Approximation by Translates of a Radial Function

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1 Introduction

The study of approximation schemes based on translates of a radial function to a set of points (centers) in $\mathbb{R}^d$, is a central subject in multivariate approximation theory. Here we study the L_∞-approximation orders of such schemes, first for centers constituting a regular grid, and then for quasi-uniformly scattered centers.

1.1 Radial functions and multivariate interpolation

The interest in radial functions approximation was initiated by the applications. Radial functions provide a convenient and simple tool for global interpolation of scattered multivariate data. Given a univariate function $g(t) : \mathbb{R}_+ \to \mathbb{R}$, and a non-negative integer m, the interpolation problem to the scattered data

(1.1) $$(x^i, f_i), \quad x^i \in \mathbb{R}^d, \quad f_i \in \mathbb{R}, \quad i = 1, \cdots, N \ ,$$

based on the radial function $\phi(x) = g(\|x\|)$, consists of finding a function of the form

(1.2) $$S(x) = \sum_{i=1}^{N} v_i \phi(x - x^i) + p_m(x) \ ,$$
$$p_m \in \pi_m, \quad \sum_{i=1}^{N} v_i q(x^i) = 0, \quad q \in \pi_m \ ,$$

satisfying

(1.3) $$S(x^i) = f_i, \quad i = 1, \cdots, N \ .$$

Here π_m is the space of all algebraic polynomials of degree at most m on $I\!R^d$, and $\|\cdot\|$ is the Euclidean norm on $I\!R^d$. This method of interpolation reproduces polynomials in π_m, whenever (1.3) is uniquely solvable. Classes of functions $g(t)$ which are well known in the literature include:

(i) "Surface splines"

$$g(t) = \begin{cases} t^{2k-d}\log t, & d \text{ even} , \\ t^{2k-d}, & d \text{ odd} , \end{cases} \tag{1.4}$$

with k an integer satisfying $2k > d$ and with $m = k-1$, studied in a series of papers (see e.g. [?],[?],[?]). The corresponding interpolant (1.2) minimizes the functional

$$\mathcal{R}_k(f) = \int_{I\!R^d} \sum_{|\alpha|=k} (D^\alpha f)^2 dx \ , \tag{1.5}$$

among all functions interpolating the data in the space

$$\chi_k = \{f \in C(I\!R^d), \quad D^\alpha f \in L_2(I\!R^d). \quad |\alpha| = k\} \ . \tag{1.6}$$

Here and hereafter we use the notations

$$\alpha = (\alpha_1, \cdots, \alpha_d) \in Z\!\!Z^d_+, \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}, \quad |\alpha| = \sum_{i=1}^d \alpha_i \ .$$

The functional $\mathcal{R}_k(f)$ is rotation invariant, and this choice reflects the assumption that there are no preferable directions in the scattered data (1.1). The functions (1.4) are fundamental solutions of the k'th iterated Laplacian,

$$\Delta^k \phi = c\delta \ . \tag{1.7}$$

For $d = 1$ the surface spline coincides with the natural spline of order $2k$.

The variational formulation yields unique solvability of (1.2), (1.3) whenever the set $X = \{x^1, \cdots, x^N\}$ satisfies the geometric condition

$$\dim \pi_m\Big|_X = \dim \pi_m \ . \tag{1.8}$$

In [?] the class of surface splines is extended by considering functionals of the type (1.5) corresponding to derivatives of fractional orders,

$$(1.9)\quad \mathcal{R}_{s,k}(f) = \int_{I\!R^d} \|\omega\|^{2s} \sum_{|\alpha|=k} (\widehat{D^\alpha f})^2(\omega) d\omega, \quad 0 < s < 1 \ , \quad |\alpha| = k \ ,$$

where $\widehat{f}$ denotes the Fourier transform of f. For $2k + 2s > d$, the solution to the variational problem determined by (1.9) and the scattered data (1.1), is given by the solution to (1.2), (1.3) with $m = k - 1$ and

$$(1.10)\qquad g(t) = \begin{cases} t^{2k+2s-d} \log t, & 2k + 2s - d \text{ even} \ , \\ t^{2k-2s-d}, & \text{otherwise} \ . \end{cases}$$

(ii) "Multiquadrics"

$$(1.11)\qquad g(t) = (t^2 + c^2)^\beta, \quad \beta = \pm\frac{1}{2}, \quad c > 0, \quad d = 2, \quad m = -1 \ ,$$

introduced in [?] (see also [?]), for geophysical applications. The value $m = -1$ in (1.11) corresponds to π_m being the empty set in (1.2).

(iii) "Shifted surface splines"

$$(1.12)\qquad g(t) = \begin{cases} (t^2 + c^2)^{(2k-d)/2} \log(t^2 + c^2)^{\frac{1}{2}}, & 2k \geq d, \quad d \text{ even} \ , \\ (t^2 + c^2)^{(2k-d)/2}, & \text{otherwise}, \ k \geq 1 \ , \end{cases}$$

with $c > 0$ and $m = k-1$, introduced in [?] for $d = 2$. These functions are the "shifted" version of the fundamental solutions of the iterated Laplacian of order $k \geq 1$. With the choice $c > 0$, $g(0)$ is well defined also for $1 \leq k \leq d/2$, in contrast to the case $c = 0$ of the surface splines.

In a comparative study [?], the quality of interpolation of scattered data in $I\!R^2$ by radial functions of classes (i), (ii) is found to be superior to other methods of interpolation.

The classes of radial functions for which the interpolation problem (1.2), (1.3) is uniquely solvable under condition (1.8) in all $I\!R^d$, is studied in [?],[?]

and [?]. This class is characterized by the strict complete monotonicity of order $m+1$ of $g(\sqrt{t})$, namely

$$(1.13)\ \frac{d^m}{dt^m} g(\sqrt{t}) \not\equiv \text{const}, \quad \varepsilon(-1)^j \frac{d^j}{dt^j} g(\sqrt{t}) \geq 0, \quad t>0, \quad j \geq m+1$$

with $\varepsilon = 1$ or $\varepsilon = -1$.

This allows to extend further the class of radial functions of interest to

$$(1.14) \qquad g(t) = \begin{cases} t^\gamma, & \gamma \in I\!R_+ \backslash 2Z\!\!Z_+ \ , \\ t^\gamma \log t, & \gamma \in 2Z\!\!Z_+ \ , \end{cases}$$

in any $I\!R^d$ independent of the parity of d, with $m > \gamma/2 - 1$ in (1.2). The corresponding "shifted" version of the class (1.14) is even wider

$$(1.15) \qquad g(t) = \begin{cases} (t^2+c^2)^{\gamma/2}, & \gamma > -d, \qquad \gamma \notin 2Z\!\!Z_+ \ , \\ (t^2+c^2)^\gamma \log(t^2+c^2)^{1/2}, & \qquad \gamma \in 2Z\!\!Z_+ \ , \end{cases}$$

with $c>0$ and where the corresponding m satisfies $m > \gamma/2 - 1$ for $\gamma \geq 0$, and $m = -1$ otherwise.

For review papers on various aspects of the theory of radial functions see e.g. [?], [?], [?] and [?].

1.2 The distributional Fourier transform of ϕ

It is the properties of the distributional Fourier transform of $\phi(x) = g(\|x\|)$, considered as a tempered distribution [?], which are relevant to the solvability of the interpolation problem as well as to the theory of approximation orders of schemes based on translates of such radial functions. While for the solvability of the interpolation problem many more radial functions can be considered, such as the Gaussian function $g(t) = e^{-t^2/a}$, $a>0$, and the radial functions of compact support introduced by Wu [?], the analysis in this chapter is confined to radial functions of the class (1.14), (1.15) and to related functions with similar properties of their distributional Fourier transform, such as fundamental solutions of homogeneous elliptic operators, which are not necessarily radial.

The distributional Fourier transform of $\phi = g(\|\cdot\|)$, with g as in (1.14), (1.15), coincides away from the origin with a function $\widehat{\phi}$ of the form [**?**]

$$\widehat{\phi}(\omega) = a_{\gamma,c}\|\omega\|^{-\gamma-d}F_{\gamma,c}(\omega) \ , \tag{1.16}$$

where $a_{\gamma,c}$ is a positive constant which depends on γ and c, and where

$$F_{\gamma,c}(\omega) = \begin{cases} 1 \ , & c = 0 \ , \\ \\ \widetilde{K}_{(d+\gamma)/2}(c\|\omega\|) \ , & c > 0 \ . \end{cases} \tag{1.17}$$

Here $\widetilde{K}_\nu(t) = t^\nu K_\nu(t)$, with K_ν the modified Bessel function. Relevant properties of $\widetilde{K}_\nu$ to our analysis are [**?**]

$$\begin{aligned} &\widetilde{K}_\nu \in C(I\!R), \quad \widetilde{K}_\nu(t) > 0, \quad t \geq 0 \ , \quad \nu > 0, \\ &\lim_{t\to\infty} \widetilde{K}_\nu(t) = 0 \text{ exponentially}, \quad \widetilde{K}_n \in C^{2n-1}(I\!R) \cap C^\infty(I\!R\backslash 0) \ , \\ &\widetilde{K}_n^{(2n)}(t) = 0(\log t), \quad t \to 0^+, \quad \widetilde{K}_{n+\frac{1}{2}} \in C^\infty(I\!R) \ , \quad n \in Z\!\!\!Z_+. \end{aligned} \tag{1.18}$$

A fundamental solution of homogeneous elliptic operator $G(D)$ of order $2m$, has a generalized Fourier transform which coincides on $I\!R^d\backslash 0$ with $1/G(\omega)$ up to a multiplicative constant.

The important features of $\widehat{\phi}$ are the order m' of its singularity at the origin, namely as $\|\omega\| \to 0$, and the rate of its decay as $\|\omega\| \to \infty$. For the fundamental solutions of homogeneous elliptic operators and for the class (1.14), the order of the singularity at the origine of $\widehat{\phi}$ equals the rate of its decay at infinity. For the class of radial functions (1.15) the decay of $\widehat{\phi}$ at infinity is exponential. For all these functions the distributional Fourier transform at the origin is a distribution of order less than m'.

1.3 Outline of the chapter

All the sections in this chapter are concerned with estimating the error measured in the L_∞-norm, incurred by approximation schemes based on translates of a radial function and its scales.

Sections 2 and 3 study two different types of approximation schemes, based on shifts of $\phi(h^{-1}\cdot)$ to the points of $h\mathbb{Z}^d$. The first type, analyzed in Section 2, consists of quasi-interpolation schemes of the from

$$Q_{\psi,h}f = \sum_{\alpha\in\mathbb{Z}^d} f(h\alpha)\psi(h^{-1}\cdot-\alpha)\ , \tag{1.19}$$

where ψ is a finite linear combimation of shifts of ϕ to points of $\mathbb{Z}^d$ near the origin. The approximation order is the power of h in the error $\|f-Q_{\psi,h}f\|_\infty$ for all f in an addmissible set of functions W. Section 2 presents the results in [**?**], which apply to a wide class of functions ϕ. Other results in this direction for specific radial functions are presented in [**?**] and [**?**]. In [**?**] approximation orders by qausi-interpolation schemes based on the Gaussian radial function are derived.

In Section 3 the approximation scheme is an optimal one which achieves the optimal aprroximation orders possible. This scheme introduced in [**?**], uses global information on the approximated function and has the form

$$L_h f = \sum_{\alpha\in\mathbb{Z}^d} \Lambda_h f(h\alpha)\psi(h^{-1}\cdot-\alpha)\ . \tag{1.20}$$

Here $\psi = \sum_{\alpha\in\mathbb{Z}^d}\mu_\alpha\phi(\cdot-\alpha)$, with $\{\mu_\alpha :\ \alpha\in\mathbb{Z}^d\}$ not necessarily of finite support, and

$$\widehat{\Lambda_h f} = \widehat{\lambda}(h\cdot)\widehat{f}, \quad \widehat{\lambda} = \eta/\widehat{\psi}\ , \tag{1.21}$$

with η any smooth function of compact support which is 1 on a ball centered at the origin. Approximation orders of optimal schemes in the $L_2(\mathbb{R}^d)$ setting are studied in [**?**].

Section 4 extends the results of the previous sections to analog approximation schemes based on translates of ϕ to quasi-uniformly scattered centers. For the quasi-interpolation schemes, the required information on the approximated function is confined to the same set of centers. Section 4 is mainly based on [?], with some results taken from [?], where the notion of quasi-uniformly scattered centers is first introduced. Analog results to those in [?] for approximation in the L_p–norms are derived in [?].

All the results presented in this chapter deal with approximation orders defined by scaling the function ψ (and therefore ϕ) by h^{-1}, and then translating it to a set of centers with distances of order h between neighbouring centers. There are other notions of approximation orders, corresponding to different types of scaling of the function ϕ (see e.g. [?] [?]).

Another important type of approximation order is based on traslates of ϕ, without any scaling, to sets of centers with increasing density. For "homogeneous" radial functions this approach yields the same orders as scaling by h^{-1}, while for the others, qualitatively different results are obtained (see e.g. [?], [?], [?]).

2 Quasi-interpolation on regular grids

In this section, we analyze the approximation order of schemes based on function values on a regular grid $h\mathbb{Z}^d$ and on the $h\mathbb{Z}^d$ translates of a scaled basis function ψ, consisting of a finite linear combination of multi-integer translates of a radial function or a related function ϕ. The schemes we study are quasi-interpolatory of the form

$$Q_{\psi,h}f = \sum_{\alpha\in\mathbb{Z}^d} f(\alpha h)\psi(h^{-1}\cdot - \alpha) \ .$$

The analysis employs two important ingredients of the scheme: the decay of $\psi(x)$ as $\|x\| \to \infty$ and polynomial reproduction. In case ψ is of compact support or decays fast enough the argument for getting the approximation order is standard and we present it here. We denote by Q_ψ the operator $Q_{\psi,1}$ and by A any constant appearing in the various bounds.

Theorem 2.1 *Assume that*

$$|\psi(x)| \leq A(1+\|x\|)^{-(d+k)}, \quad k > \ell + 1 \ , \tag{2.1}$$

and that

$$Q_\psi p = p, \quad p \in \pi_\ell \ . \tag{2.2}$$

Then for every f with bounded derivatives of order $\ell+1$

$$\|Q_{\psi,h} f - f\|_\infty \leq A\|f\|_{\infty,\ell+1} h^{\ell+1} \ , \tag{2.3}$$

where $\|f\|_{\infty,\ell+1} = \sum_{|\alpha|=\ell+1} \|D^\alpha f\|_\infty < \infty$.

Proof: First we note that (2.2) also holds for $Q_{\psi,h}$. This is easily seen for the basis of powers of π_ℓ, and hence holds for all π_ℓ.

Let $T_x f$ be the Taylor polynomial of f of degree ℓ at x, namely $D^\alpha(f - T_x f)(x) = 0$, $|\alpha| \leq \ell$. Then for $g = f - T_x f$ we get by (2.2) that

$$Q_{\psi,h} f - f = Q_{\psi,h} g - g \ , \tag{2.4}$$

while by the definition of g,

$$|g(z)| \leq A\|f\|_{\infty,\ell+1} \|z - x\|^{\ell+1} \ . \tag{2.5}$$

Thus

$$\begin{aligned} |(Q_{\psi,h} f - f)(x)| &= |Q_{\psi,h} g(x)| = \sum_{\alpha \in \mathbb{Z}^d} |\psi(h^{-1}x - \alpha)| \ |g(\alpha h)| \\ &\leq A\|f\|_{\infty,\ell+1} \sum_{\alpha \in \mathbb{Z}^d} \|\alpha h - x\|^{\ell+1} |\psi(h^{-1}x - \alpha)| \\ &\leq A h^{\ell+1} \|f\|_{\infty,\ell+1} \sum_{\alpha \in \mathbb{Z}^d} \|h^{-1}x - \alpha\|^{\ell+1} |\psi(h^{-1}x - \alpha)| \\ &\leq A h^{\ell+1} \|f\|_{\infty,\ell+1} \ , \end{aligned} \tag{2.6}$$

where in the last inequality we used (2.1) to bound the sum above by a constant independent of h and x. $\square$

For many of the radial functions and related functions that we consider in this paper Theorem 2.1 does not yield the optimal approximation orders. In fact we can show (2.1) for $k = \ell + 1$ at most, where ℓ is the maximal value possible in (2.2). A finer analysis is needed then to obtain approximation orders $O(h^{\ell+1} \log |h|)$ or sometimes even $O(h^{\ell+1})$.

2.1 The general setting

The first step in the presentation of the approximation scheme is the construction of ψ satisfying (2.1) with $k = \ell + 1$. This rate of decay is sufficient for Q_ψ to be well defined on π_ℓ. The next stage is to show that (2.2) holds. Finally the sum $\sum_{\alpha \in \mathbb{Z}^d} |\psi(h^{-1}x - \alpha)| \; |g(\alpha h)|$, has to be estimated.

The class of approximating spaces under investigation is of the form

$$S_h(\phi) = \operatorname{span}\{\phi(h^{-1}x - \alpha) : \alpha \in \mathbb{Z}^d\} \tag{2.7}$$

with the span standing for the closure of the algebraic span under the topology of uniform convergence on compact sets. Here ϕ is a function which grows at most as a power of $\|x\|$ as $\|x\| \to \infty$, and whose distributional Fourier transforms $\widehat{\phi}$ satisfies the equation

$$G\widehat{\phi} = F \ . \tag{2.8}$$

The distributional Fourier transform of ϕ as a tempered distribution is defined by the equality

$$\int_{\mathbb{R}^d} \phi(\omega) s(\omega) d\omega = \int_{\mathbb{R}^d} \widehat{\phi}(\omega) \widehat{s}(\omega) d\omega \ , \quad s \in S \ ,$$

where S is the space of all C^∞ rapidly decaying test functions[**?**].

There are several assumptions on F and G typical to the class of functions ϕ under investigation:

$$
(2.9)\qquad
\begin{array}{ll}
\text{(a)} & G(\omega) \neq 0 \text{ if } \omega \neq 0, \\
\text{(b)} & G(\omega) \text{ is a homogeneous polynomial of degree } 2m, \\
\text{(c)} & F(0) \neq 0,\ F(x) - \sum_{|\alpha| \le m_0} \frac{D^\alpha F(0)}{\alpha!} x^\alpha \in \mathcal{F}_{m_0+\theta} \text{ for some } \theta > 0, \\
\text{(d)} & F \in C^\infty(\mathbb{R}^d \backslash 0), \\
\text{(e)} & |D^\alpha (F/G)(\omega)| \le \frac{A_\alpha}{\|\omega\|^{d+\alpha+\varepsilon}} \text{ for } \|\omega\| \ge 1,\ \varepsilon > 0,\ \alpha \in \mathbb{Z}_+^d,
\end{array}
$$

where in (c) we use the notation

$$
(2.10)\qquad \mathcal{F}_r = \left\{ f \in C^\infty(\mathbb{R}^d \backslash 0) \ :\ D^\alpha f(x) = 0(\|x\|^{r-|\alpha|}) \text{ as} \|x\| \to 0,\ \alpha \in \mathbb{Z}_+^d \right\} .
$$

In case of fundamental solutions of homogeneous elliptic operators, $F \equiv 1$.

Condition (a) in (2.9) guarantees that $\widehat{\phi} = F/G$ as functions on $\mathbb{R}^d \backslash 0$. The behavior of $\widehat{\phi}$ at the origin is defined in a distributional sense by

$$
(2.11)\qquad \widehat{\phi}[s] = \int_{\mathbb{R}^d} s(\omega) \frac{F(\omega)}{G(\omega)} d\omega \ , \quad s \in S_{2m-1} \ ,
$$

where

$$
(2.12)\qquad S_{2m-1} = \{ s \in S \ :\ D^\alpha s(0) = 0 \ ,\ |\alpha| \le 2m-1 \} \ .
$$

2.2 The construction of ψ

By (2.9) the behavior of the singularity of $\widehat{\phi}$ near the origin is as the reciprocal of a polynomial of degree $2m$, hence by taking a finite linear combination of shifts of ϕ

$$
(2.13)\qquad \psi = \sum_{\alpha \in I} \mu_\alpha \phi(\cdot - \alpha) \ , \quad I \subset \mathbb{Z}^d \ ,
$$

one can get $\widehat{\psi}$ to be defined as an ordinary Fourier transform of ψ, which is well defined on $I\!R^d$ and is of the form

$$(2.14) \qquad \widehat{\psi} = \widehat{\phi} e, \quad e(\omega) = \sum_{\alpha \in I} \mu_\alpha e^{-i\alpha\cdot\omega} \ .$$

The coefficients $\{\mu_\alpha : \alpha \in I\}$ are so chosen to satisfy

$$(2.15) \qquad D^\alpha (e - G/F)(0) = 0, \quad |\alpha| \leq 2m + \ell \ ,$$

for some $\ell \in [0, m_0] \cap Z\!\!\!Z$. Thus $D^\alpha e(0) = 0$ for $\|\alpha\| \leq 2m-1$, and the zero of e at the origin cancels the singularity of $\widehat{\phi}$ there. Note that as ℓ increases in (2.15) the set I that supports the sequence $\{\mu_\alpha\}$ is bigger, since more conditions in (2.15) have to be satisfied. Conditions (2.15) together with the $(2\pi)^d$- periodicity of $e(\omega)$ and assumptions (b),(c) of (2.9), lead to

Proposition 2.2 *Let ψ be defined by (2.13)–(2.15). Then $\widehat{\psi} \in C^\ell(I\!R^d)$ and satisfies*

$$(2.16) \qquad \widehat{\psi}(0) = 1, \quad D^\alpha \widehat{\psi}(0) = 0, \quad 1 \leq |\alpha| \leq \ell \ ,$$

$$(2.17) \qquad p(-iD)\widehat{\psi}(2\pi\beta) = 0, \quad \beta \in Z\!\!\!Z^d \backslash 0, \quad p \in \mathcal{P}_G \cap \pi_{\ell+2m} \ ,$$

where P_G is the kernel of the operator $G(D)$. In particular

$$(2.18) \qquad D^\alpha \widehat{\psi}(2\pi\beta) = 0, \quad \beta \in Z\!\!\!Z^d \backslash 0, \quad 0 \leq |\alpha| \leq 2m-1 \ ,$$

since $\pi_{2m-1} \in \mathcal{P}_G$.

Proposition 2.2 is the key to both the decay of ψ and the polynomial reproduction property of Q_ψ. Using the behavior of $\widehat{\psi}$ near zero and its decay as $\|x\| \to \infty$ it is possible to estimate the decay rate of $|\psi(x)|$ as $\|x\| \to \infty$.

Theorem 2.3 *Under assumptions (2.9), (2.11), (2.13) and (2.15) with* $0 \le \ell < m_0$,

$$|\psi(x)| \le A(1 + \|x\|^{-d-\ell-1}) \text{ as } \|x\| \to \infty \ , \tag{2.19}$$

and

$$Q_\psi p = p, \quad p \in \pi_\ell \cap \mathcal{P}_G \tag{2.20}$$

with uniform convergence of $Q_\psi p$ *to* p *on compact sets.*

The polynomial reproduction property (2.20) follows from the extended version (2.16) and (2.17) of the "Strung-Fix conditions" [?], and from the uniform convergence of $Q_\psi p$ on compact sets, which allows the use of the Poisson summation formula.

Since the decay of ϕ in (2.19) is due to the choice of the sequence $\{\mu_\alpha\}$ in (2.15), such a sequence is termed hereafter "localization sequence". It defines a difference operator $\sum_{\alpha \in I} \mu_\alpha f(\cdot - \alpha)$, which vanishes on $\mathcal{P}_G \cap \pi_{2m+\ell}$.

We conclude from Theorem 2.3 that $\mathcal{P}_G \cap \pi_{m_0-1} \subset S_h(\phi)$, and that $\ell = \min(2m-1, m_0-1)$ is the maximal ℓ such that π_ℓ is reproduced by Q_ψ , and therefore by $Q_{\psi,h}$.

For two important classes of radial functions these consequences can be strengthened.

Corollary 2.4 *In case* $F = 1$, *namely* ϕ *is a fundamental solution of the elliptic operator* $G(D)$, $m_0 = \infty$ *in Theorem 2.3 and for any* $\ell \in \mathbb{Z}_+$ *there exists* ψ *such that (2.19) and (2.20) hold. Thus* $\mathcal{P}_G \subset S_h(\phi)$. *In particular* $\pi_{2m-1} \subset S_h(\phi)$, *and for* $\ell = 2m-1$, Q_ψ *reproduces* π_{2m-1}, *which is the maximal total degree polynomial space contained in* $\mathcal{P}_G$.

For later reference, we call the class of functions discussed in Corollary 2.4 class A. A second specific class of interest is that of the shifted fundamental solutions of the iterated Laplacian, referred to hereafter as class B. For this class $m_0 = 2m-1$, and near the origin $F(\omega) = H(\|\omega\|)$ has an expansion of the form

$$H(r) = \sum_{k=0}^{\infty} a_k r^{2k} + (\log r) r^{2m} \sum_{k=0}^{\infty} b_k r^{2k} \quad \text{as} \quad r \to 0^+ \ , \tag{2.21}$$

with $a_0 > 0$ and $b_0 \neq 0$.

The inverse Fourier transform of the first homogeneous $2m+1$ terms in the expansion of

$$\widehat{\psi}(\omega) = 1 + \frac{F(\omega)}{G(\omega)} \Big[e(\omega) - \frac{G(\omega)}{F(\omega)} \Big] \ ,$$

near the origin can be obtained explicitly, in view of (2.15), from which it is possible to conclude (2.19) with $\ell = m_0 = 2m-1$.

Corollary 2.5 *For ϕ in class B, condition (2.15) with $\ell = 2m-1 = m_0$ generates ψ that satisfies (2.19) and (2.20) with $\ell = 2m-1$. Thus π_{2m-1} is reproduced by Q_ψ, implying that $\pi_{2m-1} \subset S_h(\phi)$.*

An important observation about class A, which does not hold for class B, is that $Q_{\psi,h} f \in \mathrm{span}\{\phi(x-\alpha) : \alpha \in h\mathbb{Z}^d\}$. This follows since any ϕ in class A is a homogeneous function up to a polynomial of degree $\leq 2m-1$, which is cancelled in (2.13) since $\sum_{\alpha \in I} \mu_\alpha p(\cdot - \alpha) = 0$, for $p \in \pi_{2m-1}$.

2.3 The approximation orders on $\mathbb{R}^d$

The approximation order of $Q_{\psi,h}$ is now obtained from Theorem 2.3 and Corollaries 2.4, 2.5. As a first step we use Theorem 2.1 with (2.19) and (2.20).

Theorem 2.6 *Under the assumptions of Theorem 2.3 with any $0 \leq \ell \leq \min(2m, m_0 - 1)$*

$$\|f - Q_{\psi,h} f\|_\infty \leq A \|f\|_{\infty,\ell} h^\ell \ , \tag{2.22}$$

for any $f \in C^\ell(\mathbb{R}^d)$ with bounded derivatives of order ℓ. If ϕ is in class A then (2.22) holds for $\ell \le 2m$, while if ϕ is in class B, then (2.22) holds for $\ell \le 2m-1 = m_0$.

Under the same conditions as in Theorem 2.6, higher approximation orders than (2.22) for $Q_{\psi,h}$ can be achieved. This requires a finer analysis of the sum $Q_{\psi,h}g$ appearing in the proof of Theorem 2.1.

Theorem 2.7 *Let ψ satisfy (2.19) for some $\ell \ge 0$ and let*

$$Q_\psi p = p, \quad p \in \pi_\ell \ . \tag{2.23}$$

Then for $f \in C^{\ell+1}(\mathbb{R}^d)$ with bounded derivatives of orders ℓ and $\ell+1$

$$\|f - Q_{\psi,h}f\|_\infty \le A(\|f\|_{\infty,\ell} + \|f\|_{\infty,\ell+1})h^{\ell+1}|\log h| \ . \tag{2.24}$$

Proof: Let g be as in the proof of Theorem 2.1. Then by (2.23)

$$(f - Q_{\psi,h}f)(x) = Q_{\psi,h}g(x) \ . \tag{2.25}$$

To estimate the error in the approximation, we partition $Q_{\psi,h}g(x)$ into two sums:

$$Q_{\psi,h}g(x) = \sum_{\alpha\in S_{x,h}} g(\alpha h)\psi(h^{-1}x - \alpha) + \sum_{\alpha\in \mathbb{Z}^d\setminus S_{x,h}} g(\alpha h)\psi(h^{-1}x-\alpha) \ , \tag{2.26}$$

where $S_{x,h} = \mathbb{Z}^d \cap h^{-1}(x + [-1,1]^d)$.

In the first sum we use (2.5), while in the second we use a bound as (2.5) but with ℓ there replaced by $\ell - 1$, which also holds for g by the assumptions on f. Thus we get

$$\begin{aligned} |Q_{\psi,h}g(x)| &\le A\|g\|_{\infty,\ell+1}h^{\ell+1} \sum_{\alpha\in S_{x,h}} (\|h^{-1}x-\alpha\|+1)^{-d} \\ &+ A\|g\|_{\infty,\ell}h^\ell \sum_{\alpha\in\mathbb{Z}^d\setminus S_{x,h}} (\|h^{-1}x-\alpha\|+1)^{-d-1} \ . \end{aligned} \tag{2.27}$$

Applying Lemma 4.2 of [**?**], stating that the first sum is bounded by $A|\log h|$ while the second sum by Ah, we finally obtain (2.24). □

Theorem 2.7 and Corollary 2.5, applied to the radial functions of class B, yield approximation order $O(h^{2m}|\log h|)$ for the choice $\ell = m_0 = 2m - 1$. Similar approximation orders are obtained for the radial functions of class A by taking $\ell = 2m - 1$ in Theorem 2.7. Yet by Theorem 2.6 with $\ell = 2m$, which is a proper choice for functions in class A (ℓ in (2.15) can be any positive integer), one gets the better approximation order $O(h^{2m})$.

It is shown in [**?**] by quite involved analysis, that the $|\log h|$ factor in (2.24) can be removed for class A also for the choice $\ell = 2m-1$ in (2.15), but cannot be removed for class B. For the latter class there are other approximation schemes which achieve the approximation order $O(h^{2m})$, such as the cardinal interpolation scheme [**?**] or the optimal approximation scheme of [**?**]. The cardinal interpolation scheme also uses the values of f on $h\mathbb{Z}^d$, but uses the shifts of a function ψ which is an infinite linear combination of shifts of ϕ. On the other hand, the optimal approximation scheme uses global information on the approximated function f, but ψ is much simpler than in the quasi-interpolation case, with a very mild decay, independent of the approximation order (see Section 3).

2.4 The approximation orders on finite domains

The fast decay of ψ in the quasi-interpolation schemes presented here has a very important consequence: given function values on $h\mathbb{Z}^d \cap \Omega$, where Ω is an open bounded region of $\mathbb{R}^d$, it is possible to get the same approximation order that $Q_{\psi,h}$ achieves on $\mathbb{R}^d$, on any closed subdomain of Ω by the restricted scheme

$$Q_{\psi,h,\Omega} f = \sum_{\alpha \in \mathbb{Z}^d \cap h^{-1}\Omega} f(h\alpha)\psi(h^{-1}x - \alpha) \ . \tag{2.28}$$

In fact, we can get a somewhat stronger result. We denote

$$\|f\|_{\infty,\Omega} = \sup_{x\in\Omega} |f(x)|, \quad \|f\|_{\infty,\ell,\Omega} = \sum_{|\alpha|=\ell} \|D^\alpha f\|_{\infty,\Omega} \ ,$$

and

$$\Omega_\delta = \{y \in \Omega : \|y - z\|_\infty \ \le \delta \Rightarrow z \in \Omega\} \ . \tag{2.29}$$

Theorem 2.8 *Let $\Omega \subset I\!R^d$ be open and bounded, let $f \in C^{\ell+1}(\overline{\Omega})$ and let ψ and ℓ be as in Theorem 2.7. Then*

$$\|f - Q_{\psi,h,\Omega} f\|_{\infty,\Omega_{\delta(h)}} \le A(\|f\|_{\infty,\ell,\Omega} + \|f\|_{\Omega,\ell+1,\Omega}) h^{\ell+1} |\log h| \ , \tag{2.30}$$

where

$$\delta(h) \ge A|\log h|^{-1/(\ell+1)} \ . \tag{2.31}$$

Also

$$\|f - Q_{\psi,h,\Omega} f\|_{\infty,\Omega_{\delta(h)}} \le A\|f\|_{\infty,\ell,\Omega} h^\ell \tag{2.32}$$

where

$$\delta(h) \ge A h^{1/(\ell+1)} \ . \tag{2.33}$$

The idea of the proof is first to show that the main part of the approximation at $x \in \Omega_\delta$ is obtained by the local sum

$$Q_{\psi,h,\Omega,\delta} f(x) = \sum_{\{\alpha\in Z\!\!Z^d : \|h\alpha - x\|_\infty \le \delta\}} f(\alpha h)\psi(h^{-1}x - \alpha) \ , \tag{2.34}$$

and that all the other terms in (2.28) contribute to the sum a magnitude of the order $O((h/\delta)^{\ell+1})$. The second important step is to show that the local scheme (2.34) for $f \in \pi_\ell$ approximates $f(x)$ with error of the order $O((h/\delta)^{\ell+1})$.

3 Optimal approximation schemes on regular grids

The construction of the optimal approximation schemes in [?] is aimed at providing schemes which achieve the maximal possible approximation order from spaces generated by shifts of the h^{-1}-scales of a basis function to $h\mathbb{Z}^d$. The setting in [?] is quite general, and includes also other scales of the basis function. Here we present the results related to the shift invariant spaces

$$S(\phi) = \mathrm{span}\{\phi(\cdot - \alpha) : \alpha \in \mathbb{Z}^d\} \ , \tag{3.1}$$

and their scales

$$S_h(\phi) = \mathrm{span}\{\phi(h^{-1} \cdot - \alpha) : \alpha \in \mathbb{Z}^d\} \ , \tag{3.2}$$

where ϕ is a radial or a related function.

An important ingredient in the analysis is the determination of an upper bound for the approximation order from (3.2). Then an approximation scheme is termed optimal, if the approximation order provided by it is equal to the upper bound.

3.1 The general setting

The assumptions on ϕ in this section are such that all the radial functions and the related functions presented in the Introduction are included. The function ϕ grows at most as a power of $\|x\|$ as $\|x\| \to \infty$, and its distributional Fourier transform $\widehat{\phi}$ satisfies the three conditions

$$\begin{aligned} &\text{(a)} \quad \widehat{\phi} \in C(\mathbb{R}^d\backslash 0), \quad \widehat{\phi}(\omega) > 0, \quad \omega \in \mathbb{R}^d\backslash 0 \ , \\ &\text{(b)} \quad \text{for some } \delta > 0, \quad \|\widehat{\phi}(\omega)\|_\infty = 0(\|\omega\|^{-d-\delta}) \ , \quad \|\omega\| \to \infty \\ &\text{(c)} \quad \text{for some } m' \geq 0, \quad 0 < A_1 \leq \|\omega\|^{m'} |\widehat{\phi}(\omega)| \leq A_2 < \infty \ , \|\omega\| < \rho \ . \end{aligned} \tag{3.3}$$

Under these conditions there exists a continuous 2π-periodic function $\mu(\omega)$, $\mu(\omega) = \sum_{\alpha \in \mathbb{Z}^d} \mu_\alpha e^{-i\alpha\cdot\omega}$ such that the function

$$\widehat{\psi}(\omega) = \mu(\omega)\widehat{\phi}(\omega) \ , \tag{3.4}$$

is the proper Fourier transform of the $L_1(\mathbb{R}^d)$ function

$$\psi(x) = \sum_{\alpha \in \mathbb{Z}^d} \mu_\alpha \phi(x - \alpha) \ . \tag{3.5}$$

Moreover,

$$\widehat{\psi}(0) \neq 0 \quad \text{and} \quad \sum_{\alpha \in \mathbb{Z}^d} |\psi(\cdot - \alpha)| \in L_\infty(\mathbb{R}^d) \ . \tag{3.6}$$

The optimal approximation order from the scales $S_h(\psi)$ of the shift invariant space

$$S(\psi) = \text{span}\{\psi(x - \alpha) : \alpha \in \mathbb{Z}^d\} \ ,$$

is found to be independent of the localization sequence $\{\mu_\alpha\}$ in (3.5), and hence is attributed to the spaces (3.2).

The approximation orders in this setting are derived for the following classes of admissible functions, defined in terms of their distributional Fourier transforms:

Definition 3.1 *A function f of at most polynomial growth at infinity is termed k-admissible if $(1 + \|\cdot\|^k)\widehat{f}$ is a Radon measure such that*

$$\|f\|'_k = \int_{\mathbb{R}^d} (1 + \|x\|^k)|\widehat{f}(x)|dx < \infty \ . \tag{3.7}$$

The collection of these functions is denoted by $\widetilde{W}^\infty_k(\mathbb{R}^d)$.

It can be shown that any admissible f is bounded. In particular $f(x) = e^{-i\theta\cdot x}$, $\theta \in \mathbb{R}^d$, is admissible of any order k, since $\widehat{f}(\omega) = \delta_{-\theta}(\omega)$, and thus $\|f\|'_k = 1 + \|\theta\|^k$. If $\widehat{f}$ is a function then f is admissible if $(1+\|\cdot\|^k)\widehat{f} \in L_1(\mathbb{R}^d)$.

In case $k \in \mathbb{Z}_+$, f is k-admissible if and only if the distributional Fourier transforms of f and all its k'th-order derivatives are measures of finite total mass. In this case $\widetilde{W}_k^{\infty}(\mathbb{R}^d)$ is continuously embedded into the space of functions with continuous bounded derivatives of order $\leq k$, namely into $W_\infty^k(\mathbb{R}^d) \cap C^k(\mathbb{R}^d)$. In the results on approximation orders, the order of the admissibility class of f is related to the value m' in (3.3)(c).

3.2 The upper bound

The main necessary condition on the upper bound of the approximation order is obtained from the approximation of the exponential function $e^{-i\theta\cdot x}$, $\theta \in \mathbb{R}^d$ by $S_h(\psi)$, for quite a general class of functions ψ .

Theorem 3.2 *The space $S_h(\psi)$ with ψ satisfying (3.6), provides approximation order $k \geq 0$ to $f(x) = e^{-i\theta\cdot x}$, $\theta \in \mathbb{R}^d$ only if*

$$(3.8) \qquad \widehat{\psi}(h\theta+\beta) = O(h^k), \quad \beta \in 2\pi\mathbb{Z}^d\backslash 0, \quad \theta \in \mathbb{R}^d \ .$$

The upper bound for the approximation order from $S_h(\psi)$, for ψ given by (3.5), is obtained from Theorem 3.2 and assumptions (3.3).

Theorem 3.3 *The approximation order provided by the space $S_h(\psi)$ with ψ given by (3.5), is at most the order of the singularity of $\widehat{\phi}$ at the origin, as defined by m' of (3.3)(c).*

Proof: Properties (3.3) of $\widehat{\phi}$, implies that for small enough h there is a constant A dependent on β such that

$$(3.9) \qquad \left| \frac{h^{-m'}\widehat{\phi}(h\theta+\beta)}{\widehat{\phi}(h\theta)} \right| \geq A > 0 \ , \quad \beta \in 2\pi\mathbb{Z}^d \setminus 0 \ .$$

Multiplying numerator and denominator by $\mu(h\theta) = \mu(h\theta+\beta)$, for $\beta \in 2\pi\mathbb{Z}^d\backslash 0$, one obtains

$$(3.10) \qquad \left| \frac{h^{-m'}\widehat{\psi}(h\theta+\beta)}{\widehat{\psi}(h\theta)} \right| \geq A > 0 \ ,$$

and since $\widehat{\psi}(0) \neq 0$, $|\widehat{\psi}(h\theta+\beta)| \geq Ah^{m'}$ for small enough h, implying that the approximation order from $S_h(\psi)$ is at most m', independent of $\{\mu_\alpha\}$. $\square$

3.3 The optimal scheme

The scheme, which is shown later to be optimal, has the following form

$$L_h f = \sum_{\alpha \in \mathbb{Z}^d} \psi(h^{-1} \cdot -\alpha) \Lambda_h f(h\alpha) \ , \tag{3.11}$$

where

$$\widehat{\Lambda_h f} = \widehat{\lambda}(h\cdot)\widehat{f}, \quad \widehat{\lambda} = \sigma/\widehat{\psi} \ , \tag{3.12}$$

with σ any smooth function of compact support Ω, which is 1 on a ball B_ρ of radius ρ centered at the origin. The information on f required by the scheme is global, as it depends on the values of $\widehat{f}$ in the support of $\widehat{\lambda}(h\cdot)$.

The general result which yields the approximation order of the scheme L_h is

Theorem 3.4 *Assume that ψ satisfies (3.6), and that for some $k \geq 0$*

$$\sum_{\beta \in 2\pi\mathbb{Z}^d \setminus 0} \Big\| \frac{\widehat{\psi}(\cdot+\beta)}{\|\cdot\|^k \widehat{\psi}(\cdot)} \Big\|_{\infty,\Omega} < \infty \ . \tag{3.13}$$

Then for every $f \in \widetilde{W}_k^\infty(\mathbb{R}^d)$

$$\|f - L_h f\|_\infty \leq A\|f\|_k' h^k + o(h^k) \ . \tag{3.14}$$

Proof: The main idea of the proof of Theorem 3.4 is to represent the error $f - L_h f$ in terms of $\widehat{f}$, and then to decompose it into two components, namely

$$\begin{aligned} (f - L_h f)(x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} [1-\sigma(h\omega)]\widehat{f}(\omega)e^{i\omega\cdot x}d\omega + \\ &\quad + (2\pi)^{-d} \int_{\mathbb{R}^d} \sigma(h\omega) \left[1 - \frac{E_h(\omega,x)}{\widehat{\psi}(h\omega)}\right] \widehat{f}(\omega)e^{i\omega\cdot x}d\omega \ , \end{aligned} \tag{3.15}$$

with

$$E_h(\omega,x) = \sum_{\alpha\in\mathbb{Z}^d} \psi(h^{-1}x-\alpha)e^{i\omega\cdot(\alpha h - x)} \ .$$

The first integral in (3.15) vanishes for ω such that $\|\omega\| \le h^{-1}\rho$. Hence

$$(3.16) \qquad (h^{-1}\rho)^k \Big| \int_{\mathbb{R}^d} [1-\sigma(h\omega)]\widehat{f}(\omega)e^{i\omega\cdot x}d\omega\Big| \le$$

$$\le \int_{\|\omega\|\ge h^{-1}\rho} \|\omega\|^k |\widehat{f}(\omega)| \ |1-\sigma(h\omega)|d\omega \to 0 \ , \quad h\to 0 \ .$$

This proves that the first integral in (3.15) is $o(h^k)$.

To bound the second integral in (3.15), we note that the function $E_h(\omega,x)$ is h-periodic in the x variable, and can be written in terms of its Fourier series as

$$(3.17) \qquad E_h(\omega,x) = \sum_{\beta\in 2\pi\mathbb{Z}^d} \widehat{\psi}(h\omega+\beta)e^{-i\beta\cdot h^{-1}x} \ ,$$

since (3.17) is uniformly convergent by (3.6). Thus

$$(3.18) \qquad \frac{E_h(\omega,x)}{\widehat{\psi}(h\omega)} - 1 = \sum_{\beta\in 2\pi\mathbb{Z}^d\setminus 0} \frac{\widehat{\psi}(h\omega+\beta)}{\widehat{\psi}(h\omega)}e^{-i\beta\cdot h^{-1}x} \ ,$$

which together with (3.13) and the k-admissibility of f implies that the second integral in (3.15) is bounded by $A\|f\|'_k h^k$. □

As a direct consequence of Theorem 3.4, we get

Corollary 3.5 *Let ψ be given by (3.5) with ϕ satisfying (3.3). Then the approximation scheme (3.11)-(3.12) provides the optimal approximation order m' for m'-admissible functions.*

Proof: The first assumption in Theorem 3.4 is satisfied by ψ . To show (3.13) with k replaced by m', we use the relation $\widehat{\psi} = \mu\widehat{\phi}$ with μ 2π-periodic, and assumption (3.3)(c). Thus for $\beta \in 2\pi\mathbb{Z}^d\setminus 0$

$$(3.19) \qquad \Big\|\frac{\widehat{\psi}(\cdot+\beta)}{\|\cdot\|^m\widehat{\psi}(\cdot)}\Big\|_{\infty,\Omega} = \Big\|\frac{\widehat{\phi}(\cdot+\beta)}{\|\cdot\|^m\widehat{\phi}(\cdot)}\Big\|_{\infty,\Omega} \le A\|\widehat{\phi}(\cdot+\beta)\|_{\infty,\Omega} \ .$$

This together with (3.3)(b) yields (3.13), and hence (3.14) with k replaced by m'. $\square$

4 Approximation on quasi-uniformly scattered centers

In this section we construct analog approximation schemes to those studied in the previous sections, based on the translates of $\phi(h^{-1}\cdot)$ to sets of scatterd points $\{\Xi_h\}_{h>0}$. These schemes achieve the same approximation orders as their regular-grid analogs, provided that the sets $\{\Xi_h\}_{h>0}$ satisfy certain quasi-uniformity conditions.

4.1 Quasi-uniform sets of points

We start by stating the quasi-uniformity conditions on the sets of points $\{\Xi_h\}_{h>0}$.

Definition 4.1 *A set Ξ_h of points (centers) is called quasi-uniform of type ρ at level h if*

(4.1) $$\{y : \|y-x\| \le \rho\} \cap h^{-1}\Xi_h \ne \emptyset \quad \text{for all } x \in I\!R^d .$$

In fact a weaker implicit condition on Ξ_h is the key proprety needed in the forthcoming theory.

Definition 4.2 *A set Ξ_h is called k-approximating of type R, E at level h , if there exists a matrix $\{K(\alpha,\xi) : \alpha \in Z\!\!Z^d,\ \xi \in \Xi_h\}$ with the properties*

(4.2)
$$\begin{aligned}
&\text{(a)} \quad \sum_{\xi\in\Xi_h} |K(\alpha,\xi)| < E , \quad \alpha \in Z\!\!Z^d , \\
&\text{(b)} \quad K(\alpha,\xi) = 0 \quad \text{if} \quad \|\alpha - h^{-1}\xi\| > R,\ \alpha \in Z\!\!Z^d,\ \xi \in \Xi_h, \\
&\text{(c)} \quad \sum_{\xi\in\Xi_h} K(\alpha,\xi)p(h^{-1}\xi) = p(\alpha), \quad p \in \pi_k , \quad \alpha \in Z\!\!Z^d .
\end{aligned}$$

It is shown in [?] that a quasi-uniform set of points of type ρ at level h is also k-approximating of type R, E at level $\widetilde{h}$ for any k, with R , E , and $\widetilde{h}$ depending on ρ and k.

In [?] condition (b) is replaced by the weaker condition

$$\sum_{\xi\in\Xi_h} |K(\alpha,\xi)|(1+\|h^{-1}\xi-\alpha\|^j) < E_j \ , \quad \alpha\in\mathbb{Z}^d \ , \ j=1,\dots,s \ , \tag{4.3}$$

with $s > k$.

In the following we assume that the set Ξ_h under investigation is k-approximating of type R, E at level h for the required k, with fixed R and E. We denote the "active" subset of Ξ_h

$$\{\xi\in\Xi_h : K(\alpha,\xi)\neq 0 \ \text{ for some } \alpha\in\mathbb{Z}^d\} \ , \tag{4.4}$$

as our set Ξ_h.

4.2 The approximation scheme

The approximation scheme to be constructed is of the form

$$\mathcal{L}_{\Xi_h} f(x) = \sum_{\xi\in\Xi_h} \psi_\xi(h^{-1}x)\Gamma_h f(\xi) \ , \tag{4.5}$$

where Γ_h is either the identity for the quasi-interpolatory schemes, or $\Gamma_h = \Lambda_h$ for the optimal schemes, and where

$$\psi_\xi(x) = \sum_{\eta\in\Xi_h} N(\xi,\eta)\phi(x-h^{-1}\eta), \quad \xi\in\Xi_h \ . \tag{4.6}$$

The matrix N is defined in terms of the matrix K as

$$N(\xi,\eta) = \sum_{\alpha,\beta\in\mathbb{Z}^d} K(\alpha,\xi)\mu_{\beta-\alpha}K(\beta,\eta), \quad \xi,\eta\in\Xi_h \ , \tag{4.7}$$

with $\{\mu_\alpha\}$ an appropriate localization sequence, used in the analog scheme on the regular grid.

The method for deriving the approximation order provided by $\mathcal{L}_{\Xi_h}$ is by comparison to the corresponding scheme on the regular grid $\mathcal{L}_h$ given by

$$\mathcal{L}_h f(x) = \sum_{\alpha\in\mathbb{Z}^d} \psi(h^{-1}x-\alpha)\Gamma_h f(h\alpha) \ , \tag{4.8}$$

with Γ_h as above, and with

$$\psi = \sum_{\alpha\in\mathbb{Z}^d} \mu_\alpha \phi(\cdot-\alpha) \ . \tag{4.9}$$

The comparison is aimed at showing that

$$\|\mathcal{L}_{\Xi_h} f - \mathcal{L}_h f\|_\infty \le A h^\ell \ , \tag{4.10}$$

with ℓ not smaller than the known approximation order provided by $\mathcal{L}_h$. In many cases ℓ is greater or equal to the optimal approximation order provided by the given ϕ (see Section 3). In fact this method of comparison works for any scheme derived from one on a regular grid, if the schemes (4.5) and (4.8) are defined by the same localization sequence $\{\mu_\alpha\}$ and by the same operator Γ_h. This is the general setting in which the results in [**?**] are derived.

4.3 The pseudo-shifts

To reveal the similarity between $\mathcal{L}_h$ and $\mathcal{L}_{\Xi_h}$ we first introduce the "pseudo-shifts" $\{\phi_\alpha : \ \alpha\in\mathbb{Z}^d\}$, which approximate the shifts $\{\phi(\cdot-\alpha) : \ \alpha\in\mathbb{Z}^d\}$ in a relevant sense to the comparison (4.10). The pseudo-shifts have the form

$$\phi_\alpha = \sum_{\xi\in\Xi_h} K(\alpha,\xi)\phi(\cdot-h^{-1}\xi), \quad \alpha\in\mathbb{Z}^d \ , \tag{4.11}$$

and their relation to the shifts is of the following nature

Theorem 4.3 *Let ϕ be one of the radial or related functions introduced in the Introduction, and denote by $m'\in\mathbb{R}_+$ the order of the singularity of $\widehat{\phi}$ at the origin. Let K be the matrix satisfying (4.2). Then*

$$|\Phi_\alpha(x)| = |\phi_\alpha(x)-\phi(x-\alpha)| \le A(1+\|x-\alpha\|)^{-n_\Phi}, \quad \alpha\in\mathbb{Z}^d \ , \tag{4.12}$$

with A dependent on E, R, k but not on α, and with

$$n_\Phi = k - m' + d + 1 \ . \tag{4.13}$$

The proof of this theorem is based on the explicit form of the expansion of the Fourier transform of Φ_α near the origin and its decay behavior as $\|\omega\| \to \infty$. Since

$$\widehat{\Phi}_\alpha(\omega) = e_\alpha(\omega)\widehat{\phi}(\omega) \ , \quad e_\alpha(\omega) = \sum_{\xi\in\Xi_h} K(\alpha,\xi)e^{-i\omega\cdot h^{-1}\xi} - e^{-i\omega\cdot\alpha} \ , \tag{4.14}$$

it is the zero of order $k+1$ of $e_\alpha(\omega)$ at the origin which cancels the singularity of $\widehat{\phi}$ there in case $k \geq [m']$. For the comparison analysis we assume that in (4.2) $k = [m']$ so that $n_\Phi > d$.

With the pseudo-shifts of ϕ we also define the pseudo-shifts of ψ, in an analog form to (4.9)

$$\psi_\alpha = \sum_{\beta\in\mathbb{Z}^d} \mu_{\beta-\alpha}\phi_\beta \ . \tag{4.15}$$

In case the localization sequence $\{\mu_\alpha\}$ is of finite support, as in Section 2, we conclude from (4.12) and (4.15) that

$$|\psi_\alpha(x) - \psi(x-\alpha)| \leq A(1+\|x-\alpha\|)^{-n_\Phi}, \quad \alpha\in\mathbb{Z}^d \ . \tag{4.16}$$

A bound as in (4.16) also holds for more general localization sequences $\{\mu_\alpha\}$.

Theorem 4.4 *Assume*

$$|\mu_\alpha| \leq A(1+\|\alpha\|)^{-n_\mu}, \qquad n_\mu > d \ , \tag{4.17}$$

$$|\psi(x)| \leq A(1+\|x\|)^{-n_\psi}, \quad n_\psi > d \ , \tag{4.18}$$

and that the sum in (4.9) is uniformly convergent on compact sets of $\mathbb{R}^d$. Then

$$|\Psi_\alpha(x)| = |\psi_\alpha(x) - \psi(x-\alpha)| \leq A(1+\|x-\alpha\|)^{-n_\Psi} \tag{4.19}$$

with $n_\Psi = \min\{n_\mu, n_\Phi\} > d$ *. Also*

$$|\psi_\alpha(x)| \le A(1 + \|x - \alpha\|)^{-n'_\psi} \ , \tag{4.20}$$

with $n'_\psi = \min\{n_\mu, n_\Phi, n_\psi\} > d$ *.*

In the proof of (4.19), the sum defining Ψ_α

$$\Psi_\alpha(x) = \psi_\alpha(x) - \psi(x - \alpha) = \sum_{\beta \in \mathbb{Z}^d} \mu_{\beta-\alpha}[\phi_\beta(x) - \phi(x - \beta)] \ , \tag{4.21}$$

is estimated, in view of (4.12), by the discrete convolution

$$\sum_{\beta \in \mathbb{Z}^d} |\mu_\beta|(1 + \|(y + \alpha') - \beta\|)^{-n_\Phi}, \quad \|y\|_\infty \le \frac{1}{2}, \quad \alpha' \in \mathbb{Z}^d \ .$$

The estimate (4.20) is a direct consequence of (4.18) and (4.19).

Using the pseudo-shifts $\{\psi_\alpha\}$, we rewrite ψ_ξ of (4.6) as

$$\psi_\xi = \sum_{\alpha \in \mathbb{Z}^d} K(\alpha, \xi)\psi_\alpha \ , \tag{4.22}$$

which together with (4.20) leads to

Corollary 4.5 *Under the conditions of Theorem 4.4,*

$$|\psi_\xi(x)| \le A(1 + \|x - h^{-1}\xi\|)^{-n'_\psi} \ , \tag{4.23}$$

and the approximation scheme (4.5) is well defined for $f \in C(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$*.*

4.4 Comparison theorem for $C^\ell(\mathbb{R}^d) \cap W^\ell_\infty(\mathbb{R}^d)$

One more observation is needed before we can prove the first comparison result.

Lemma 4.6 *Let* $\{\mu_\alpha\}$ *be a localization sequence such that* ψ *satisfies (4.18), and such that the linear functional*

$$\mu p = \sum_{\alpha \in \mathbb{Z}^d} \mu_\alpha p(-\alpha) \ , \tag{4.24}$$

is well defined for $p \in \pi_\ell$. *Then* $\mu p = 0$, $p \in \pi_\ell$ *whenever* $\ell < m'$. *Moreover if* μ *is well defined on* $\pi_{\ell+1}$, *then for* $f \in C^{\ell+1}(\mathbb{R}^d) \cap W_\infty^{\ell+1}(\mathbb{R}^d)$

$$|\mu f(h(\beta + \cdot))| \le Ah^{\ell+1} \|f\|_{\infty,\ell+1}, \quad \beta \in \mathbb{Z}^d \ . \tag{4.25}$$

Proof: Assume $\ell < m'$. By the conditions on μ , $\widehat{\mu}(\omega) = \sum_{\alpha \in \mathbb{Z}^d} \mu_\alpha e^{-i\omega\cdot\alpha}$ is ℓ-time differentiable. It has a zero of order $\ell + 1$ at the origin, since $\widehat{\psi} = \widehat{\mu}\widehat{\phi}$ is continuous everywhere. This implies that $\mu p = 0$ for $p \in \pi_\ell$, from which (4.25) follows by considering the operation of μ on $(f - T_{h\beta} f)(h(\beta + \cdot))$, with $T_x f$ the Taylor polynomial of f at x of degree ℓ. □

With the results of Lemma 4.6, it is possible to compare $\mathcal{L}_{\Xi_h} f$ with $\mathcal{L}_h f$ in the case where $\mathcal{L}_h$ is a quasi-interpolation scheme or a cardinal interpolation scheme. For the schemes considered in Section 2 $m' = 2m$, $\{\mu_\alpha\}$ is of compact support and Lemma 4.6 holds for $0 \le \ell \le 2m - 1$. For the case of cardinal interpolation schemes see [?], [?] for the relevant properties of $\{\mu_\alpha\}$, depending on the properties of ϕ. For these two types of schemes the relevant comparison theorem is

Theorem 4.7 *Let* $\{\mu_\alpha\}$ *be a localization scheme satisfying the conditions of Lemma 4.6, and let* $\Gamma_h f = f$. *Then for* $f \in C^{\ell+1}(\mathbb{R}^d) \cap W_\infty^{\ell+1}(\mathbb{R}^d)$

$$\|(\mathcal{L}_{\Xi_h} - \mathcal{L}_h) f\|_\infty \le A \|f\|_{\infty,\ell+1} h^{\ell+1} \ . \tag{4.26}$$

In case ℓ *is the maximal integer smaller than* m' , $\mathcal{L}_{\Xi_h}$ *provides the same approximation order as* $\mathcal{L}_h$.

Proof: Let us introduce an intermediate scheme between $\mathcal{L}_h$ and $\mathcal{L}_{\Xi_h}$,

$$\widetilde{\mathcal{L}}_h f = \sum_{\alpha \in \mathbb{Z}^d} f(\alpha h)\psi_\alpha(h^{-1}\cdot) \ . \tag{4.27}$$

Then

$$\begin{aligned}(\widetilde{\mathcal{L}}_h - \mathcal{L}_h)f(x) &= \sum_{\alpha \in \mathbb{Z}^d} f(\alpha h)\Psi_\alpha(h^{-1}x) \\ &= \sum_{\alpha \in \mathbb{Z}^d} f(\alpha h) \sum_{\beta \in \mathbb{Z}^d} \mu_{\beta-\alpha}\Phi_\beta(h^{-1}x) \ .\end{aligned} \tag{4.28}$$

Since Φ_β satisfies (4.12), we can change the order of summation in the right hand side of (4.28) and apply (4.25) to obtain

$$|(\widetilde{\mathcal{L}}_h - \mathcal{L}_h)f(x)| = \Big| \sum_{\beta \in \mathbb{Z}^d} \Phi_\beta(h^{-1}x)\mu f(h(\beta + \cdot))\Big| \le Ah^{\ell+1}\|f\|_{\infty,\ell+1} \ . \tag{4.29}$$

To conclude (4.26) we have still to estimate

$$\begin{aligned}(\mathcal{L}_{\Xi_h} - \widetilde{\mathcal{L}}_h)f(x) &= \sum_{\xi \in \Xi_h} f(\xi) \sum_{\alpha \in \mathbb{Z}^d} K(\alpha,\xi)\psi_\alpha(h^{-1}x) \\ &\quad - \sum_{\alpha \in \mathbb{Z}^d} f(\alpha h)\psi_\alpha(h^{-1}x) \ .\end{aligned} \tag{4.30}$$

Again, since ψ_α satisfies (4.20) with $n'_\psi > d$, we can change the order of summation in the first sum in (4.30) to obtain

$$(\mathcal{L}_{\Xi_h} - \widetilde{\mathcal{L}}_h)f(x) = \sum_{\alpha \in \mathbb{Z}^d} \left[\sum_{\xi \in \Xi_h} K(\alpha,\xi)f(\xi) - f(\alpha h)\right] \psi_\alpha(h^{-1}x) \ . \tag{4.31}$$

Using properties (4.2) of the matrix K with k=[m'], we obtain in analogy to (4.25)

$$\Big|f(\alpha h) - \sum_{\xi \in \Xi_h} K(\alpha,\xi)f(\xi)\Big| \le A\|f\|_{\infty,\ell+1}h^{\ell+1}, \quad \ell \le [m'] \ . \tag{4.32}$$

This leads to the estimate

$$\|(\mathcal{L}_{\Xi_h} - \widetilde{\mathcal{L}}_h)f\|_\infty \le A\|f\|_{\infty,\ell+1}h^{\ell+1} \ , \tag{4.33}$$

which together with (4.29) yields (4.26). In case ℓ is the greatest integer less than m' then $\ell+1$ is equal or greater to the optimal approximation order possible m', and the last claim of the theorem follows. $\square$

The proof of Theorem 4.7 gives another interpretation to the scheme $\mathcal{L}_{\Xi_h} f$, namely

$$\mathcal{L}_{\Xi_h} f = \sum_{\xi \in \Xi_h} f(\xi)\psi_\xi(h^{-1}x) = \sum_{\alpha \in \mathbb{Z}^d} \widetilde{f}(h\alpha)\psi_\alpha(h^{-1}x) \ , \tag{4.34}$$

where $\widetilde{f}(\alpha h) = \sum_{\xi \in \Xi_h} K(\alpha, \xi) f(\xi)$ approximates $f(\alpha h)$ with an appropriate order.

Remark

(i) For the quasi-interpolation schemes in Section 2, $\{\mu_\alpha\}$ is of compact support and ℓ in Theorem 4.7 coincides with ℓ of Section 2, thus guaranteeing that $\mathcal{L}_{\Xi_h}$ provides the same approximation order as $\mathcal{L}_h$ for $0 \le \ell \le m' - 1$. More complicated situations, where $\ell + 1$ in (4.26) is not the optimal approximation order but equals or exceeds the one achieved by $\mathcal{L}_h$, are analyzed in [**?**] in the context of cardinal interpolation.

(ii) The class of approximated functions f in Theorem 4.7 is smaller than that considered in the regular grid case, when polynomial reproduction arguments are emplyed. This is due to the mild requirement on the decay of Φ_α (and hence on ψ_α and ψ_ξ) in the comparison analysis. It is possible to obtain by similar arguments the results of Theorem 4.7 for unbounded $f \in C^{\ell+1}(\mathbb{R}^d)$, with bounded derivatives of order ℓ and $\ell+1$, if Φ_α is assumed to decay as fast as ψ in (4.9). This can be obtained by taking a large enough k in condition (4.2) on the matrix K. In this setting, $\mathcal{L}_{\Xi_h}$ reproduces polynomials in π_ℓ for $\ell < m'$, although this property is not needed in the comparison analysis (see details in [**?**]).

(iii) The polynomial reproduction property of $\mathcal{L}_{\Xi_h}$ is the key property used in [**?**] for obtaining the approximation order provided by quasi-interpolation schemes of the form $\mathcal{L}_{\Xi_h} f = \sum_{\xi\in\Xi_h} f(\xi)\psi_\xi(h^{-1}\cdot)$, for ϕ satisfying the assumptions of Section 2. There the decay of ψ_ξ is obtained directly from properties of the matrix N in (4.6), (4.7) and from similar arguments to those in [**?**]. The proof of the polynomial reproduction property in [**?**] considers a sequence of sets $\{\Xi_{h,M} : M \in \mathbb{Z}_+\}$ such that $\Xi_{h,M}$ coincides with Ξ_h for $\|h^{-1}\xi\|_\infty \le M$ and with an appropriate regular grid for $\|h^{-1}\xi\|_\infty \ge M + A$, and shows that $\mathcal{L}_{\Xi_h} p$, $p \in \pi_\ell$ depends mostly on that part of the set of points which is regular.

4.5 Comparison theorem for $\widetilde{W}^m_\infty(\mathbb{R}^d)$

A second comparison theorem for schemes defined on the function spaces $\widetilde{W}^m_\infty(\mathbb{R}^d)$, as in Section 3, is based on the following analog of Lemma 4.6.

Lemma 4.8 *Assume $\{\mu_\alpha\}$ is such that for some $m \in \mathbb{R}_+$, $\|\cdot\|^{-m}|\widehat{\mu}|$ is bounded. Let $\widehat{\lambda} \in L_\infty(\mathbb{R}^d)$, and define $\widehat{\Gamma_h f} = \widehat{f}\widehat{\lambda}$. Then for any $f \in \widetilde{W}^m_\infty(\mathbb{R}^d)$*

$$|\mu\Gamma_h f(h(\beta+\cdot))| \le Ah^m |f|'_{\infty,m}, \quad \beta \in \mathbb{Z}^d \ , \tag{4.35}$$

where $|f|'_m = \int\limits_{\mathbb{R}^d} \|\omega\|^m |\widehat{f}(\omega)| d\omega < \infty$.

The proof of (4.35) is carried in the Fourier domain and is quite involved. Lemma 4.8 leads to

Theorem 4.9 *Assume $\{\mu_\alpha\}$ and λ satisfy the requirements of Lemma 4.8. Then for every $f \in \widetilde{W}^m_\infty(\mathbb{R}^d)$, $m \le m'$*

$$|(\mathcal{L}_{\Xi_h} - \mathcal{L}_h) f\|_\infty \le Ah^m |f|'_{\infty,m} \ . \tag{4.36}$$

In particular for $f \in \widetilde{W}^{m'}_\infty(\mathbb{R}^d)$, $\mathcal{L}_{\Xi_h}$ provides the same approximation order as $\mathcal{L}_h$.

Proof: The proof is similar to that of Theorem 4.7, with Lemma 4.8 replacing Lemma 4.6. One has also to show that for $m \le m'$

$$\Big|\sum_{\xi\in\Xi_h} K(\alpha,\xi) f(\xi) - f(\alpha h)\Big| \le Ah^m |f|'_{\infty,m}, \quad \alpha \in \mathbb{Z}^d \ , \tag{4.37}$$

for the estimation of $(\mathcal{L}_{\Xi_h} - \widetilde{\mathcal{L}}_h)f$. This follows from the observation that

$$\sum_{\xi\in\Xi_h} K(\alpha,\xi)f(\xi) - f(\alpha h) = \frac{1}{(2\pi)^d}\int_{I\!R^d} e_\alpha(h\omega)\widehat{f}(\omega)d\omega \ , \tag{4.38}$$

with $e_\alpha(\omega)$ defined as in (4.14). Since $e_\alpha \in C^{[m']+1}(I\!R^d)$ and has a zero of order $[m']+1$ at the origin (by conditions (4.2) on K with $k=[m']$), then $\|\omega\|^{-m}e_\alpha(\omega)$ is bounded for $m \le m'$ and (4.37) follows from (4.38). □

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