Approximation of Univariate Set-Valued Functions- an Overview

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Abstract

The paper is an updated survey of our work on the approximation of univariate set-valued functions by samples-based linear approximation operators, beyond the results reported in our previous overview. Our approach is to adapt operators for real-valued functions to set-valued functions, by replacing operations between numbers by operations between sets. For set-valued functions with compact convex images we use Minkowski convex combinations of sets, while for those with general compact images metric averages and metric linear combinations of sets are used. We obtain general approximation results and apply them to Bernstein polynomial operators, Schoenberg spline operators and polynomial interpolation operators.

Key words: compact sets, set-valued functions, linear approximation operators, Minkowski sum of sets, metric average, metric linear combinations

1 Introduction

In this paper we present the progress of our work on the approximation of univariate set-valued functions (multifunctions) by linear approximation operators, beyond the results reported in [11]. We adapt linear samples-based approximation operators for real-valued functions to set-valued functions (SVFs) with compact images in $\mathbb{R}^n$, by replacing operations between numbers by operations between sets. For this purpose, the well-known Minkowski sum of sets is a proper substitute for addition of numbers, only in case of SVFs with convex images. For such multifunctions, the representation of convex compact sets in terms of their support functions allows to reduce approximation of SVFs by linear positive operators to the approximation of the corresponding support functions. The application of known approximation results from the case of real-valued functions to the case of SVFs with convex compact images is studied in [19, 6, 2, 8]. The positivity of the operators is necessary for the approximants to be well defined.

It was noticed by Vitale [19] that positive approximation operators with Minkowski sums of sets fail to approximate multifunctions with general compact images (not necessarily convex). Vitale also observed that the images of the Bernstein approximants of increasing degree tend to convex sets. Similarly, limits of spline subdivision schemes are convex-valued

SVFs for any initial data [10]. The obvious conclusion is that operators with Minkowski sums of sets are not appropriate for the approximation of SVFs with general compact images.

In [1] a binary operation between sets, the "metric average", is introduced, and the piecewise linear interpolant based on it is shown to approximate continuous SVFs with general images. The use of this operation in the adaptation of known positive approximation operators to SVFs, requires a representation of the approximation operators by repeated binary averages. Such a representation exists for any samples-based linear operator, which reproduces constants, but is not unique [20]. This non-uniqueness leads to different operators for SVFs which are not necessarily approximating. Yet, spline subdivision schemes represented by repeated binary averages [9], and the Schoenberg operators defined in terms of the de Boor algorithm [13], approximate SVFs with general compact images. On the other hand, for the adaptation of the Bernstein operators based on the de Casteljau algorithm, an approximation result was obtained only for a certain clas of SFVs with images in $\mathbb{R}$ [13].

The lack of associativity of the metric average is the reason why it is hard to extend this binary operation to an average of three or more sets. Yet, in [12] a set-operation on a finite sequence of compact sets, termed "metric linear combination", which extends the metric average, is devised. With this operation, linear approximation operators are successfully adapted to univariate SVFs. It should be emphasized that this adaptation method is not restricted to positive operators. To the best of our knowledge so far only positive operators were applied to SVFs.

We apply the different adaptations to two classes of positive operators, Bernstein operators and Schoenberg spline operators. Adaptation of polynomial interpolation operators is constructed only with metric linear combinations of sets. Such interpolation operators at the zeros of the Tchebyshev polynomials of growing degree are shown to converge to Lipschitz continuous SFVs [12].

The outline of the paper is as follows. Section 2 contains definitions and notation. Section 3 discusses operators based on Minkowski averages of sets; their applicability for the approximation of convex-valued multifunctions, and their failure in the case of SVFs with general compact images. In Section 4 two metric operations on sets are presented, and used in Section 5 to construct approximating operators for multifunctions with general compact images. In Section 6 error estimates for specific approximation operators are presented.

2 Preliminaries

First we introduce some notation. The collection of all nonempty compact sets in $\mathbb{R}^n$ is denoted by $\mathcal{K}_n$, $\mathcal{C}_n$ denotes the collection of convex sets in $\mathcal{K}_n$. $\langle \cdot , \cdot \rangle$ is the inner product, $|\cdot|$ is the Euclidean norm and S_{n-1} is the unit sphere in $\mathbb{R}^n$. We use co$A$ for the convex hull of $A \in \mathcal{K}_n$ and $\operatorname{dist}(x, A) = \inf_{a\in A} |x - a|$ for the distance from a point $x \in \mathbb{R}^n$ to A.
We define the set of *metric pairs* of $A, B \in \mathcal{K}_n$ by

$$\Pi(A, B) = \big\{(a, b) \in A \times B : \ |a - b| = \operatorname{dist}(a, B) \text{ or } |a - b| = \operatorname{dist}(b, A)\big\}.$$

For $A, B \in \mathcal{K}_n$ the Hausdorff metric is

$$\operatorname{haus}(A, B) = \sup\{ \ |a - b| : (a, b) \in \Pi(A, B) \ \}.$$

The space $\mathcal{K}_n$ is a complete metric space with respect to this metric. [17].
The support function $\delta^*(A,\cdot):\mathbb{R}^n \to \mathbb{R}$ is defined for $A \in \mathcal{K}_n$ as

$$\delta^*(A,l) = \max_{a\in A}\langle l,a\rangle, \quad l \in \mathbb{R}^n.$$

A linear Minkowski combination of sets is

$$\sum_{i=1}^{k} \lambda_i A_i = \{\sum_{i=1}^{k} \lambda_i a_i \ : \ a_i \in A_i, \ i=1,...,k\}, \quad A_i \in \mathcal{K}_n, \ \lambda_i \in \mathbb{R}.$$

In particular, $A+B=\{a+b, \ a\in A, \ b\in B\}$ is the Minkowski sum of two sets. A Minkowski average (a Minkowski convex combination) of sets is a linear Minkowski combination with λ_i non-negative, summing up to 1.

We consider functions defined on $[0,1]$ with images in a metric space (X,ρ), with X either $\mathbb{R}^n$ or $\mathcal{K}_n$, and ρ either the Euclidean metric or the Hausdorff metric respectively. The notions of convergence, continuity, Hölder/Lipschitz continuity are to be understood with respect to the appropriate metric, e.g. $f(\cdot)$ is Hölder continuous with exponent α if

$$\rho(f(x),f(y)) \le \mathcal{L}|x-y|^\alpha, \quad x,y\in[0,1],$$

where the constant $\mathcal{L}$ depends on f. The collection of Hölder continuous multifunctions with exponent α and constant $\mathcal{L}$ is denoted by $\mathcal{H}_\alpha(\mathcal{L})$. For $\alpha=1$ the notation is $Lip(\mathcal{L})$.

We recall that the modulus of continuity (see e.g. [5], Chapter 2) of a function $f:[0,1]\to X$ with a step $\delta \ge 0$ is

$$\omega(f,\delta) = \sup_{0<h\le\delta} \|\Delta_h(f,\cdot)\|_\infty,$$

where

$$\Delta_h(f,x) = \begin{cases} \rho(f(x+h),f(x)) & for \ x,x+h\in[0,1], \\ 0 & otherwise. \end{cases}$$

$\omega(f,\delta)$ is also known as the first modulus of smoothness. The $k-$th modulus of smoothness is defined by

$$\omega_k(f,\delta) = \sup\{\|\Delta_h^k(f,\cdot)\|_\infty : 0<h\le\delta\},$$

with $\Delta_h^1=\Delta_h$ and $\Delta_h^k(f,x)=\Delta_h\big(\Delta_h^{k-1}(f,x),x\big)$.

Note that for $f\in\mathcal{H}_\alpha(\mathcal{L})$, $\omega_k(f,\delta)=O(\delta^\alpha)$, $k\ge 1$ and for f $k-$times continuously differentiable $\omega_k(f,\delta)=O(\delta^k)$.

In this paper we discuss the adaptation to univariate SVFs of certain linear operators approximating real-valued functions. We consider linear operators based on samples at a set of points $\chi=\{x_0,...,x_N\}$, $0\le x_0<x_1<...<x_N\le 1$ of the form

$$\mathcal{A}_\chi(f,x) = \sum_{i=0}^{N} c_i(x) f(x_i). \tag{1}$$

We restrict this class to operators which approximate continuous functions in $[0,1]$ or in most of it. Thus, we require that $\sum_{i=0}^N c_i(x) = 1$ either in $[0,1]$, or in most of it. We denote by $|\chi| = \max\{x_{i+1} - x_i : i = 0, ..., N-1\}$. χ_N denotes the set of equidistant points $\{i/N : 0 \le i \le N\}$, with $|\chi_N| = 1/N$. An operator based on χ_N we denote by $\mathcal{A}_N$.

We recall that a linear operator $L(f, x)$ is called positive if for a non-negative f, $L(f, x)$ is non-negative. Obviously, $\mathcal{A}_\chi$ is a positive linear operator if $c_i(x) \ge 0$, $i = 0, ..., N$. It reproduces the constant functions, namely $\mathcal{A}_\chi(f, x) = f(x)$ for $f(x) = \text{Const}$ at all x such that $\sum_{i=0}^{N} c_i(x) = 1$. At all such points, $\mathcal{A}_\chi(f, x)$ is a weighted average of the function values $f(x_i), i = 0, 1, ..., N$.

3 Approximation based on Minkowski averages

The first adaptations of operators of type (1) to SFVs were done with the help of Minkowski sum of sets. In this section we survey some general results for such adaptations [19, 2, 6, 8].

For given data points χ, a positive operator of the form (1) with Minkowski sums of sets replacing addition of numbers is

$$\mathcal{A}_\chi(F, x) = \sum_{i=0}^{N} c_i(x) F(x_i), \; x \in [0, 1], \quad c_i(x) \ge 0. \tag{2}$$

3.1 The case of convex-valued multifunctions

Here we consider SVFs with images in $\mathcal{C}_n$, and the operation of $\mathcal{A}_\chi$ defined by (2) on such multifunctions. It is clear, that

$$\mathcal{A}_\chi(\lambda F + \mu G, \cdot) = \lambda \mathcal{A}_\chi(F, \cdot) + \mu \mathcal{A}_\chi(G, \cdot), \quad \lambda, \mu \ge 0. \tag{3}$$

Moreover, by the positivity of $c_i(x)$, the images of $\mathcal{A}_\chi(F, \cdot)$ remain in the cone $\mathcal{C}_n$.

The approximation and the shape-preservation properties of such operators follow from the parametrization of convex compact sets by their support functions. The well known properties of the support functions δ^*, relevant to our investigation, are [16]:

for $A, B \in \mathcal{C}_n$,

1. $\delta^*(A + B, \cdot) = \delta^*(A, \cdot) + \delta^*(B, \cdot)$,
2. $\delta^*(\lambda A, \cdot) = \lambda \delta^*(A, \cdot), \quad \lambda \ge 0$,
3. $A \subseteq B \iff \delta^*(A, l) \le \delta^*(B, l)$ for each $l \in \mathbb{R}^n$,
4. $\text{haus}(A, B) = \max_{l \in S_{n-1}} |\delta^*(A, l) - \delta^*(B, l)|$.

Thus, the operator $\mathcal{A}_\chi$ in (2) is related to the operator $\mathcal{A}_\chi$ in (1) by,

$$\delta^*(\mathcal{A}_\chi(F, t), l) = \mathcal{A}_\chi(\delta^*(F, l), t), \quad l \in \mathbb{R}^n. \tag{4}$$

Also, $F \in \mathcal{H}_\alpha(\mathcal{L})$ iff $\delta^*(F(\cdot), l) \in \mathcal{H}_\alpha(\mathcal{L})$, uniformly in $l \in S_{n-1}$.

By the above two observations, approximation results for positive operators can be extended from the case of real-valued functions to the case of set-valued functions with compact convex images. Here we formulate a general result of this type.

Theorem 3.1. *Let $\mathcal{A}_\chi$ approximate continuous real-valued functions with the error estimate*

$$|\mathcal{A}_\chi(f,x) - f(x)| \leq C\omega_k(f, \psi(x, |\chi|)),$$

where $\psi : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous real-valued function, non-decreasing in its second argument, satisfying $\psi(x,0) = 0$.

Then for a continuous convex-valued multifunction $F : [0,1] \to \mathcal{C}_n$

$$haus(\mathcal{A}_\chi(F,x), F(x)) \leq C \sup_{l \in S_{n-1}} \omega_k(\delta^*(F,l), \psi(x, |\chi|)).$$

As in the real-valued case, the adapted positive operators (2) have shape preservation properties in the convex-valued case. The positivity of $\mathcal{A}_\chi$ preserves the order between two multifunctions in the sense of set-inclusion: $F(x) \subseteq G(x) \Rightarrow \mathcal{A}_\chi(F,x) \subseteq \mathcal{A}_\chi(G,x)$.

Moreover, by Property 3 of support functions, if $\mathcal{A}_\chi$ preserves monotonicity of real-valued functions, then $\mathcal{A}_\chi$ preserves monotonicity of multifunctions. Here monotonicity of SVFs is in the sense of set-inclusion, namely if $F(x) \subseteq F(x+h)$ for all $h > 0$, then $\mathcal{A}_\chi(F,x) \subseteq \mathcal{A}_\chi(F,x+h)$ for all $h > 0$ (see [8] for more details).

3.2 The general case – convexification

Vitale [19] noticed that for the constant multifunction $F(x) = \{0,1\}$, the piecewise-linear approximation constructed with Minkowski sums does not converge to $F(x)$ when $|\chi| \to 0$. He also observed that the Bernstein approximants of a multifunction with general compact images converge, when increasing their degree, to a convex-valued multifunction.

More generally, if the number of summands in (2) grows with N, as for the Bernstein operators, the Shapley-Folkman-Starr Theorem (see Appendix 2 in [18] and Theorem 2 in [4]) yields

$$\text{haus}(\mathcal{A}_\chi(F,x), \text{co}\mathcal{A}_\chi(F,x)) \leq \sqrt{n} \max_{0 \leq i \leq N} c_i(x) \max_{s \in [0,1]} \sup\{|y| : y \in F(s)\},$$

for any multifunction with compact images in $\mathbb{R}^n$. Since $\text{co}\mathcal{A}_\chi(F,x) = \mathcal{A}_\chi(\text{co}F,x)$, by Theorem 3.1, $\lim_{N\to\infty} \mathcal{A}_\chi(\text{co}F,x) = \text{co}F(x)$. Thus, if $\lim_{N\to\infty} \max_i c_i(x) = 0$, as in the case of Bernstein operators, then $\lim_{N\to\infty} \mathcal{A}_\chi(F,x) = \text{co}F(x)$ (see [10] for other operators with this property).

Another type of operators for which convexification occurs are spline subdivision schemes. For these operators the Shapley-Folkman-Starr Theorem is not applicable. Subdivision schemes are recursive averaging procedures with a fixed finite number of summands and fixed weights. For this case an inequality, involving a measure of non-convexity of sets, introduced in [4], is used to prove that spline subdivision schemes with Minkowski averages applied to arbitrary initial compact sets in $\mathbb{R}^n$ converge to a multifunction with convex images [10].

The convexification occuring with Minkowski averages, motivated the search for alternative operations on sets.

4 Metric operations on general sets

The lack of approximation by operators of type (2) in case of SVFs with general images, is due to the fact that the Minkowski averages of non-convex sets are too big. For example, the convex combination $\lambda A + (1-\lambda)A$, $\lambda \in [0,1]$ equals A if A is convex, but is a superset of A for general A. Two operations on sets are introduced in [1] and [12] which produce subsets of the Minkowski average or linear Minkowski combination respectively. With these operations it is possible to avoid convexification and to achieve approximation for SVFs with general images.

4.1 The metric average of two sets

A binary operation between sets was constructed in [1] and used for piecewise-linear approximation of SVFs with compact (not necessarily convex) images. This binary operation is termed in [9] "metric average".

Definition 4.1. *Let $A, B \in \mathcal{K}_n$, $t \in [0,1]$. The* ***t-weighted metric average*** *of A and B is*

$$A \oplus_t B = \{ta + (1-t)b : \ (a,b) \in \Pi(A,B)\}.$$

The following properties of the metric average are important for our applications. The first three are easy to observe [9], and the fourth is the metric property proved in [1].

Let $A, B, C \in \mathcal{K}_n$ and $0 \le t \le 1$, $0 \le s \le 1$. Then

1. $A \oplus_0 B = B, \quad A \oplus_1 B = A, \quad A \oplus_t B = B \oplus_{1-t} A.$
2. $A \oplus_t A = A.$
3. $A \cap B \subseteq A \oplus_t B \subseteq tA + (1-t)B.$
4. $\mathrm{haus}(A \oplus_t B, A \oplus_s B) = |t-s| \mathrm{haus}(A,B).$

Note that the analogues of properties 2 and 4 are true in the case of Minkowski averages only for convex sets, while with the metric average these essential properties are valid for general compact sets.

Although the metric average is a non associative binary operation, there exists an extension of this operation to a finite number of ordered sets.

4.2 The metric linear combination of sets

In [12] a new operation on a finite sequence of sets is introduced. It is based on the notion of a metric chain, which is an extension of a metric pair.

Definition 4.2. *For $\{A_0, ..., A_N\}$ with $A_i \in \mathcal{K}_n$, a vector $(a_0, ..., a_N)$ is called a* ***metric chain*** *of $\{A_0, ..., A_N\}$, if $a_i \in A_i$, $i = 0, ..., N$, and there exists j, $0 \le j \le N$ such that*

$$a_{i-1} \in \Pi_{A_{i-1}}(a_i),\ 1 \le i \le j \ \text{ and } \ a_{i+1} \in \Pi_{A_{i+1}}(a_i),\ j \le i \le N-1.$$

Here $\Pi_A(b) = \{\, a \in A \ : \ |a-b| = \mathrm{dist}(b,A) \,\}$ *for* $b \in \mathbb{R}^n$.

An illustration of such a metric chain is given in Figure 4.1.

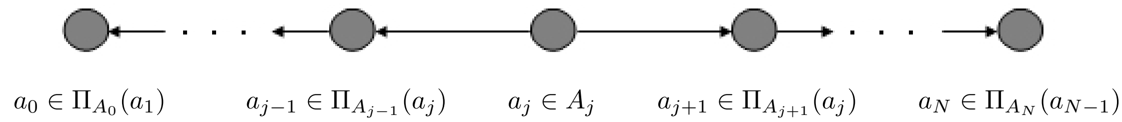

Figure 4.1.

Thus each element of each set A_i, $i = 0, ..., N$ generates at least one metric chain. We denote by $CH(A_0, ..., A_N)$ the collection of all metric chains of $\{A_0, ..., A_N\}$. The set $CH(A_0, ..., A_N)$ depends on the order of the sets A_i, $i = 0, ..., N$. With this notion of metric chains we can define,

Definition 4.3. *A **metric linear combination** of a sequence of sets $A_0, ..., A_N$ with coefficients $\lambda_0, ..., \lambda_N \in \mathbb{R}$, is*

$$\bigoplus_{i=0}^{N} \lambda_i A_i = \left\{ \sum_{i=0}^{N} \lambda_i a_i \; : (a_0, ..., a_N) \in CH(A_0, ..., A_N) \right\}. \tag{5}$$

The following distributive laws are easily derived from the definition,

$$\text{(i)} \quad \bigoplus_{i=0}^{N} \lambda_i A = \left(\sum_{i=0}^{N} \lambda_i \right) A \,, \qquad \text{(ii)} \quad \bigoplus_{i=0}^{N} \lambda A_i = \lambda \left(\bigoplus_{i=0}^{N} 1 \cdot A_i \right).$$

Note that $\lambda_0, ..., \lambda_N$ can be any real numbers, and that if $\sum_{i=0}^{N} \lambda_i = 1$, then by (i), $\bigoplus_{i=0}^{N} \lambda_i A = A$.

5 Metric approximation operators

In this section we describe our general approach to the adaptation of operators of type (1) to the set-valued setting, based on the metric operations of Section 4. The discussion of the adaptation of specific operators is postponed to Section 6.

5.1 Operators based on the metric average

The metric average was successfully used in [9] for the construction of set-valued subdivision schemes and in [13] for the adaptation of the Schoenberg spline operators to multifunctions. Also in [13] the Bernstein operators based on the metric average are shown to approximate a certain class of SVFs with images in $\mathbb{R}$.

The main disadvantage of the metric average, as an operation on sets, is the lack of associativity. Hence it is not directly extendable to several sets. This is the reason why the adaptation of (1) based on the metric average requires to represent it in terms of repeated binary averages. Let us note that a representation by repeated binary averages exists for any samples-based linear operator, which reproduces constants, but it is not unique [20]. The non-uniqueness leads to a variety of operators, which are not necessarily approximating. Therefore general approximation results are not available. Yet, the representations chosen in [9, 13], for concrete approximation operators, are proved to be adequate theoretically and experimentally.

5.2 Operators based on the metric linear combinations

All the results of this subsection are cited from [12].

We use the metric linear combination (5) to define the **metric analogue** of the linear operator (1).

Definition 5.1. *For $F : [0,1] \to \mathcal{K}_n$, we define a* ***metric linear operator*** A_χ^M *by*

$$A_\chi^M(F,x) = \bigoplus_{i=0}^{N} c_i(x) F(x_i). \tag{6}$$

In contrast to the adaptations of positive operators based on the metric average, the metric analogues (6) of two linear operators of the form (1), which are identical on single-valued functions, are identical on SVFs.

Here we formulate a general error estimate for these operators.

Theorem 5.2. *Let A_χ be of the form (1), then for a continuous $F : [0,1] \to \mathcal{K}_n$*

$$\mathrm{haus}(A_\chi^M(F,x), F(x)) \leq 2\,\omega(F, |\chi|) + \sup_{\varphi \in CH} |A_\chi(s(\chi,\varphi), x) - s(\chi,\varphi)(x)|, \tag{7}$$

where $s(\chi,\varphi)$ is a piecewise-linear single-valued function interpolating the data (x_i, f_i), $i = 0, ..., N$, with $\varphi = (f_0, ..., f_N) \in CH(F(x_0), ..., F(x_N))$.

In case $F \in Lip(\mathcal{L})$, then also $s(\chi,\varphi) \in Lip(\mathcal{L})$, and we have

Corollary 5.3. *Let $F \in Lip(\mathcal{L})$ and let A_χ be of the form (1), satisfying*

$$|A_\chi(f,x) - f(x)| \leq C\,\mathcal{L}\psi(x, |\chi|), \quad f \in Lip(\mathcal{L}),$$

where ψ is as in Theorem 3.1. Then

$$\mathrm{haus}(A_\chi^M(F,x), F(x)) \leq 2\,\mathcal{L}|\chi| + C\mathcal{L}\psi(x, |\chi|), \tag{8}$$

6 Adaptation of specific approximation operators

The approximation results from Sections 3,5 are specialized here to two classes of positive operators: the Schoenberg spline operators and the Bernstein polynomial operators. We also present the adaptation of polynomial interpolation operators to SVFs as examples of non-positive operators.

Error estimates for the various types of adapted approximation operators are provided, using C as a generic constant.

6.1 Bernstein operators

The Bernstein operator $B_N(f,x)$ for a real-valued function $f:[0,1]\to\mathbb{R}$ is

$$B_N(f,x)=\sum_{i=0}^{N}\binom{N}{i}x^i(1-x)^{N-i}f\left(\frac{i}{N}\right). \tag{9}$$

It is known (see [5], Chapter 10) that there exists a constant C independent of f such that for a continuous f

$$|f(x)-B_N(f,x)|\le C\omega_{[0,1]}(f,\sqrt{x(1-x)/N}).$$

The value $B_N(f,x)$ can be calculated recursively by repeated binary averages, using the de Casteljau algorithm [15],

$$\begin{aligned} f_i^0=f(i/N),\ i &= 0,...,N, \\ f_i^k=(1-x)f_i^{k-1}+xf_{i+1}^{k-1},\ i &= 0,1,...,N-k,\ k=1,...,N, \\ B_N(f,x) &= f_0^N. \end{aligned} \tag{10}$$

This algorithm is commonly used in CAGD.

Next we present three different adaptations of the Bernstein operators to SVFs .

The adapted Bernstein operator of the form (2) is

$$B_N^{Mn}(F,x)=\sum_{i=0}^{N}\binom{N}{i}x^i(1-x)^{N-i}F\left(\frac{i}{N}\right), \tag{11}$$

and by Theorem 3.1 we have,

Theorem 6.1. *For a convex-valued multifunction $F\in\mathcal{H}_\alpha(\mathcal{L})$*

$$\text{haus}\left(F(x),B_N^{Mn}(F,x)\right)\le C\left(\frac{x(1-x)}{N}\right)^{\frac{\alpha}{2}}.$$

In the adaptation of (9), based on the metric average with the de Casteljau algorithm, starting with $F_i^0=F(i/N)$, we replace in (10) the average $f_i^k=(1-x)f_i^{k-1}+xf_{i+1}^{k-1}$ by the metric average $F_i^k=F_i^{k-1}\oplus_{1-x}F_{i+1}^{k-1}$ and obtain the approximant $B_N^{MA}(F,x)=F_0^N$, $x\in[0,1]$. It is not known whether these operators approximate multifunctions with general compact images in $\mathbb{R}^n$, yet for a certain class of SVFs with compact images in $\mathbb{R}$, the following approximation result holds [13],

Theorem 6.2. *Let $F\in Lip(\mathcal{L})$ be such that for $\forall x\in[0,1]$, $F(x)=\bigcup_{j=1}^{J}F_j(x)$, where $F_j(x)$ are disjoint compact intervals. Then for sufficiently large N*

$$\text{haus}\left(B_N^{MA}(F,x),F(x)\right)\le C/\sqrt{N},\quad x\in[0,1].$$

The metric analogue of the Bernstein operator for set-valued functions is [12],

$$\begin{aligned} B_N^M(F,x) &= \bigoplus_{i=0}^{N}\binom{N}{i}x^i(1-x)^{N-i}F\left(\frac{i}{N}\right) \\ &= \left\{\sum_{i=0}^{N}\binom{N}{i}x^i(1-x)^{N-i}f_i\ :\ (f_0,...,f_N)\in CH\right\}, \end{aligned}$$

where $CH = CH(F(0), F(1/N), ..., F(1))$.

It follows from Corollary 5.3 that

Corollary 6.3. *Let $F \in Lip(\mathcal{L})$, then*

$$\mathrm{haus}\left(B_N^M(F,x), F(x)\right) \leq 2\mathcal{L}/N + C\mathcal{L}\sqrt{x(1-x)/N}.$$

6.2 Schoenberg operators

For the Schoenberg operators we have four successful adaptations to SVFs. The approximation results in this case are numerous.

6.2.1 The real-valued case

The Schoenberg spline operator of order m with uniform sampling points χ_N, for a real-valued function f, is

$$S_{m,N}(f,x) = \sum_{i=0}^{N} f(i/N) b_m\left(Nx - i\right),\ x \in [0,1], \tag{12}$$

where $b_m\left(x\right)$ is the B-spline of order m (degree $m-1$) with integer knots and support $[0,m]$. By the known approximation result (see [3], Chapter XII),

$$|S_{m,N}(f,x) - f(x)| \leq \left\lfloor \frac{m+1}{2} \right\rfloor \omega_{[0,1]}(f, 1/N), \quad x \in \left[\frac{m-1}{N}, 1\right], \tag{13}$$

where $\lfloor x \rfloor$ is the maximal integer not greater than x.

Note, that the rate of approximation of the Schoenberg operators can be improved if b_m in (12) is replaced by the centered B-spline $\tilde{b}_m = b_m(\cdot + m/2)$. We omit the details here.

In [3], Chapter X it is shown that (12) can be evaluated recursively in terms of repeated binary averages. For $x \in [j, j+1]$ let

$$\begin{aligned} f_i^1 &= f\left(i/N\right), \quad & i &= j-m+1, ..., j, \\ f_i^k &= \lambda_i^k f_{i-1}^{k-1} + \left(1-\lambda_i^k\right) f_i^{k-1}, \quad & i &= j-m+k, ..., j,\ k = 2, ..., m, \\ S_{m,N}\left(f,x\right) &= f_j^m. \end{aligned} \tag{14}$$

with $\lambda_i^k = \dfrac{i+m+1-k-Nt}{m+1-k}$, $i = j-m+k, ..., j$, $k = 2, ..., m$.

For real-valued functions the Schoenberg operators can be also evaluated by subdivision schemes (see e.g. [7]). Given the initial sequence $f_i^0 = f(\frac{i}{N})$, $i = 0, \ldots, N$ of values in $\mathbb{R}$, with $f_i^0 = 0$ for $i \in \mathbb{Z} \setminus \{0, 1, ..., N\}$, the spline subdivision scheme for the evaluation of $S_{m,N}(f, \cdot)$ is given by the refinement steps

$$f_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{i-2j}^{[m]} f_j^k, \quad i \in \mathbb{Z}, \quad k = 0, 1, 2, ... \tag{15}$$

where $a_i^{[m]} = \binom{m+1}{i}/2^m,\ i = 0, 1, ..., m+1$ and $a_i^{[m]} = 0$ for $i \in \mathbb{Z} \setminus \{0, 1, ..., m+1\}$. At the $k-$th refinement level one defines the piecewise-linear function

$$f^{[k]}(x) = \sum_{i \in \mathbb{Z}} f_i^k \tilde{b}_2(2^k x - i), \quad x \in \mathbb{R}, \tag{16}$$

where $\{f_i^k,\ i \in \mathbb{Z}\}$ are the values generated by the subdivision scheme at refinement level k.

The scheme (15) is uniformly convergent, namely the sequence $\{f^{[k]}(\cdot)\}_{k \geq 0}$ is a Cauchy sequence, and its limit function is of the form (see e.g. [7])

$$f^{\infty}(x) = \sum_{i=0}^{N} f_i^0 b_m(x - i), \quad x \in \mathbb{R}.$$

Therefore

$$S_{m,N}(f, x) = f^{\infty}(Nx),\ x \in [0, 1]. \tag{17}$$

The refinement step (15) can be computed by repeated binary averages as follows:

$$\begin{aligned} f_{2i}^{k+1,0} &= f_i^k, \quad f_{2i-1}^{k+1,0} = (1/2) f_{i-1}^k + (1/2) f_i^k, \quad i \in \mathbb{Z}, \\ f_i^{k+1,j} &= (1/2) f_i^{k+1,j-1} + (1/2) f_{i+1}^{k+1,j-1},\ j = 1, ..., m-1 \\ f_i^{k+1} &= f_{i+\lfloor \frac{m-1}{2} \rfloor}^{k+1,m-1}, \qquad i \in \mathbb{Z}. \end{aligned} \tag{18}$$

6.2.2 The convex-valued case

To define Schoenberg operators for a multifunction with convex images F, one can use the direct formula (12), the evaluation procedure (14) or spline subdivision schemes with a refinement step given by (15) or by (18), replacing f by F and sums of numbers by Minkowski sums of sets. By the results obtained for the real-valued case and by (4), all methods of computation lead to the same SVF, denoted by $S_{m,N}(F, \cdot)$ [8],[11]. By Theorem 3.1 we have for $F \in \mathcal{H}_\alpha(\mathcal{L})$

$$\text{haus}\left(S_{m,N}(F, x), F(x)\right) \leq \left\lfloor \frac{m+1}{2} \right\rfloor \frac{1}{N^\alpha}, \quad x \in \left[\frac{m-1}{N}, 1\right]. \tag{19}$$

6.2.3 Schoenberg operators based on the metric average

In [13] the Schoenberg operator for a multifunction F, $S_{m,N}^{MA}(F, \cdot)$, is defined in terms of algorithm (14) with the binary averages between numbers replaced by the corresponding metric averages between sets. It is shown there that

Theorem 6.4. *For a set-valued function $F : [0, 1] \to \mathcal{K}_n$, $F \in \mathcal{H}_\alpha(\mathcal{L})$, the Schoenberg operator $S_{m,N}^{MA}(F, x)$ satisfies*

$$\text{haus}\left(S_{m,N}^{MA}(F, x), F(x)\right) \leq \frac{C}{N^\alpha}. \tag{20}$$

Another way to adapt the Schoenberg operators using the metric average as a basic operation is to adapt the m-th degree spline subdivision scheme, represented by the sequence of repeated binary averages (18). Starting with $F_i^0 = F(i/N)$, $i = 0, ..., N$ and $F_i^0 = \{0\}$ otherwise, we replace in (18) the averages of numbers by corresponding metric averages of sets to obtain $\{F_i^{k+1} : i \in \mathbb{Z}\}$ from $\{F_i^k : i \in \mathbb{Z}\}$

At the $(k+1)-$th refinement level, a metric piecewise-linear SVF, $F^{[k+1]}(t)$ is defined by

$$F^{[k+1]}(t) = F_i^{k+1} \oplus_{\lambda(t)} F_{i+1}^{k+1}, \qquad i2^{-(k+1)} \le t \le (i+1)2^{-(k+1)}, \qquad i \in \mathbb{Z} \tag{21}$$

with $\lambda(t) = (i+1) - t2^{k+1}$.

The following results are proved in [9].

Theorem 6.5. *Let $\{F_i^0 : i \in \mathbb{Z}\}$ be compact sets with $\mathcal{L} = \sup\{\mathrm{haus}(F_i^0, F_{i+1}^0) : i \in \mathbb{Z}\} < \infty$. Then the sequence $\{F^{[k]}(\cdot)\}_{k \in \mathbb{Z}_+}$ in (21) converges uniformly on $\mathbb{R}$ to a set-valued function $F^\infty(\cdot) \in Lip(\mathcal{L})$.*

Theorem 6.6. *Let the initial sets for the subdivision be the samples $F_i^0 = F(i)$, $i \in \mathbb{Z}$, with $F \in Lip(\mathcal{L})$ on $\mathbb{R}$, and let $F^\infty(\cdot)$ be as in Theorem 6.5. Then*

$$\max_{x \in \mathbb{R}} \mathrm{haus}(F^\infty(x), F(x)) \le \mathcal{L}(7+m)/2.$$

Applying these results to the initial data relevant to the evaluation of the Schoenberg operator of functions defined on $[0, 1]$ we obtain,

Corollary 6.7. *Let $F \in Lip(\mathcal{L})$ on $[0, 1]$, and let*

$$F_i^0 = \begin{cases} F(i/N) & 0 \le i \le N, \\ \{0\} & \textit{otherwise.} \end{cases}$$

Then $F^\infty \in Lip(\mathcal{L}/N)$ on $\mathbb{R}$, and

$$\mathrm{haus}\big(F^\infty(Nx), F(x)\big) \le \frac{\mathcal{L}(7+m)}{2N}, \qquad x \in \left[\frac{m-1}{N}, 1\right].$$

Corollary 6.7 can be extended to $F \in \mathcal{H}_\alpha(\mathcal{L})$ to obtain error of order $O(N^{-\alpha})$.

6.2.4 Metric analogues of Schoenberg operators

The metric analogue of the Schoenberg operator of order m for a multifunction F and a set of equidistant points χ_N is

$$S_{m,N}^M(F, x) = \bigoplus_{i=0}^{N} b_m (Nx - i)\, F\left(\frac{i}{N}\right) = \left\{ \sum_{i=0}^{N} b_m (Nx - i)\, f_i : \; (f_0, ..., f_N) \in CH \right\},$$

where $CH = CH(F(0), F(1/N), ..., F(1))$.

By Corollary 5.3 and the known approximation result (13), we have for Lipschitz continuous SVFs

Corollary 6.8. *Let $F \in Lip(\mathcal{L})$ on $[0, 1]$. Then*

$$\mathrm{haus}\big(S_{m,N}^M(F, x), F(x)\big) = \left(2 + \left\lfloor \frac{m+1}{2} \right\rfloor\right) \frac{\mathcal{L}}{N}, \quad x \in \left[\frac{m-1}{N}, 1\right].$$

6.3 Polynomial Interpolants

For a real-valued function f the polynomial interpolation operator at the set of points χ is

$$P_\chi(f,x) = \sum_{i=0}^{N} l_i(x) f(x_i), \quad \text{with } l_i(x) = \prod_{j=0, j\neq i}^{N} \frac{x - x_j}{x_i - x_j}, \; i = 0, ..., N.$$

For $N > 1$, P_χ is not a positive operator. Thus the only possible adaptation of P_χ to SVFs is the metric analogue of Definition 5.1.
For a multifunction F, the metric polynomial interpolation operator at χ, is given by

$$P_\chi^M(F,x) = \bigoplus_{i=0}^{N} l_i(x) F(x_i) = \left\{ \sum_{i=0}^{N} l_i(x) f_i \, : \, (f_0, ..., f_N) \in CH \right\},$$

with $CH = CH(F(x_0), ..., F(x_N))$, $i = 0, 1, ..., N$.

To illustrate the metric set-valued polynomial interpolants, and to see the geometry of metric linear combinations of sets with negative coefficients, we present in Figure 6.1 an example of a metric parabolic interpolant to three sets in $\mathbb{R}$. The parabolic interpolant interpolates the data (x_i, A_i), $i = 0, 1, 2$ with $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$ and $A_0 = [1/4, 1/2] \cup [3/4, 1]$, $A_1 = [9/16, 11/16]$, $A_2 = A_0$.

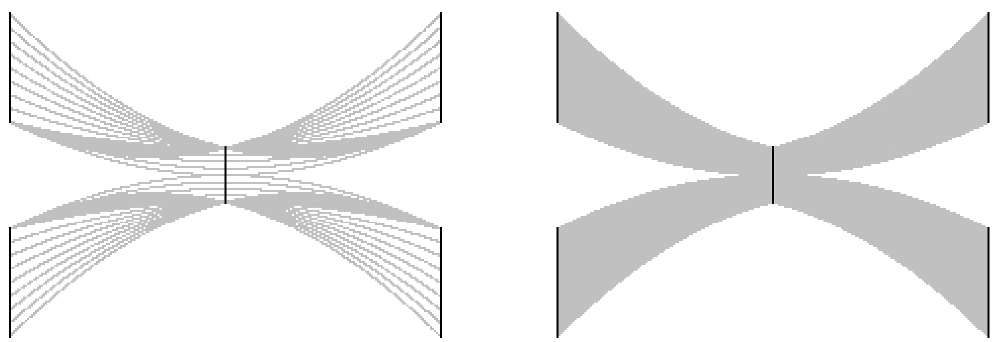

Figure 6.1. *Metric parabolic interpolant*

In the above figure the interpolated sets are depicted in black. The gray curves in the left figure are parabolic interpolants to the data (x_i, a_i), $i = 0, 1, 2$ for some metric chains $(a_0, a_1, a_2) \in CH(A_0, A_1, A_2)$. The right figure is the graph of the set-valued interpolant.

Next we consider a specific sequence of interpolation operators which, when operating on $F \in Lip(\mathcal{L})$, converges to F. Let the interpolation points χ be the roots of the Tchebyshev polynomial of degree $N+1$ on $[0, 1]$. It is known (see e.g. [14]) that for these points

$$\sum_{i=0}^{N} |l_i(x)| \leq C \log N.$$

For a real-valued function f,

$$|f(x) - \sum_{i=0}^{N} l_i(x) f(x_i)| \leq (1 + \sum_{i=0}^{N} |l_i(x)|) E_N(f),$$

with $E_N(f)$ the error of the best approximation by polynomials of degree N on $[0,1]$. Since $E_N(f) \leq C\omega(f, 1/N)$ (see [5], Chapter 7), we obtain for a Lipschitz continuous function f

$$|f(x) - \sum_{i=0}^{N} l_i(x) f(x_i)| \leq \frac{C \log N}{N}, \qquad x \in [0,1],$$

and the error in the interpolation of such a function at the roots of the Tchebyshev polynomials tends to zero as $N \to \infty$.

When adapting these interpolation operators to Lipschitz continuous SVFs, we get by Corollary 5.3 and by the observation that $|\chi| \leq \pi/(2N)$,

Corollary 6.9. *Let $F : [0,1] \to \mathcal{K}_n$, $F \in Lip(\mathcal{L})$, and let the points χ be the roots of the Tchebyshev polynomial of degree $N+1$ on $[0,1]$, then*

$$\mathrm{haus}(P_\chi^M(F,x), F(x)) \leq 2\mathcal{L}|\chi| + \frac{C \log N}{N} = O\left(\frac{\log N}{N}\right).$$

To the best of our knowledge this result is the first convergence result of non-positive operators to the approximated set-valued functions.

Although we get approximation results for adapted operators based on metric linear combinations, the direct computation of the approximants according to definitions (5), (6) is of high complexity. From Figure 6.1 it is clear that such a computation is redundant. Our aim is to devise efficient algorithms for the computation of these operators.

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References

[1] Z. Artstein. Piecewise linear approximations of set-valued maps. *Journal of Approx. Theory* **56** (1989), 41-47.

[2] R. Baier. Mengenwertige Integration und die diskrete Approximation erreichbarer Mengen. *Bayreuth. Math. Schr.* **50** (1995), 1-248.

[3] C. de Boor. A practical guide to spline. Springer-Verlag, 2001.

[4] J. W. S. Cassels. Measures of the non-convexity of sets and the Shapley–Folkman–Starr theorem. *Math. Proc. Camb. Phil. Soc.* **78** (1975), 433–436.

[5] R. DeVore and G. Lorentz. Constructive Approximation. Springer-Verlag, Berlin, 1993.

[6] T. Donchev and E. Farkhir. Moduli of smoothness of vector-valued functions of a real variable and applications. *Numer. Funct. Anal. Optimiz.* **11(586)** (1990), 497–509.

[7] N. Dyn. Subdivision schemes in computer-aided geometric design. In: Advances in Numerical Analysis, Vol. II, Wavelets, Subdivision Algorithms and Radial Basis Functions, (Ed W. Light), Clarendon Press, Oxford, 1992, 36–104.

[8] N. Dyn and E. Farkhi. Spline subdivision schemes for convex compact sets. *Journal of Comput. Appl. Mathematics* **119** (2000), 133–144.

[9] N. Dyn and E. Farkhi. Spline subdivision schemes for compact sets with metric averages. In: Trends in Approximation Theory, (Eds K. Kopotun, T. Lyche and M. Neamtu), Vanderbilt Univ. Press, Nashville, TN, 2001, 95–104.

[10] N. Dyn and E. Farkhi. Set-valued approximations with Minkowski averages - convergence and convexification rates. *Numer. Funct. Anal. Appl.* **25** (2004), 363–377.

[11] N. Dyn and E. Farkhi. Approximation of set-valued functions with compact images - an overview. In: Approximation and Probability, Banach Center Public., Vol 72, 2006, 71–84.

[12] N. Dyn, E. Farkhi and A. Mokhov. Approximations of set-valued functions by metric linear operators. *Constr. Approx.* **25** (2007), 193–209.

[13] N. Dyn and A. Mokhov. Approximations of set-valued functions based on the metric average. *Rendic. di Matem. S.VII* **26** (2006), 249–266.

[14] I.P. Natanson. Constructive Function Theory. Vol. III, Frederick Ungar, New York, 1965.

[15] H. Prautzsch, W. Boehm and M. Paluszny. Bezier and B-Spline Techniques. Springer, 2002.

[16] R. T. Rockafellar. Convex Analysis. Princeton University Press, Princeton, 1970.

[17] R. Schneider. Convex Bodies: the Brunn–Minkowski Theory. Cambridge University Press, Cambridge, 1993.

[18] R. Starr. Quasi-equilibria in markets with non-convex preferences. *Econometrica* **37** (1969), 25–38.

[19] R. Vitale. Approximation of convex set-valued functions. *J. Approximation Theory* **26** (1979), 301–316.

[20] J. Wallner and N. Dyn. Convergence and C^1 analysis of subdividision schemes on manifolds by proximity. *CAGD* **22** (2005), 593-622.