uniquely determined by the factors of the symbol.

To be more specific, let us consider a symbol of the form

$$a(z) = (1+z)\frac{1+x_1 z}{1+x_1} \cdots \frac{1+x_m z}{1+x_m}\ . \tag{1.24}$$

Note that this symbol corresponds to an affine invariant scheme since

$a(1) = 2$, and $a(-1) = 0$. In fact, the form of the symbol in (1.24) is general for converging factorizable schemes.

Let $\mathcal{P}^k$ denote the control polygon at refinement level k, corresponding to the control points $\{P_i^k\}$. Each simple step in the execution of the refinement $S_{\mathbf{a}}\mathcal{P}^k$ corresponds to one factor in (1.24). The first step in calculating the control points at level $k+1$ corresponds to the factor $1+z$, and consists of elementary refinement:

$$P_{2i}^{k+1,0} = P_{2i+1}^{k+1,0} = P_i^k \ . \qquad (1.25)$$

This step is followed by m averaging steps corresponding to the factors $\frac{1+x_j z}{1+x_j}$, $j = 1, \ldots, m$. The averaging step dictated by the factor $\frac{1+x_j z}{1+x_j}$ is,

$$P_i^{k+1,j} = \frac{1}{1+x_j}(P_i^{k+1,j-1} + x_j P_{i-1}^{k+1,j-1}) \, , \quad i \in \mathbb{Z} \, . \qquad (1.26)$$

The control points at level $k+1$ are $P_i^{k+1} = P_i^{k+1,m}$, $i \in \mathbb{Z}$.

The execution of the refinement step by several simple global steps is equivalent to the observation that

$$S_{a(z)}\mathcal{P}^k = R_{\frac{1+x_1 z}{1+x_1}} \cdots R_{\frac{1+x_m z}{1+x_m}} S_{(1+z)}\mathcal{P}^k \, , \qquad (1.27)$$

with $(R_{b+cz}\mathcal{P})_i = bP_i + cP_{i-1}$. The equality (1.27) follows from the representation of (1.1) as the formal equality

$$\sum_i (S_{\mathbf{a}}\mathcal{P})_i z^i = a(z) \sum_j P_j z^{2j} \, , \qquad (1.28)$$

with $\mathcal{P} = \{P_i\}$. The formal equality in (1.28) is in the sense of equality between the coefficients of the same power of z in both sides of (1.28).

We term the execution of the refinement step by several simple global steps, based on the factorization of the symbol to linear factors, *global refinement procedure by repeated averaging*.

An important family of factorizable schemes is that of the B-spline schemes, with symbols given by (1.18). Note that the factors in (1.18) corresponding to repeated averaging are all of the form $\frac{1+z}{2}$. Thus the global refinement procedure by repeated averaging is equivalent to the algorithm of Lane & Riesenfeld (1980). Also, it follows from the fact that all factors in (1.18) except for the factor $1+z$ are smoothing factors, that any B-spline scheme is optimal, in the sense that it has a mask of minimal support among all schemes with the same smoothness.

The interpolatory 4-point scheme given by the insertion rule (1.9) with

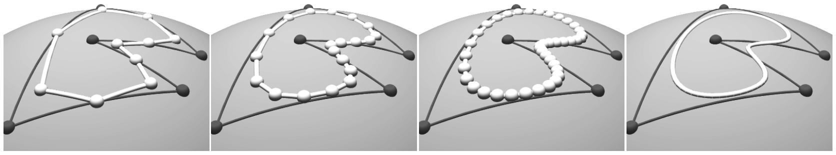

Fig. 1.2. Geodesic B-Spline subdivision of degree three. From left to right: Tp, T^2p,T^3p, $T^\infty p$.

$w = \frac{1}{16}$ (which is the DD_2 scheme) is also factorizable. Its symbol has the form (see e.g. Dyn (2005))

$$a(z) = (1+z)^4(3+\sqrt{3}z)(3-\sqrt{3}z)/48. \tag{1.29}$$

1.3.2 Construction of subdivision schemes on manifolds

The two constructions of nonlinear schemes on manifolds in Wallner & Dyn (2005), start from a convergent linear scheme, S, given either by local refinement rules in terms of repeated binary averages, or by a global refinement procedure in terms of repeated binary averages.

The first construction of a subdivision T on a manifold M, *analogous to* S, replaces every binary average in the representation of S, by a corresponding geodesic average on M. Thus $Av_\alpha(P,Q)$ is replaced by $gAv_\alpha(P,Q)$, where $gAv_\alpha(P,Q) = c(\alpha\tau)$, with $c(t)$ the geodesic curve on M from P to Q, satisfying $c(0) = P$ and $c(\tau) = Q$. The resulting subdivision scheme is termed geodesic subdivision scheme.

The second construction uses a smooth projection mapping onto M, and replaces every binary average by its projection onto M. The resulting nonlinear scheme is termed a projection subdivision scheme. In case of a surface in $\mathbb{R}^3$, a possible choice of the projection mapping is the orthogonal projection onto the surface.

Note that for a factorizable scheme the analoguous manifold schemes obtained from its representation in terms of the global refinement procedure by repeated averaging, depend on the order of the linear factors corresponding to binary averages in (1.24). Yet for the B-spline schemes there is one geodesic analoguous scheme, and one projection analogous scheme obtained from this representation, since all the factors in (1.18), except the factor $(1+z)$, are identical.

Example. In this example the linear scheme is the Chaikin algorithm of (1.8)

$$P_{2j}^{k+1} = Av_{\frac{1}{4}}(P_{j-1}^k, P_j^k) \ , \quad P_{2j+1}^{k+1} = Av_{\frac{3}{4}}(P_{j-1}^k, P_j^k) \ , \tag{1.30}$$

with the symbol

$$a(z) = (1+z)^3/4 \ . \tag{1.31}$$

The different adaptations to the manifold case are:

(i) Chaikin geodesic scheme, derived from (1.30):

$$P_{2j}^{k+1} = gAv_{\frac{1}{4}}(P_{j-1}^k, P_j^k) \ , \quad P_{2j+1}^{k+1} = gAv_{\frac{3}{4}}(P_{j-1}^k, P_j^k) \ .$$

(ii) Chaikin geodesic scheme, derived from (1.31):

$$P_{2i}^{k+1,0} = P_{2i+1}^{k+1,0} = P_i^k \ , \quad P_i^{k+1,j} = gAv_{\frac{1}{2}}(P_i^{k+1,j-1}, P_{i-1}^{k+1,j-1}) \ , \ j = 1, 2 \ .$$

(iii) Chaikin projection scheme derived from (1.30):

$$P_{2j}^{k+1} = G(Av_{\frac{1}{4}}(P_{j-1}^k, P_j^k)) \ , \quad P_{2j+1}^{k+1} = G(Av_{\frac{3}{4}}(P_{j-1}^k, P_j^k)) \ .$$

(iv) Chaikin projection scheme derived from (1.31):

$$P_{2i}^{k+1,0} = P_{2i+1}^{k+1,0} = P_i^k \ , \quad P_i^{k+1,j} = G(Av_{\frac{1}{2}}(P_i^{k+1,j-1}, P_{i-1}^{k+1,j-1}) \ , \ j = 1, 2 \ .$$

In the above G is a specific smooth projection mapping to the manifold M. Figure 1.2 displays a curve on a sphere, created from a finite number of initial control points on the sphere, by a geodesic analoguous scheme to a third degree B-spline scheme.

1.3.3 Analysis of convergence and smoothness by proximity

The analysis of convergence and smoothness of the geodesic and the projection schemes we present, is based on their proximity to the linear scheme from which they are derived, and on the smoothness properties of this linear scheme. We limit the discussion to C^1 and C^2 smoothness. To formulate the proximity conditions we introduce some notation.

For a control polygon $\mathcal{P} = \{P_i\}$, we define

$$\Delta\mathcal{P} = \{P_i - P_{i-1}\}, \ \Delta^\ell\mathcal{P} = \Delta(\Delta^{\ell-1}\mathcal{P}), \ d_\ell(\mathcal{P}) = \max_i \|(\Delta^\ell\mathcal{P})_i\|. \tag{1.32}$$

The difference between two control polygons $\mathcal{P} = \{P_i\}$, $\mathcal{Q} = \{Q_i\}$, is $\mathcal{P} - \mathcal{Q} = \{P_i - Q_i\}$.

With this notation the two proximity relations of interest to us are the following:

Definition 1.1

(i) Two schemes S and T are in 0-proximity if

$$d_0(S\mathcal{P} - T\mathcal{P}) \leq C d_1(\mathcal{P})^2,$$

for all control polygons $\mathcal{P}$ with $d_1(\mathcal{P})$ small enough.

(ii) Two schemes S and T are in 1-proximity if

$$d_1(S\mathcal{P} - T\mathcal{P}) \leq C[d_1(\mathcal{P})d_2(\mathcal{P}) - d_1(\mathcal{P})^3],$$

for all control polygons $\mathcal{P}$ with $d_1(\mathcal{P})$ small enough.

In these definitions C is a generic constant.

From the 0-proximity condition we can deduce the convergence of T from the convergence of the linear scheme S. Furthermore, under mild conditions on S, we can also deduce that if S generates C^1 limit curves then T generates C^1 limit curves whenever it converges.

In Wallner (2006), results on C^2 smoothness of the limit curves generated by T are obtained, based on 0-proximity and 1-proximity of T and S, and some mild conditions on S, in addition to S being C^2.

The 0-proximity and the 1-proxiomity conditions hold for the manifold schemes of subsection 1.3.2, when S is a convergent linear subdivision scheme and when M is a smooth manifold. Moreover, for M a compact manifold or a surface with bounded normal curvatures, the two proximity conditions hold uniformly for all $\mathcal{P}$ such that $d_1(\mathcal{P}) < \delta$, with a global δ.

Examples. The B-spline schemes with symbol (1.18) for $m \geq 3$, generate C^2 curves and satisfy the mild conditions necessary for deducing that the limit curves of their manifold analoguous schemes are also C^2. On the otherhand the linear 4-point scheme generates limit curves which are only C^1, a property that is shared by its manifold analoguous schemes.

Further analysis of manifold-valued subdivision schemes can be found in a series of papers by Wallner and his collaborators, see e.g. Wallner, Nava Yazdani & Grohs (2007), Grohs & Wallner (2008), and in the works Xie & Yu (2007), Xie & Yu (2008), Xie & Yu (2008a).

1.4 Geometric 4-point interpolatory subdivision schemes

The refinement step of a linear scheme $(S_{\mathbf{a}}\mathcal{P})_j = \sum_i a_{j-2i}P_i\,,\ j \in \mathbb{Z}\,,$ is applied separately to each component of the control points. Therefore these schemes are insensitive to the geometry of the control polygons.

For control polygons with edges of similar length, this insensitivity is not problematic. Yet, the limit curves generated by linear schemes, in case the initial control polygon has edges of significantly different length, have artifacts, namely geometric features which do not exist in the initial control polygon. This can be seen in the upper right figure of Figure 1.3 and in the second column of Figure 1.4. Data dependent schemes can cure this problem.

Here we present two geometric versions of the 4-point scheme, which are data dependent. The first is based on adapting the tension parameter to the geometry of the 4-points involved in the insertion rule, the second is based on a geometric parametrization of the control polygon at each refinement level.

1.4.1 Adaptive tension parameter

In this section we present a nonlinear version of the linear 4-point interpolatory scheme, introduced in Marinov, Dyn & Levin (2005), which adapts the tension parameter to the geometry of the control points.

It is well known that the linear 4-point scheme with the refinement rules

$$P_{2j}^{k+1} = P_j^k, \quad P_{2j+1}^{k+1} = -w(P_{j-1}^k + P_{j+2}^k) + (\frac{1}{2} + w)(P_j^k + P_{j+1}^k) \,, \quad (1.33)$$

where w is a fixed tension parameter, has the following attributes:

- It generates "good" curves when applied to control polygons with edges of comparable length.
- It generates curves which become smoother (have greater Hölder exponent of the first derivative), the closer the tension parameter is to 1/16.
- Only for very small values of the tension parameter, it generates a curve which preserve the shape of an initial control polygon with edges of significantly different length. (Recall that the control polygon itself corresponds to the generated curve with zero tension parameter.)

We first write the refinement rules in (1.33) in terms of the edges $\{e_j^k = P_{j+1}^k - P_j^k\}$ of the control polygon, and relate the inserted point P_{2j+1}^{k+1} to the edge e_j^k. The insertion rule can be written in the form,

$$P_{e_j^k} = M_{e_j^k} + w_{e_j^k}(e_{j-1}^k - e_{j+1}^k) \quad (1.34)$$

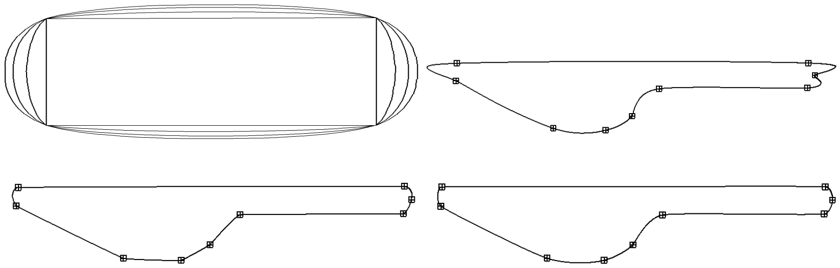

Fig. 1.3. Curves generated by the linear 4-point scheme: (Upper left) the effect of different tension parameters, (upper right) artifacts in the curve generated with $w = \frac{1}{16}$, (lower left) artifact-free but visually non-smooth curve generated with $w = 0.01$. Artifact-free and visually smooth curve generated in a nonlinear way with adaptive tension parameters (lower right).

with $M_{e_j^k}$ the midpoint of e_j^k and $w_{e_j^k}$ the adaptive tension parameter. Defining $d_{e_j^k} = w_{e_j^k}(e_{j-1}^k - e_{j+1}^k)$ as the displacement from $M_{e_j^k}$, we control its size by choosing $w_{e_j^k}$ according to a geometrical criterion.

In Marinov, Dyn & Levin (2005) there are various geometrical criteria, all of them guaranteeing that the inserted control point $P_{e_j^k}$ is different from the boundary points of the edge e_j^k, and that the length of each of the two edges replacing e_j^k is bounded by the length of e_j^k. This is achieved if $w_{e_j^k}$ is chosen so that

$$\|d_{e_j^k}\| \leq \frac{1}{2}\|e_j^k\| \ . \tag{1.35}$$

All the criteria restrict the value of the tension parameter $w_{e_j^k}$ to the interval $(0, \frac{1}{16}]$, such that a tension close to $1/16$ is assigned to *regular stencils* namely, stencils of four points with three edges of almost equal length, while the *less regular* the stencil is, the closer to zero is the tension parameter assigned to it.

A *natural* choice of an adaptive tension parameter obeying (1.35) is

$$w_{e_j^k} = \min\left\{\frac{1}{16}, c\,\frac{\|e_j^k\|}{\|e_{j-1}^k - e_{j+1}^k\|}\right\}\ , \quad \text{with a fixed } c \in [\frac{1}{8}, \frac{1}{2})\ . \tag{1.36}$$

In (1.36) c is restricted to the interval $[\frac{1}{8}, \frac{1}{2})$ to guarantee that $w_{e_j^k} = \frac{1}{16}$ for stencils with $\|e_{j-1}^k\| = \|e_j^k\| = \|e_{j+1}^k\|$. Indeed in this case, $\|e_{j-1}^k - e_{j+1}^k\| = 2\sin\frac{\theta}{2}\|e_j^k\|$, with θ, $0 \leq \theta \leq \pi$, the angle between the two vectors e_{j-1}^k, e_{j+1}^k. Thus $\|e_j^k\|/\|e_{j-1}^k - e_{j+1}^k\| = (2\sin\frac{\theta}{2})^{-1} \geq \frac{1}{2}$, and if $c \geq \frac{1}{8}$ then the minimum in (1.36) is $\frac{1}{16}$. The choice (1.36) defines irregular stencils (corresponding to small $w_{e_j^k}$) as those with $\|e_j^k\|$ much smaller than at least one of $\|e_{j-1}^k\|, \|e_{j+1}^k\|$, and such that when these two edges are of comparable length, the angle between them is not close to zero.

The convergence of this geometric 4-point scheme, and the continuity of the limits generated, follow from a result in Levin (1999). There it is proved that the 4-point scheme with variable tension parameter is convergent, and that the limits generated are continuous, whenever the tension parameters are restricted to the interval $[0, \tilde{w}]$, with $\tilde{w} < \frac{1}{8}$.

But we cannot apply the result in Levin (1999), on C^1 limits of the 4-point scheme with variable tension parameter to the geometric 4-point scheme defined by (1.34) and (1.36), since the tension parameters used during this subdivision process, are not bounded away from zero.

Nevertheless, many simulations indicate that the curves generated by this scheme are C^1 (see Marinov, Dyn & Levin (2005)).

1.4.2 Geometric parametrization of the control polygons

In this subsection we present a geometric 4-point scheme, which is introduced and investigated in Dyn, Floater & Hormann (2008). The idea for the geometric insertion rule of the point P_{2i+1}^k, comes from the insertion rule of the DD_2 scheme (see §1.2.1).

The insertion rule of the DD_2 scheme is obtained by sampling the vector cubic polynomial, interpolating the data $\{((i+j), P_{i+j}^k),\ j = -1, 0, 1, 2\}$, at the point $i + \frac{1}{2}$. From this point of view, the linear scheme corresponds to a uniform parametrization of the control polygon at each refinement level. This approach fails when the initial control polygon has edges of significantly different length. Yet the use of the centripetal parametrization, instead of the uniform parametrization, leads to a geometric 4-point scheme with artifact-free limit curves, as can be seen in Figure 1.4.

The centripetal parametrization, which is known to be effective for interpolation of control points by a cubic spline curve (see Floater (2008)), has the form $\mathbf{t_{cen}}(\mathcal{P}) = \{t_i\}$, with

$$t_0 = 0, \quad t_i = t_{i-1} + \|P_i - P_{i-1}\|_2^{\frac{1}{2}}, \tag{1.37}$$

where $\|\cdot\|_2$ is the Euclidean norm, and $\mathcal{P} = \{P_i\}$.

Let $\mathcal{P}^k$ be the control polygon at refinement level k, and let $\{t_i^k\} =$

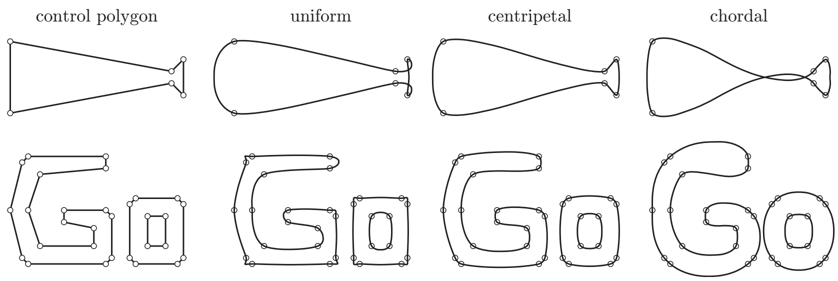

Fig. 1.4. Comparisons between 4-point schemes based on different parametrizations.

$\mathbf{t_{cen}}(\mathcal{P}^k)$. The refinement rules for the geometric 4-point scheme, based on the centripetal parametrization are:

$$P_{2i}^{k+1} = P_i^k, \quad P_{2i+1}^{k+1} = \pi_{k,i}\Big(\frac{1}{2}(t_i^k + t_{i+1}^k)\Big),$$

with $\pi_{k,i}$ the vector of cubic polynomials, satisfying the interpolation conditions

$$\pi_{k,i}(t_{i+j}^k) = P_{i+j}^k, \quad j = -1, 0, 1, 2.$$

Note that this construction can be done with any parametrization. In fact in Dyn, Floater & Hormann (2008) the chordal parametrization ($t_{i+1} - t_i = \|P_{i+1} - P_i\|_2$) is also investigated, but found to be inferior to the centripetal parametrization (see Figure 1.4).

In contrast to the analysis of the schemes on manifolds, the method of analysis of the geometric 4-point schemes, based on the chordal and centripetal parametrizations is rather ad-hoc. It is shown in Dyn, Floater & Hormann (2008) that both schemes are well defined, in the sense that any inserted point is different from the end points of the edge to which it corresponds, and that both schemes are convergent to continuous limit curves. Although numerical simulations indicate that both schemes generate C^1 curves, as the linear 4-point scheme, there is no proof of such a property there.

Another type of information on the limit curves, which is relevant to the absence or presence of artifacts, is available in Dyn, Floater & Hormann (2008). Bounds on the Hausdorff distance from sections of a limit curve to their corresponding edges in the initial control polygon are derived. These bounds give a partial qualitative understanding why the limit curves corresponding to the centripetal parametrization are artifact free.

Let C denote a curve generated by the scheme based on the centripetal parametrization from an initial control polygon $\mathcal{P}^0$. Since the scheme is interpolatory, C passes through the initial control points. Denote by $C|_{e_i^0}$ the section of C starting at P_i^0 and ending at P_{i+1}^0. Then

$$haus(C|_{e_i^0}, e_i^0) \le \frac{5}{7}\|e_i^0\|_2.$$

Thus the section of the curve cannot be too far from a short edge. On the other hand the corresponding bound in the linear case has the form

$$haus(C|_{e_i^0}, e_i^0) \le \frac{3}{13}\max\{\|e_j^0\|_2 \ , \quad |j-i| \le 2\},$$

and a section of the curve can be rather far from its corresponding short edge, if this edge has a long neighboring edge. In case of the chordal parametrization the bound is even worse

$$haus(C|_{e_i^0}, e_i^0) \le \frac{11}{5}\max\{\|e_j^0\|_2 \ , \quad |j-i| \le 2\}.$$

Comparisons of the performance of the three 4-point schemes, discussed in this section, are given in Figure 1.4.

1.5 Geometric refinement of curves

The scheme discussed in this section is designed and invesigated in Dyn, Elber & Itai (2008). It is a nonlinear extension of the quadratic B-spline scheme $S_{\mathbf{a}}$, corresponding to the symbol given by (1.18) with $m = 2$. $S_{\mathbf{a}}$ is a linear scheme generating a curve from a set of initial control points, yet its extension presented here generates surfaces, as the refined objects are not control points but control curves. This new nonlinear scheme repeatedly refines a set of control curves, taking into account the geometry of the curves, so as to generate a limit surface, which is related to the geometry of the initial control curves.

Infact, a surface can be generated from an initial set of curves $\{C_i\}_{i=0}^n$, using $S_{\mathbf{a}}$ in a linear way. The initial curves have to be parametrized in some reasonable way to yield the set $\{C_i(s), \ s \in [0,1]\}_{i=0}^n$, and then $S_{\mathbf{a}}$ is applied to each control polygon of the form $\mathcal{P}_s = \{C_i(s)\}_{i=0}^n$ corresponding to a fixed s in $[0,1]$. This is equivalent to refining the curves with $S_{\mathbf{a}}$, to obtain the refined set of control curves $S_{\mathbf{a}}\mathcal{P}_s, \ s \in [0,1]$. The limit surface is obtained by repeated refinements of the control curves, and has the form

$$(S_{\mathbf{a}}^{\infty}\mathcal{P}_s)(t), \quad s \in [0,1]\,, \ t \in \mathbb{R}$$

The quality of the generated surface depends on the quality of the parametrization of the initial curves, as a reasonable prametrization for each set of refined curves at each refinement level:

$$\{S^k_{\mathbf{a}}\mathcal{P}_s\,,\ s \in [0,1]\} \quad k = 0,1,2,\dots$$

In the nonlinear scheme the control curves at each refinement level are parametrized, taking into account their geometry, and $S_{\mathbf{a}}$ is applied to all control polygons generated by points on the curves corresponding to the same parameter value.

The parametrization of the curves at refinement level k, $\{C^k_i\}_{i=0}^{n_k}$ is in terms of a vector of *correspondences* $\tau = (\tau_0, \dots, \tau_{n_k-1})$ between pairs of consecutive curves. A correspondence between two curves is a one-to-one and onto, continuous map from the points of one curve to the points of the other. Thus τ determines a parametrization of the curves in terms of the points of C^k_0, in the sense that all the points $P_0, \dots, P_{n_k}$, with $P_0 \in C^k_0$ and $P_{i+1} = \tau_i(P_i) \in C_{i+1}$, correspond to the same parameter value.

The convergence of the nonlinear scheme is proved for initial curves contained in a compact set in $\mathbb{R}^3$. This condition is also satisfied by all control curves generated by the nonlinear scheme, due to the refinement rules of the curves. The correspondence used is a geometrical correspondence τ^* defined by,

$$\tau^*_i = \arg \min_{\tau \in T^k(C^k_i, C^k_{i+1})} \max\{\|\tau(P) - P\|_2\,, \quad P \in C^k_i\},$$

where $T^k(C^k_i, C^k_{i+1})$ is a set of allowed correspondences. This set depends on all the curves at refinement level k in a rather mild way. We ommit here the technical details. It is shown in Dyn, Elber, & Itai (2008) that if the initial curves are *admissible for subdivision*, namely the sets $T^0(C^0_i, C^0_{i+1})$ for $i = 0, \dots, n_0 - 1$ are nonempty, then the sets $T^k(C^k_i, C^k_{i+1})$ for $i = 0, \dots, n_k - 1$ are also nonempty for all $k > 0$.

With the above notation, the refinement step at refinement level k can be written as:

- for $i = 0, \dots, n_k - 1$,
  - (i) compute τ^*_i
  - (ii) for each $P \in C^k_i$, define

$$Q_i(P) = \frac{3}{4}P + \frac{1}{4}\tau^*_i(P), \quad R_i(P) = \frac{1}{4}P + \frac{3}{4}\tau^*_i(P)$$

(iii) define two refined curves

$$C_{2i}^{k+1} = \{Q_i(P), \quad P \in C_i\}, \quad C_{2i+1}^{k+1} = \{R_i(P), \quad P \in C_i\}$$

- $n_{k+1} = 2n_k - 1$

For the convergence proof we need an analoguous notion to the control polygon in the case of schemes refining control points. This notion is the control piecewise-ruled-surface, defined at refinement level k by

$$PR^k = \cup_{i=0}^{n_k-1}\{P = \lambda P_i + (1-\lambda)\tau_i^*(P_i), \quad P_i \in C_i^k, \lambda \in [0,1]\}$$

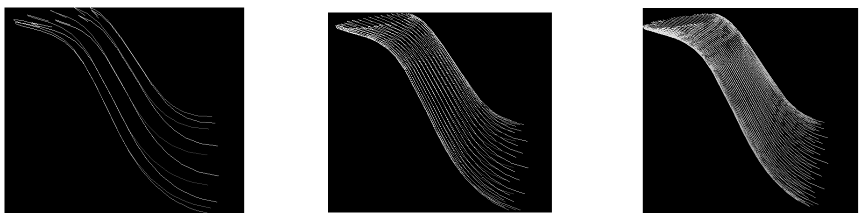

Fig. 1.5. Geometric refinement of curves. From left to right: initial curves, the refined curves after two and three refinement steps.

It is shown in Dyn, Elber & Itai (2008) that if the initial curves are simple, nonintersecting, and admissible for subdivision, then the sequence of control piecewise-ruled-surfaces $\{PR^k\}_{k\geq 0}$ is well defined, and converges in the Hausdorff metric to a set in $\mathbb{R}^3$. Also it is shown there, that if the initial curves are sampled densly enough from a smooth surface, then the limit of the scheme approximates the surface.

From the computational point of view, the refinements are executed only a small number of times (at most 5), so the "limit" is represented by the surface PR^k with $3 \leq k \leq 5$. All computations are done discretely. The curves are sampled at a finite number of points, and τ^* is computed by dynamical programming.

An example demonstrating the performance of this scheme on an initial set of curves, sampled from a smooth surface, is given in Figure 1.5.

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