Approximations of Set-Valued Functions by Metric Linear Operators

Nira Dyn, Elza Farkhi, Alona Mokhov
School of Mathematical Sciences
Tel-Aviv University, Israel

Abstract. In this work, we introduce new approximation operators for univariate set-valued functions with general compact images. We adapt linear approximation methods for real-valued functions by replacing linear combinations of numbers with new metric linear combinations of finite sequences of compact sets, thus obtaining "metric analogues" operators for set-valued functions. The new metric linear combination extends the binary metric average of Artstein. Approximation estimates for the metric analogue operators are derived. As examples we study metric Bernstein operators, metric Shoenberg operators and metric polynomial interpolants.

1 Introduction

In this work, we adapt approximating operators for real-valued functions to set-valued functions (multifunctions, SVFs), by replacing an operation between numbers with an operation between sets. The known approximation methods, based on Minkowski sums of sets, fail to approximate, when the images of a multifunction are not convex. In case of Bernstein-type operators and subdivision operators there is a phenomenon of "convexification" ([9, 6]).

In [1] a binary operation between sets, the "metric average", is introduced and the metric piecewise linear interpolant based on it is shown to approximate continuous SVFs with general images. The use of this operation in the adaptation of known approximation methods to SVFs, requires a representation of the approximation operators by repeated binary averages. Such a representation exists for any operator which reproduces constants, but is not unique [10]. This non-uniqueness leads to different operators and it is not clear what are the appropriate adaptations. Spline subdivision schemes represented by repeated averages [5] and the Schoenberg operators defined in terms of the de Boor algorithm [7] are proved to approximate SVFs with general compact images. Yet, for the adaptation of the Bernstein operators based on the de Casteljau algorithm we could obtain an approximation result only for SFVs with images in R all consisting of the same number of disjoint intervals [7].

In this paper we introduce a set-operation on a finite sequence of compact sets, termed "metric linear combination", which extends the metric average. We adapt approximation methods for real-valued functions to SVFs, replacing linear combinations of numbers by the metric linear combinations of sets. We prove that this adaptation of any linear operator, approximating continuous real-valued functions, approximates continuous SVFs of bounded variation. In particular for Lipschitz continuous SVFs, sharper error estimates are obtained. Approximation results for set-valued functions which are only continuous, are given for a limited class of operators. It should be noted that our adaptation method is not restricted to positive operators. The approximation results are specialized to the Shoenberg spline operators and the Bernstein polynomial operators. Also the adaptation of polynomial interpolation to SVFs is presented as examples of non-positive operators. This adaptation is illustrated by two metric parabolic interpolants.

An outline of the paper is as follows. The next section contains basic definitions, notation and known results. In Section 3 we introduce the metric linear combination between a finite number of ordered sets and define metric linear operators for multifunctions based on it. In Section 4 properties of the metric piecewise linear intorpolant are considered. In particular a representation of it by a specific set of selections is studied. Similar selections are used in [8] and [2] to prove the existence of a continuous selection for a continuous SVF of bounded variation, and of a representation of a Lipschitz SVF respectively. In Section 5 we derive approximation results for the metric linear approximation operators, based on the results in Section 4. Finally, in Section 6, we specialize these results to some classical approximation operators.

2 Preliminaries

First we present some definitions and notation.

- $K(R^n)$ is the collection of all compact nonempty subsets of R^n.
- A linear Minkowski combination of two sets A and B from $K(R^n)$ is

$$\lambda A + \mu B = \{\lambda a + \mu b : a \in A, b \in B\},$$

with $\lambda, \mu \in R$.

- The Euclidean distance from a point $a \in R^n$ to a set $B \in K(R^n)$ is defined as

$$\mathrm{dist}(a, B) = \inf_{b \in B} |a - b|,$$

where $|\cdot|$ is the Euclidean norm in R^n.

- The Hausdorff distance between two sets $A, B \in K(R^n)$ is defined by

$$\mathrm{haus}(A, B) = \max\left\{\sup_{a \in A} \mathrm{dist}(a, B), \sup_{b \in B} \mathrm{dist}(b, A)\right\}.$$

- The set of all projections of $a \in R^n$ into a set $B \in K(R^n)$ is

$$\Pi_B(a) = \{b \in B : |a - b| = \mathrm{dist}(a, B)\}.$$

• For $A, B \in K(R^n)$ and $0 \le t \le 1$, the t-weighted metric average of A and B is [1]

$$A \oplus_t B = \{ta + (1-t)b : \ (a,b) \in \Pi(A,B)\} \qquad (1)$$

with $\Pi(A,B) = \{(a,b) \in A \times B : \ a \in \Pi_A(b) \ \text{or} \ b \in \Pi_B(a)\}$.
The metric average has the metric property [1]

$$\begin{aligned} \text{haus}(A \oplus_t B, A \oplus_s B) &= |t-s| \, \text{haus}(A,B), \\ \text{haus}(A \oplus_t B, A) &= (1-t) \, \text{haus}(A,B), \\ \text{haus}(A \oplus_t B, B) &= t \, \text{haus}(A,B). \end{aligned} \qquad (2)$$

• The modulus of continuity of $f : [a,b] \to X$ with images in a metric space (X, ρ) is

$$\omega_{[a,b]}(f, \delta) = \sup\{\, \rho(f(x), f(y)) : \ |x-y| \le \delta, \ x, y \in [a,b]\,\}, \ \ \delta > 0. \qquad (3)$$

In this paper X is either R^n or $K(R^n)$, and ρ is either the Euclidean distance or the Hausdorff distance respectively.
The property of the modulus that we use is

$$\omega_{[a,b]}(f, \lambda\delta) \le (1+\lambda)\, \omega_{[a,b]}(f, \delta). \qquad (4)$$

• By $Lip\,([a,b], \mathcal{L})$ we denote the set of all Lipschitz functions $f : [a,b] \to X$ satisfying

$$\rho(f(x), f(y)) \le \mathcal{L}|x-y|, \quad \forall\, x, y \in [a,b],$$

where $\mathcal{L}$ is a constant independent of x and y.

• A variation of $f : [a,b] \to X$ on a partition $\chi = \{x_0 < ... < x_N : \ x_i \in [a,b], \ i = 0, ..., N\}$ is defined by

$$V(f, \chi) = \sum_{i=1}^{N} \rho(f(x_i), f(x_{i-1})),$$

The total variation of f on $[a,b]$ is

$$V_a^b(f) = \sup_{\chi} V(f, \chi).$$

We say that f is of bounded variation if $V_a^b(f) < \infty$ and define in this case

$$v_f(x) = V_a^x(f), \ x \in [a,b]. \qquad (5)$$

It is obvious that v_f is nondecreasing. If f is also continuous then v_f is continuous as well. For completeness we prove it.

Proposition 2.1. *A function $f : [a,b] \to X$ is continuous and of bounded variation on $[a,b]$ if and only if v_f is a continuous function on [a,b].*

Proof. The sufficiency follows from

$$\rho(f(x), f(y)) \leq V_x^y(f) = v_f(y) - v_f(x), \quad \text{for } x < y. \tag{6}$$

To prove the other direction, fix $x \in [a, b]$ and $\varepsilon > 0$. By the uniform continuity of f on $[a, b]$, $\rho(f(z), f(y)) < \varepsilon/2$ if $|z - y| < \delta$ for some $\delta > 0$. First we show that v_f is continuous from the left. We can always choose $\chi = \{a = x_0 < x_1 < ... < x_N = x\}$ such that

$$V_a^x(f) < V(f, \chi) + \varepsilon/2 = \sum_{i=1}^{N} \rho(f(x_i), f(x_{i-1})) + \varepsilon/2,$$

and $x - x_{N-1} < \delta$. Thus

$$V_a^x(f) < \sum_{i=1}^{N-1} \rho(f(x_i), f(x_{i-1})) + \varepsilon,$$

implying that $v_f(x) - v_f(x_{N-1}) < \varepsilon$. By the monotonicity of v_f we get for every $x_{N-1} < y < x$

$$v_f(x) - v_f(y) < \varepsilon.$$

Similarly one can show the continuity of v_f from the right. Thus we obtain that v_f is continuous at x and consequently it is continuous on $[a, b]$. □

From (6) we conclude that

$$\omega_{[a,b]}(f, \delta) \leq \omega_{[a,b]}(v_f, \delta). \tag{7}$$

• By CBV we denote the set of all functions $f : [a, b] \to X$ which are continuous and of bounded variation.

• For a set-valued function $F : [a, b] \to K(R^n)$, any single-valued function $f : [a, b] \to R^n$ with $f(x) \in F(x)$, $\forall x \in [a, b]$ is called a selection of F.

Definition 2.2. *A set of selections of F, $\{f^\alpha : \alpha \in \mathcal{A}\}$, is termed a **representation** of F if*

$$F(x) = \{f^\alpha(x) : \alpha \in \mathcal{A}\}, \quad \forall\, x \in [a, b].$$

We denote this shortly by $F = \{f^\alpha : \alpha \in \mathcal{A}\}$.

3 Linear operators on SVFs based on a metric linear combination of ordered sets

In this section we introduce a new operation on a finite number of ordered sets. Using this operation we present a new adaptation of linear operators to multifunctions.

Definition 3.1. *Let* $\{A_0, A_1, ..., A_N\}$ *be a finite sequence of compact sets. A vector* $(a_0, a_1, ..., a_N)$ *with* $a_i \in A_i$ *,* $i = 0, ..., N$ *, for which there exists* j *,* $0 \le j \le N$ *such that*

$$a_{i-1} \in \Pi_{A_{i-1}}(a_i),\ 1 \le i \le j \ \ and\ \ a_{i+1} \in \Pi_{A_{i+1}}(a_i),\ j \le i \le N-1$$

is called a ***metric chain*** *of* $\{A_0, ..., A_N\}$ *.*

An illustration of such a metric chain is given in Figure 3.2.

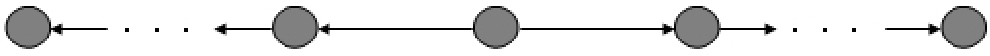

Figure 3.2.

Thus each element of each set A_i , $i = 0, ..., N$ generates at least one metric chain. We denote by $CH(A_0, ..., A_N)$ the collection of all metric chains of $\{A_0, ..., A_N\}$. The set $CH(A_0, ..., A_N)$ depends on the order of the sets A_i , $i = 0, ..., N$.

With this notion of metric chains we can introduce a new operation between sets.

Definition 3.3. *A* ***metric linear combination*** *of a sequence of sets* $A_0, ..., A_N$ *with coefficients* $\lambda_0, ..., \lambda_N \in R$ *, is*

$$\bigoplus_{i=0}^{N} \lambda_i A_i = \left\{ \sum_{i=0}^{N} \lambda_i a_i \ : (a_0, ..., a_N) \in CH(A_0, ..., A_N) \right\}. \tag{8}$$

Since for two sets $CH(A, B) = \Pi(A, B)$, in the special case $N = 1$ and $\lambda_0, \lambda_1 \in [0, 1]$, $\lambda_0 + \lambda_1 = 1$, the metric linear combination is the metric average. The following are two important properties of the metric linear combination which can be easily seen from the definition.

(i) $\bigoplus_{i=0}^{N} \lambda_i A_i = \bigoplus_{i=0}^{N} \lambda_{N-i} A_{N-i}$,

(ii) For $\lambda_0, ..., \lambda_N$ such that $\sum_{i=0}^{N} \lambda_i = 1,\ \bigoplus_{i=0}^{N} \lambda_i A = A$.

With this operation, a large class of linear operators can be adapted to SVFs.
Let A_χ , $\chi = \{x_0, ..., x_N\}$ be a linear operator of the form

$$A_\chi(f, x) = \sum_{i=0}^{N} c_i(x) f(x_i), \tag{9}$$

defined on real-valued functions, with domain containing χ .

Definition 3.4. *Let* $F : [a, b] \to K(R^n)$ *,* $\chi \subset [a, b]$ *and let* $\{F(x_i), i = 0, ..., N\}$ *be samples of* F *at* χ *. For* A_χ *of the form (9), we define a* ***metric linear operator*** A_χ^M *on* F *by*

$$A_\chi^M F(x) = A_\chi^M (F, x) = \bigoplus_{i=0}^{N} c_i(x) F(x_i). \tag{10}$$

We term this operator the ***metric analogue*** *of (9).*

Note that due to property (ii), the metric analogue of a linear operator which preserves constants, preserves constant multifunctions. The analogue of property (ii) does not hold for Minkowski linear combinations with some negative coefficients, even for convex sets. This is one reason why only positive operators, based on Minkowski sum, were applied to set-valued functions. As is shown in the sequel, Definition 3.4 allows to define also non-positive operators.

The analysis of the approximation properties of $A_\chi^M F$ is based on properties of the metric piecewise linear approximation operator.

4 Metric piecewise linear approximations of SVFs

From now on $F : [a, b] \to K(R^n)$, $\{ F_i = F(x_i) \}_{i=0}^N$, where $a = x_0 < ... < x_N = b$ and $\chi = (x_0, ..., x_N)$ denotes a partition of $[a, b]$. We use the notation $CH = CH(F_0, ..., F_N)$, and $\delta_{max} = \max \{ \delta_i : \ 0 \le i \le N-1 \}$, $\delta_{min} = \min \{ \delta_i : \ 0 \le i \le N-1 \}$ with values δ_i defined as $\delta_i = (x_{i+1} - x_i)$, $i = 0, ..., N-1$. In case of a uniform partition, we have $\delta_{max} = \delta_{min} = h = (b-a)/N$ and denote such a partition by χ_N.

Definition 4.1. *The* ***metric piecewise linear*** *approximation to* F *is*

$$S_\chi^M(F, x) = \{ \lambda_i(x) f_i + (1 - \lambda_i(x)) f_{i+1} : \ (f_0, ..., f_N) \in CH \}, \quad x \in [x_i, x_{i+1}],$$

where

$$\lambda_i(x) = (x_{i+1} - x)/(x_{i+1} - x_i). \tag{11}$$

By construction, the set-valued function $S_\chi^M F$ has a representation by selections

$$S_\chi^M F = \{ s(\chi, \varphi) : \ \varphi \in CH(F_0, ..., F_N) \}, \tag{12}$$

where $s(\chi, \varphi)$ is a piecewise linear single-valued function interpolating the data (x_i, f_i), $i = 0, ..., N,$ with $\varphi = (f_0, ..., f_N)$.

Recall the piecewise linear interpolant based on the metric average, introduced in [1]:

$$S_\chi^{MA}(F, x) = F_i \oplus_{\lambda_i(x)} F_{i+1}, \quad x \in [x_i, x_{i+1}]$$

with $\lambda_i(x)$ defined by (11).

It is easy to see by the triangle inequality for the Hausdorff metric and by (2) that for a continuous set-valued function F

$$\text{haus}(F(x), S_\chi^{MA}(F, x)) \le 2\, \omega_{[a,b]}(F, \delta_{max}), \ x \in [a, b]. \tag{13}$$

Remark 4.2. *It is not unexpected that* $S_\chi^{MA} F \equiv S_\chi^M F$.

Indeed, for a fixed $x \in [x_i, x_{i+1}]$ *and for any* $y \in S_\chi^{MA}(F, x)$,

$$y = \lambda_i(x) f_i + (1 - \lambda_i(x)) f_{i+1}$$

with $(f_i, f_{i+1}) \in \Pi(F_i, F_{i+1})$. *Then there exists a metric chain* $\varphi = (f_0, ..., f_i, f_{i+1}, ..., f_N)$, $\varphi \in CH$, *such that* $y = s(\chi, \varphi)(x)$. *Also it is obvious that for any* $x \in [a, b]$ *and any* $\varphi \in CH$, $s(\chi, \varphi)(x) \in S_\chi^{MA}(F, x)$.

In the following we show that $S^M_\chi F$, and its piecewise linear selections (12) "inherit" some continuity properties of a continuous multifunction F. The following lemma and corollary consider Lipschitz continuous SVFs.

Lemma 4.3. *Let $F \in Lip([a,b],\mathcal{L})$, and let χ be a partition of $[a,b]$. Then*

$$S^M_\chi F \in Lip([a,b],\mathcal{L}).$$

Proof. For $x,y \in [x_j,x_{j+1}]$ the claim of the lemma follows from the metric property (2). Now, let $x \in [x_j,x_{j+1}]$ and $y \in [x_k,x_{k+1}]$, where $0 \le j \le k \le N-1$. Using the triangle inequality, (2) and the Lipschitz continuity of F, we get

$$\begin{aligned}
&\mathrm{haus}(S^M_\chi(F,x), S^M_\chi(F,y)) \\
&\le \frac{x_{j+1}-x}{x_{j+1}-x_j}\,\mathrm{haus}(F_j,F_{j+1}) + \mathrm{haus}(F_{j+1},F_k) + \frac{y-x_k}{x_{k+1}-x_k}\,\mathrm{haus}(F_k,F_{k+1}) \\
&\le \mathcal{L}(x_{j+1}-x+x_k-x_{j+1}+y-x_k) \le \mathcal{L}|y-x|.
\end{aligned}$$

□

Corollary 4.4. *Under the conditions of Lemma 4.3, for any $s(\chi,\varphi)$ in 12*

$$s(\chi,\varphi) \in Lip([a,b],\mathcal{L}).$$

The proof of this corollary is similar to the proof of the previous lemma and uses the observation that for $k \ge j$

$$\begin{aligned}
&|s(\chi,\varphi)(x_{j+1}) - s(\chi,\varphi)(x_k)| \le \sum_{l=j+1}^{k-1} |s(\chi,\varphi)(x_l) - s(\chi,\varphi)(x_{l+1})| \\
&\le \sum_{l=j+1}^{k-1} \mathrm{haus}(S^M_\chi(F,x_l), S^M_\chi(F,x_{l+1})) \le \mathcal{L}\sum_{l=j+1}^{k-1}(x_{l+1}-x_l) = \mathcal{L}|x_k-x_{j+1}|.
\end{aligned}$$

Now we consider the case when F is a general continuous function.

Lemma 4.5. *Let $F:[a,b]\to K(R^n)$ be a continuous set-valued function. Then for any partition χ of $[a,b]$*

$$\omega_{[a,b]}(S^M_\chi F,\delta) \le 5\,\omega_{[a,b]}(F,\delta).$$

Proof. By definition, for any $\delta > 0$

$$\omega_{[a,b]}(S^M_\chi F,\delta) = \sup\{\,\mathrm{haus}(S^M_\chi(F,x), S^M_\chi(F,y)) :\ |x-y| \le \delta,\ x,y \in [a,b]\,\}.$$

In case $x,y \in [x_j,x_{j+1}]$, $|x-y| \le \delta$, the claim of the lemma is obtained using (11), the metric property (2) and (4),

$$\begin{aligned}
\mathrm{haus}(S^M_\chi(F,x), S^M_\chi(F,y)) &= \frac{|x-y|}{\delta_j}\,\mathrm{haus}(F_j,F_{j+1}) \le \frac{\delta}{\delta_j}\omega_{[a,b]}(F,\delta_j) \\
&\le \frac{\delta}{\delta_j}\left(1+\frac{\delta_j}{\delta}\right)\omega_{[a,b]}(F,\delta) \le 2\,\omega_{[a,b]}(F,\delta).
\end{aligned} \tag{14}$$

Now, let $x \in [x_j, x_{j+1}]$, $y \in [x_k, x_{k+1}]$, $0 \le j < k \le N-1$ and $|x-y| \le \delta$. By the triangle inequality

$$\begin{aligned} \text{haus}(S_\chi^M(F,x), S_\chi^M(F,y)) \le\ & \text{haus}(S_\chi^M(F,x), S_\chi^M(F,x_{j+1})) \\ & + \text{haus}(S_\chi^M(F,x_{j+1}), S_\chi^M(F,x_k)) \\ & + \text{haus}(S_\chi^M(F,x_k), S_\chi^M(F,y)), \end{aligned} \tag{15}$$

while by the interpolation property of $S_\chi^M F$ and since $|x_k - x_{j+1}| \le \delta$, we have

$$\text{haus}(S_\chi^M(F,x_{j+1}), S_\chi^M(F,x_k)) \le \omega_{[a,b]}(F,\delta). \tag{16}$$

Applying (14) and (16) to (15) we obtain the claim of the lemma. □

Corollary 4.6. *For any* $s(\chi,\varphi)$ *in (12) and any* $x, y \in [x_j, x_{j+1}]$, $0 \le j \le N-1$

$$|s(\chi,\varphi)(x) - s(\chi,\varphi)(y)| \le 2\,\omega_{[a,b]}(F, |x-y|) \tag{17}$$

Also,

$$\omega_{[a,b]}(s(\chi,\varphi),\delta) \le 4\,\omega_{[a,b]}(F,\delta), \quad \delta \le \delta_{min}. \tag{18}$$

The proof of this corollary is similar to the proof of assertion (14).

We cannot generalize (18) for arbitrary $0 < \delta \le b-a$ if F is only continuous. Yet we can get an estimate for $\omega_{[a,b]}(s(\chi,\varphi),\delta)$ if F is continuous and of bounded variation.

Lemma 4.7. *Let* $F \in CBV([a,b])$. *Then for any* $s(\chi,\varphi)$ *in (12),*

$$\omega_{[a,b]}(s(\chi,\varphi),\delta) \le 4\,\omega_{[a,b]}(F,\delta) + \omega_{[a,b]}(v_F,\delta) \le 5\,\omega_{[a,b]}(v_F,\delta).$$

Proof. Denote $s = s(\chi,\varphi)$. For a given $\delta > 0$, let $x \in [x_j, x_{j+1}]$, $y \in [x_k, x_{k+1}]$, $0 \le j \le k \le N-1$, such that $|x-y| \le \delta$. Using the definition of $s(\chi,\varphi)$ and of $S_\chi^M F$ we get

$$\begin{aligned} |s(x)-s(y)| &\le |s(x)-s(x_{j+1})| + \sum_{l=j+1}^{k-1} |s(x_{l+1}) - s(x_l)| + |s(y)-s(x_k)| \\ &\le \frac{x_{j+1}-x}{\delta_j}|s(x_{j+1})-s(x_j)| + \sum_{l=j+1}^{k-1} \text{haus}(F(x_{l+1}),F(x_l)) + \frac{y-x_k}{\delta_k}|s(x_{k+1})-s(x_k)|. \end{aligned}$$

Now, (17) yields

$$|s(x)-s(y)| \le 4\,\omega_{[a,b]}(F,\delta) + V_{x_{j+1}}^{x_k}(F) \le 4\,\omega_{[a,b]}(F,\delta) + \omega_{[a,b]}(v_F,\delta).$$

Taking the supremum over $|x-y| \le \delta$ and using (7), we complete the proof. □

5 Approximation by metric linear operators

We use the metric piecewise linear approximation to obtain error estimates for metric linear operators.

Let $A^M_\chi F$ be defined by (10) and $S^M_\chi F$ be a metric piecewise linear multifunction as defined in Section 4. By Definition 3.4

$$A^M_\chi F \equiv A^M_\chi (S^M_\chi F). \tag{19}$$

Moreover by (9), (10) and (12)

$$A^M_\chi (S^M_\chi F) = \{A_\chi s(\chi, \varphi) : \varphi \in CH(F_0, ..., F_N)\}. \tag{20}$$

Remark 5.1. *In contrast to our previous definition of positive operators for SVFs based on the metric average [5, 7], the metric analogues (10) of two linear operators of the form (9), which are identical on single-valued functions, are identical on SVFs. For example, in [5, 7] spline subdivision schemes are not identical to the Schoenberg spline operators for SVFs.*

The metric analogues of linear operators of the form (9), which approximate real-valued functions, are approximating SVFs. By (19), (20) the approximation results depend on the way A_χ approximates piecewise linear real-valued functions.

In what follows $\phi : [a, b] \times R_+ \to R_+$ is a continuous real-valued function, non-decreasing in its second argument, satisfying $\phi(x, 0) = 0$, and $\mathcal{S}_\chi$ denotes the set of piecewise linear continuous single-valued functions, with values in R^n and knots at χ.

Theorem 5.2. *Let A_χ be of the form (9), such that for any $s \in \mathcal{S}_\chi \bigcap Lip\,([a, b], \mathcal{L})$*

$$|A_\chi(s, x) - s(x)| \le C\,\mathcal{L}\phi(x, \delta_{max}). \tag{21}$$

Then if $F \in Lip\,([a, b], \mathcal{L})$,

$$\text{haus}(A^M_\chi (F, x), F(x)) = 2\,\mathcal{L}\delta_{max} + C\mathcal{L}\phi(x, \delta_{max}).$$

Proof. By (19)

$$\text{haus}(A^M_\chi (F, x), F(x)) \le \text{haus}(A^M_\chi (S^M_\chi F, x), S^M_\chi (F, x)) + \text{haus}(S^M_\chi (F, x), F(x)), \tag{22}$$

while by (20)

$$\text{haus}(A^M_\chi (S^M_\chi F, x), S^M_\chi (F, x)) \le \sup_{\varphi \in CH} |A_\chi(s(\chi, \varphi), x) - s(\chi, \varphi)(x)|.$$

In view of Corollary 4.4 and (21)

$$\sup_{\varphi \in CH} |A_\chi(s(\chi, \varphi), x) - s(\chi, \varphi)(x)| \le C\,\mathcal{L}\phi(x, \delta_{max}) \tag{23}$$

The proof is completed by substituting (23) and (13) in (22). □

For general continuous SVFs we cannot prove an analogous approximation result. Yet for continuous multifunctions of bounded variation we get a weaker approximation result, by applying Lemma 4.7 instead of Corollary 4.4 in the proof of Theorem 5.2.

Theorem 5.3. *Let* $F \in CBV([a,b])$ *, and let* A_χ *be of the form (9), satisfying*

$$|A_\chi(s,x) - s(x)| \le C\,\omega_{[a,b]}(s, \phi(x, \delta_{max})), \quad s \in \mathcal{S}_\chi\,. \tag{24}$$

Then

$$\text{haus}(A_\chi^M(F,x), F(x)) = 2\,\omega_{[a,b]}(F, \delta_{max}) + 5C\omega_{[a,b]}(v_F, \phi(x, \delta_{max})).$$

For continuous SVFs which are not of bounded variation we can prove an approximation result only for uniform partitions and for a limited class of linear operators.

Theorem 5.4. *Let* A_N *be a linear operator of the form (9), defined on a uniform partition* χ_N *and let* $h = (b-a)/N$ *. If*

$$|A_N(s,x) - s(x)| \le C\,\phi(x, \omega_{[a,b]}(s,h)), \quad s \in \mathcal{S}_\chi\,, \tag{25}$$

then for a continuous F

$$\text{haus}(A_N^M(F,x), F(x)) = 2\,\omega_{[a,b]}(F,h) + C\phi(x, 4\,\omega_{[a,b]}(F,h)).$$

The proof of this result repeats the proof of Theorem 5.2, but replaces Corollary 4.4 by (18) of Corollary 4.6.

6 Examples

In this section we present metric analogues for SVFs of the Schoenberg spline operators and the Bernstein polynomial operators and give approximation results. We conclude by two examples demonstrating the operation of metric analogues of parabolic interpolants. To our knowledge so far only positive operators were applied to SVFs. The two examples we present assert that such interpolation between sets is reasonable.

6.1 Metric Bernstein operators

The Benstein operator $B_N(f,x)$ for $f \in C[0,1]$ is

$$B_N(f,x) = \sum_{i=0}^{N} \binom{N}{i} x^i (1-x)^{N-i} f\left(\frac{i}{N}\right). \tag{26}$$

It is known (see [4], Chapter 10) that there exists a constant C independent of f such that

$$|f(x) - B_N(f,x)| \le C\omega_{[0,1]}(f, \sqrt{x(1-x)/N}). \tag{27}$$

The classical Bernstein operator for $F : [0,1] \to K(R^n)$ with sums of numbers replaced by Minkowski sums of sets is

$$B_N^{Mn}(F,x) = \sum_{i=0}^{N} \binom{N}{i} x^i (1-x)^{N-i} F\left(\frac{i}{N}\right). \tag{28}$$

It was shown in [9] that for $x \in (0,1)$ the limit of $B_N^{Mn}(F,x)$ when $N \to \infty$, is the convex hull of $F(x)$, therefore these operators cannot approximate SVFs with general images.

In [7] Bernstein operators for set-valued functions are defined procedurally in terms of the de Casteljau algorithm, with the metric average as a basic binary operation,

$$\begin{gathered} F_i^0 = F(i/N), \quad i = 0, ..., N, \\ F_i^k = F_i^{k-1} \oplus_{1-x} F_{i+1}^{k-1}, \quad i = 0, 1, ..., N-k, \ k = 1, ..., N, \\ B_N^{MA}(F,x) = F_0^N. \end{gathered} \tag{29}$$

We do not know whether these operators approximate multifunctions with general compact images in R^n, yet they approximate multifunctions with compact images in R all consisting of the same number of disjoint intervals [7].

Here we investigate the metric analogue of the Bernstein operators for SVFs.

Definition 6.1. *For $F : [0,1] \to K(R^n)$ the metric Bernstein operator is*

$$\begin{aligned} B_N^M(F,x) &= \bigoplus_{i=0}^{N} \binom{N}{i} x^i (1-x)^{N-i} F\left(\frac{i}{N}\right) \\ &= \left\{ \sum_{i=0}^{N} \binom{N}{i} x^i (1-x)^{N-i} f_i \ : \ (f_0, ..., f_N) \in CH \right\}, \end{aligned}$$

where $CH = CH(F(0), F(1/N) ..., F(1))$.

By Theorem 5.2 and by (27) we conclude that

Corollary 6.2. *Let $F \in Lip([0,1], \mathcal{L})$, then*

$$\text{haus}(B_N^M(F,x), F(x)) \leq 2\mathcal{L}/N + C\mathcal{L}\sqrt{x(1-x)/N}.$$

Moreover by Theorem 5.3 and by (27)

Corollary 6.3. *Let $F \in CBV([0,1])$, then*

$$\text{haus}(B_N^M(F,x), F(x)) \leq 2\,\omega_{[0,1]}(F, 1/N) + 5C\omega_{[0,1]}(v_F, \sqrt{x(1-x)/N}).$$

Since (25) does not hold for these operators, Theorem 5.4 cannot be applied.

6.2 Metric Schoenberg operators

For a uniform partition χ_N , the "classical" set-valued analogues of the Schoenberg spline operators for $F:[0,1]\to K(R^n)$ is

$$S^{Mn}_{m,N}(F,x)=\sum_{i=0}^{N}F(i/N)b_m\left(Nx-i\right), \tag{30}$$

where $b_m\left(x\right)$ is the B-spline of order m (degree $m-1$) with integer knots and support $[0,m]$, and where the linear combination is in the Minkowski sense. An example, given in [9], shows that the operators in (30) with $m=2$ and $N\to\infty$ cannot approximate F with general compact images, in any point of $[0,1]\setminus\chi_N$.

A Shoenberg operator based on the metric average is introduced in [7], by a procedural definition in terms of repeated binary averages according to the de Boor algorithm. It is proved that for Hölder continuous set-valued functions, the approximation rate is the Hölder exponent.

Here we consider the metric analogue of the Schoenberg operators.

Definition 6.4. *The metric Shoenberg operator of order* m *for a set-valued function* $F:[0,1]\to K(R^n)$ *and a uniform partition* χ_N *is defined by*

$$S^{M}_{m,N}(F,x)=\bigoplus_{i=0}^{N}b_m\left(Nx-i\right)F\left(\frac{i}{N}\right)=\left\{\sum_{i=0}^{N}b_m\left(Nx-i\right)f_i:\ (f_0,...,f_N)\in CH\right\},$$

where $CH=CH(F(0),F(1/N)...,F(1))$.

By Theorem 5.4 and the known approximation result in case of single-valued functions (see [3], Chapter XII), we obtain

Corollary 6.5. *Let* F *be a continuous SFV defined on* $[0,1]$ *. Then*

$$\mathrm{haus}(S^{M}_{m,N}(F,x),F(x))=2\left(1+2\left\lfloor\frac{m+1}{2}\right\rfloor\right)\omega_{[0,1]}(F,1/N),\quad x\in\left[\frac{m-1}{N},1\right]$$

with $\lfloor t\rfloor$ *the maximal integer not greater than* t *.*

The approximation result in the specific case of Lipschitz continuous SVFs, can be further improved by applying Theorem 5.2.

Corollary 6.6. *For* $F\in Lip\,([0,1],\mathcal{L})$ *,*

$$\mathrm{haus}(S^{M}_{m,N}(F,x),F(x))=\left(2+\left\lfloor\frac{m+1}{2}\right\rfloor\right)\frac{\mathcal{L}}{N}.$$

6.3 Metric polynomial interpolants

Definition 6.7. *Let* $F : [a, b] \to K(R^n)$ *, and let* χ *be a partition of* $[a, b]$ *. The metric polynomial interpolation operator is given by*

$$P_\chi^M(F, x) = \bigoplus_{i=0}^{N} l_i(x)F(x_i) = \left\{ \sum_{i=0}^{N} l_i(x)f_i \, : \, (f_0, ..., f_N) \in CH(F(x_0), ..., F(x_N)) \right\},$$

with $l_i(x)$ *the* i *-th Lagrange polynomial,*

$$l_i(x) = \prod_{j=0, j \neq i}^{N} \frac{x - x_j}{x_i - x_j} .$$

To illustrate our method we apply the metric parabolic interpolation operator to three sets in R. We consider two different examples.
The first example: $x_0 = 0$, $x_1 = 2$, $x_2 = 6$;

$$F(x_0) = [2, 8], \qquad F(x_1) = \{5\}, \qquad F(x_2) = \{5\}.$$

The second example: $x_0 = 0$, $x_1 = 4$, $x_2 = 8$;

$$F(x_0) = [2, 4] \cup [6, 8], \qquad F(x_1) = [4.5, 5.5], \qquad F(x_2) = [2, 4] \cup [6, 8].$$

The two set-valued interpolants are illustrated in Figure 6.8 and Figure 6.9 respectively.

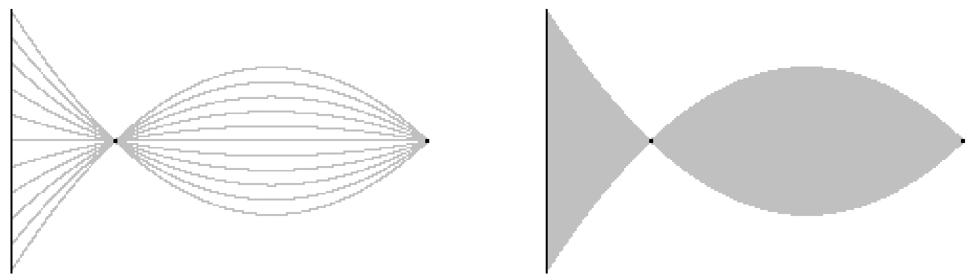

Figure 6.8. Parabolic interpolation - first example.

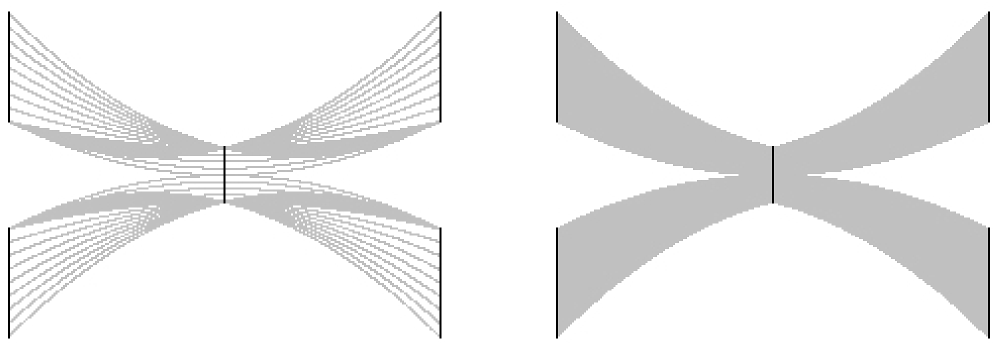

Figure 6.9. Parabolic interpolation - second example.

In the above figures the sets in black are $F(x_0)$, $F(x_1)$, $F(x_2)$ and the gray curves are the parabolic interpolants to the selections in (12).

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