Interpolatory subdivision schemes

Nira Dyn[1,2]

[1] School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv 69978, Israel
[2] written while the author was visiting the IMA, Minneapolis, Minnesota, USA.

Abstract. Interpolatory subdivision schemes are refinement rules, which refine data by inserting values corresponding to intermediate points, using linear combinations of neighbouring points. Here we consider only refinements of regular meshes, which in the univariate case (subdivision for curve generation) are uniformly distributed points on the real line, and in the bivariate case (subdivision for surface generation) are either square grids or regular triangulations.

1 The univariate stationary case

1.1 Definitions and basic results

Let us first consider the simple case of refining univariate data of the form $(ih, f_i), i = 0, \ldots, N$, for some positive h. To get the values at the refined mesh $ih/2, i = 0, \ldots, 2N$, we need a method that "predicts" well the intermediate values at the points $(2i+1)h/2, i = 0, \ldots, N-1$. Such predictions make sense, if the data is sampled from a smooth function. We can then approximate the function by a good approximating system of functions (e.g. polynomials) and read the value of the approximating function at the intermediate point. If we want our refinement to be local, we have to use different approximating functions in different intervals. Further, if we want our rule of insertion to be the same everywhere, and based on the function values available, we should use polynomial interpolation (or interpolation by exponentials), as a method of approximation, since then the coefficients of the function values in the interpolating polynomial (interpolating sum of exponentials) depend only on the relative distances between the interpolation points, which is the same because of the regularity of the mesh of points. Polynomial interpolation is advantageous in the sense that the rule does not change with the refinement level i.e. with h.

Example 1. Suppose we use linear interpolation between the two endpoints of an interval to get the intermediate value, corresponding to the midpoint of the interval, then the rule is

$$f_{i+\frac{1}{2}} = \frac{1}{2}(f_i + f_{i+1}), \quad i = 0, \ldots, N-1.$$

To reiterate this rule we use the notation of levels of refinement; we denote the level of refinement by superscript and denote the initial data as values at level zero. Then the iterations become

$$f_{2i}^{k+1} = f_i^k \ , \quad i = 0, \ldots, 2^k N \ ,$$
$$f_{2i+1}^{k+1} = \frac{1}{2}(f_i^k + f_{i+1}^k) \ , \quad i = 0, \ldots, 2^k N - 1 \ , \quad k = 0, 1, 2, \ldots \ .$$

The value f_i^k is attached to the parameter value $i2^{-k}$. Note that this subdivision scheme is the degree 1 spline subdivision scheme. (See Section 2 in the previous chapter).

It is easy to see that the limit of this subdivision, i.e. the limit of the sequence of control polygons through the data of the different refinement levels,

$$(i2^{-k}, f_i^k), \quad i = 0, \ldots, 2^k N,$$

is the polygonal line going through the initial points $(i, f_i^0), \quad i = 0, \ldots, N$.

We can now state, in a formal definition, the notion of convergence of subdivision schemes. For that let us denote by π_m the space of univariate polynomials of degree not exceeding m.

Definition 1. A univariate subdivision scheme S, is termed (uniformly) convergent, if for any initial data $f^0 = \{f_i^0 \ : \ i \in \mathbb{Z}\}$, the sequence of polygonal lines $\{f^k(t) \ : \ k \in \mathbb{Z}_+\}$, converges uniformly, where

$$f^k(t) \in \pi_1, \ t \in (i2^{-k}, (i+1)2^{-k}), \quad f^k(2^{-k}i) = f_i^k, \quad i \in \mathbb{Z},$$

with $f^k = S^k f^0 = \{f_i^k \ : \ i \in \mathbb{Z}\}$, the data generated by the scheme at refinement level k. The limit function, which is necessarily continuous, is denoted by $S^\infty f^0$. It is also required that there exists f^0 such that $S^\infty f^0 \not\equiv 0$.

Remark 1. The first requirement in the above definition of convergence is equivalent to (see the Bibliographical notes) the existence of a continuous function f, such that for any closed interval $I \subset \mathbb{R}$

$$\lim_{k \to \infty} \sup_{i \in 2^k I} |f_i^k - f(2^{-k}i)| = 0 \ .$$

Obviously $f = S^\infty f^0$.

If we want to get a smoother limit than the one obtained in Example 1, through the initial data, we have to use more points in the "prediction" formula for the inserted values. As in the spline case, higher smoothness can be obtained only with a bigger mask, namely with a greater influence range of the points used to predict.

Example 2. We construct an insertion rule, based on 4 points: the endpoints of the interval where the midpoint is to be inserted, and the other two boundary points of its two neighbouring intervals. We interpolate the data (j, f_j^k), $j = i-1, i, i+1, i+2$, by a cubic polynomial $p_i \in \pi_3$, satisfying

$$p_i(j) = f_j^k, \quad j = i-1, i, i+1, i+2,$$

and then predict

$$f_{2i+1}^{k+1} = p_i(i + \frac{1}{2}).$$

This procedure yields the following subdivision scheme: the same rule as in the previous example at the "old" locations,

$$f_{2i}^{k+1} = f_i^k, \quad i = 0, \ldots, 2^k N,$$

and the new insertion rule

$$f_{2i+1}^{k+1} = \frac{9}{16}(f_i^k + f_{i+1}^k) - \frac{1}{16}(f_{i-1}^k + f_{i+2}^k), \quad i = 0, \ldots, 2^k N - 2, \quad k = 0, 1, 2, \ldots .$$

Note that there is a problem near the boundaries: with the data (i, f_i^0), $i = 0, \ldots, N$, we can insert only values at intervals which are not boundary intervals. Thus the limit function is defined only in the interval $(2, N-2)$. In case of no boundaries (see the periodic case in the solution of Exercise 7 in Section 6), or in case of a finite set of initial data, the limit function can be proved to be C^1 in the domain of its definition, as we shall verify in the next chapter.

On the other hand, it is easy to verify that the scheme is exact for cubic polynomials, (reproduces cubic polynomials), namely if the initial data is sampled from a cubic polynomial, then the values generated by the scheme at all refinement levels are on the same polynomial, and the limit function generated by the scheme is that same cubic polynomial.

Exercise 1. Verify the last statement.

We use in the above example local interpolation by cubic polynomials, but in order to get C^1 limit functions, it is necessary that the interpolatory scheme is exact for linear polynomials only. This follows from the result

Theorem 1. *An interpolatory subdivision scheme generates C^m limit functions, only if it is exact for polynomials of degree m.*

Proof: Let the interpolatory subdivision scheme S be given by the insertion rule

$$f_{2i+1}^{k+1} = \sum_j \alpha_j f_{i-j}^k, \tag{1}$$

and denote by $f = S^\infty f^0$ the limit function of the scheme from the initial data f^0. Consider the following n-th order divided difference of f

$$\delta_\epsilon^n f(x) = [x+\epsilon, x + 2^{-1}\epsilon, \cdots, x + 2^{-n}\epsilon] f = \epsilon^{-n} \sum_{i=0}^{n} b_i f(x + 2^{-i}\epsilon),$$

where $b_i^{-1} = \Pi_{j=0,j\neq i}^{n}(2^{-i} - 2^{-j}), i = 0,\dots,n$. For fixed $x \in 2^{-k}\mathbb{Z}$, $\epsilon = 2^{-\ell}$, $\ell > k$, we get after substituting $f(x+2^{-i-\ell}) = f^{i+\ell}_{2^{i+\ell}x+1}$, by its expression in terms of values at level $\ell + i - 1$,

$$\begin{aligned}\delta^n_{2^{-\ell}} f(x) &= 2^{\ell n}\sum_{i=0}^{n} b_i \sum_j \alpha_j f^{i+\ell-1}_{2^{\ell+i-1}x-j} \\ &= 2^{\ell n}\sum_{i=0}^{n} b_i \sum_j \alpha_j f(x - j2^{-\ell-i+1}) \\ &= 2^n \sum_j \alpha_j (-j)^n \delta^n_{j2^{-\ell+1}} f(x).\end{aligned}$$

Taking the limit as $\ell \to \infty$, and recalling the assumption that $f \in C^m$, we get for $n \leq m$,

$$f^{(n)}(x) = 2^n \sum_j \alpha_j (-j)^n f^{(n)}(x). \tag{2}$$

Since equation (2) holds for all $x \in 2^{-k}\mathbb{Z}$, $k \in \mathbb{Z}_+$, which is a dense set in $\mathbb{R}$, and since $f^{(n)}$ is continuous for $n \leq m$, equation (2) holds for $x \in \mathbb{R}$. Moreover, $f^{(n)}$ cannot be identically zero for all initial data. Thus, by choosing initial data such that for some x, $f^{(n)}(x) \neq 0$, we can divide equation(2) by $f^{(n)}(x)$, and obtain

$$(\frac{1}{2})^n = \sum_j \alpha_j (-j)^n, \quad n \leq m, \tag{3}$$

which proves that the scheme is exact for polynomials of degree not exceeding m. □

Exercise 2. Verify that (3) implies that the interpolatory scheme given by the insertion rule (1) is exact for polynomials of degree not exceeding m.

The last theorem implies that schemes that generate C^1 limit functions, must be exact for linear polynomials. The scheme in Example 1 is exact for linear polynomials, but generates only continuous functions. The scheme in Example 2 is exact for cubic polynomials, but generates only C^1 functions. The following example presents a one-parameter family of schemes, which are exact for linear polynomials, and generate C^1 limit functions.

Example 3. Here we construct a 4-point insertion rule, which is exact for linear polynomials, and depends on one parameter. This rule is based on the two rules in the previous two examples. Taking a convex combination of the insertion rules in the two previous examples, with weights $16w$ for the rule of Example 2 and $1 - 16w$ for the rule of Example 1, we get the general symmetric 4-point insertion rule which is exact for linear polynomials. The resulting subdivision scheme is,

$$f_{2i}^{k+1} = f_i^k, \quad i \in \mathbb{Z},$$
$$f_{2i+1}^{k+1} = -w(f_{i+2}^k + f_{i-1}^k) + (\frac{1}{2} + w)(f_i^k + f_{i+1}^k), \quad i \in \mathbb{Z}.$$

It can be shown by the analysis tools of the next chapter, that the above 4-point subdivision scheme generates continuous limit functions for $|w| < \frac{1}{4}$, and C^1 limit functions for $0 < w < \frac{1}{8}$. Note that the ranges of w given here, are not the best possible. Also note that for the special values $w = 0, w = \frac{1}{16}$ the scheme is the one of Examples 1 and 2 respectively. It can be also shown that the limit functions, generated by the the scheme of Example 2 do not have a second derivative at all the diadic points $\cup_{k \in \mathbb{Z}} 2^{-k}\mathbb{Z}$.

Exercise 3. Use the Eigenanalysis (presented in Chapter **??**) to verify the last statement.

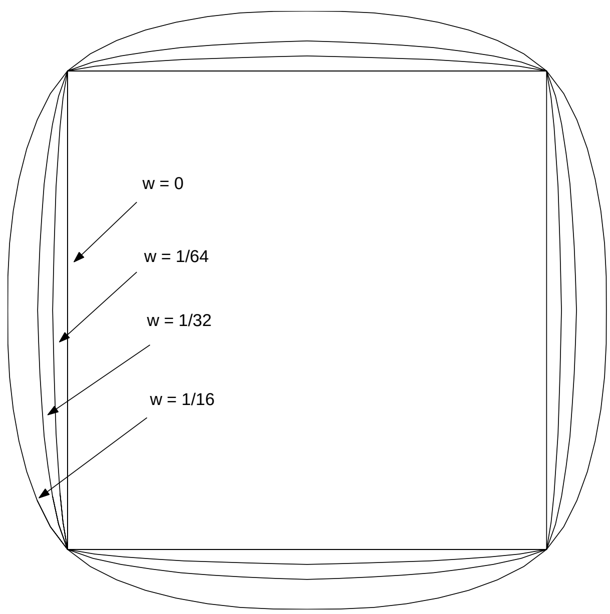

Fig. 1. Curves generated by the 4-point scheme (the polygonal lines after 4 iterations)

In Examples 1, 2 and 3 the insertion rules are independent of the level of refinement. Subdivision schemes with rules for defining the values at the next refinement level, which are independent of the refinement level are termed stationary schemes. (See Section 2 for examples of non-stationary subdivision schemes). Except for Section 2, all the schemes considered in the first four chapters of this book are stationary, and if not stated otherwise, all discussions refer to stationary subdivision schemes without stating it explicitly.

We are now ready to define formally the notion of interpolatory subdivision scheme.

Definition 2. A subdivision scheme is called interpolatory, if it is of the form

$$f_{2i}^{k+1} = f_i^k, \quad f_{2i+1}^{k+1} = \sum_j \alpha_j f_{i-j}^k, \quad i \in \mathbb{Z}, \quad k \in \mathbb{Z}_+.$$

There is a well studied one parametric family of interpolatory subdivision schemes, with a symmetric insertion rule, and with maximal polynomial exactness for the number of points involved in the insertion rule. This number is the parameter defining the schemes in the family. The scheme with insertion rule based on $2N$ points, is exact for polynomials of degree $\leq 2N-1$. The insertion rule is obtained by taking the value at the inserted point of an interpolating polynomial of degree $2N-1$ to the data at $2N$ symmetric points to the inserted one. These interpolatory subdivision schemes, termed the Dubuc-Deslauriers schemes, include the scheme of Example 1 for $N=1$, and the scheme of Example 2 for $N=2$.

The smoothness of the limit functions generated by these schemes increases with N, and is asymptotically $0.4N$ The method of proof of this result is based on Fourier analysis, and is beyond the scope of this chapter.

Remark 2.

1. Note that only a symmetric insertion rule makes sense, in the absence of any additional information on the initial data.
2. The values generated by a convergent interpolatory subdivision scheme S, given by $f^k = S^k f^0$, are on the limit function, namely

$$S^\infty f^0(2^{-k} i) = f_i^k, \quad i \in \mathbb{Z}, \quad k \in \mathbb{Z}_+.$$

Exercise 4.

1. Construct the Dubuc-Deslauriers subdivision scheme, based on 6 points.
2. Construct a 6-point insertion rule, by taking a convex combination of the 6-point insertion rule constructed in part 1 of the exercise, and the 4-point insertion rule of Dubuc-Deslauriers constructed in Example 2.
3. What is the maximal possible smoothness of the limit functions generated by the subdivision rule constructed in part 2 of the exercise?

1.2 The mask

The mask of a subdivision scheme consists of a set of coefficients, which measure the influence of a value at a location (point) on the values at neighbouring locations after subdivision. Since the schemes we consider are interpolatory and uniform (the same insertion rule everywhere), the coefficients of the mask are uniform, and determined by the insertion rule. There is always the coefficient 1, due to the interpolatory rule $f_{2i}^{k+1} = f_i^k$. The coefficient 1 indicates the location of the influencing point relative to the other influenced points.

Here are few examples:

1. The mask of the scheme in Example 1: $\frac{1}{2}, 1, \frac{1}{2}$
2. The mask of the scheme in Example 2: $-\frac{1}{16}, 0, \frac{9}{16}, 1, \frac{9}{16}, 0, \frac{1}{16}$
3. The mask of the scheme in Example 3: $-w, 0, \frac{1}{2} + w, 1, \frac{1}{2} + w, 0, -w$

We use here the convention that the coefficients which are not specified in the mask are all zero.

In general, the mask of an interpolatory subdivision scheme corresponding to the insertion rule

$$f_{2i+1}^{k+1} = \sum_{j=-L}^{U} \alpha_j f_{i-j}^k,$$

is

$$\alpha_{-L}, 0, \alpha_{-L+1}, 0, \cdots, \alpha_{-1}, 1, \alpha_0, 0, \alpha_1, 0, \cdots, 0, \alpha_U,$$

namely

$$a_0 = 1, \quad a_{2j} = 0, \quad a_{2j+1} = \alpha_j, \quad j = -L, -L+1, \ldots, U. \tag{4}$$

It is the mask $\mathbf{a} = \{a_i\}$ that encompasses all the information about the subdivision scheme, and on which the analysis of the properties of the scheme is based (See next two chapters).

Exercise 5. Determine the masks of the schemes constructed in Exercise 4.

1.3 The basic limit function

Most of the material in this section applies to all stationary subdivision schemes (also multivariate), and not only to univariate interpolatory schemes.

With each convergent subdivision scheme, there is associated a basic limit function, which for a spline subdivision scheme is the corresponding B-spline.

Definition 3. Denote by δ, the sequence which is zero everywhere except at 0 where it is 1. Then the basic limit function of a scheme S is $\phi_S = S^\infty \delta$.

By definition the basic limit function of S has compact support, if the mask of the scheme is of finite support. It is easy to follow the progress of

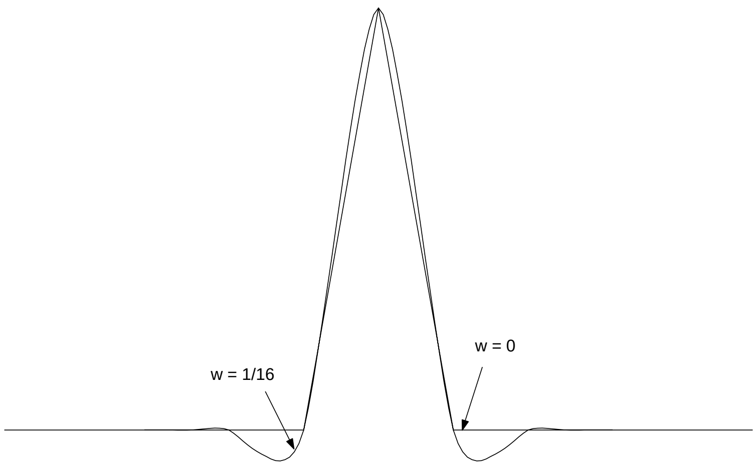

Fig. 2. The basic limit function of the 4-point scheme

the nonzero values in $S^k\delta$ as k increases. If $\{K_1, K_1+1, \dots, K_2\}$ denotes the support of the mask of S, then the support of ϕ_S is contained in $[K_1, K_2]$.

For a convergent interpolatory subdivision scheme (which is necessarily exact for π_0 by Theorem 1),

$$f^0 \equiv 1 \Longrightarrow f^k = S^k f^0 \equiv 1. \tag{5}$$

This property is also true for any convergent subdivision scheme (see Theorem **??** in Chapter **??**). A property of ϕ_S, which is valid for interpolatory schemes only, is

$$\phi_S(i) = \delta_i, \quad i \in \mathbb{Z}.$$

This follows from the fact that the initial data is interpolated by the limit function.

With the basic limit function of a subdivision scheme S, we can express any limit function generated by the scheme, in terms of the initial data. This is due to the linearity and uniformity of the scheme. Thus

$$S^\infty f^0(x) = \sum_j f_j^0 \phi_S(x-j) \tag{6}$$

The two relations (5)and (6), when combined, lead to

$$\sum_i \phi_S(\cdot - i) \equiv 1. \tag{7}$$

This property, termed **partition of unity**, plays a crucial role in ensuring the approximating nature of the scheme. More specifically, for initial data sampled from a continuous function F: $f^0 = \{f_j^0 = F(jh) : \ j \in \mathbb{Z}\}$, the limit of the subdivision scheme $S^\infty f^0 = \sum_{j\in\mathbb{Z}} f_j^0 \phi_S(\frac{\cdot}{h} - j)$ tends to F as $h \to 0$ (See Section 1.4).

The application of a convergent interpolatory subdivision scheme S, with insertion rule (1) to a set of control points in $\mathbb{R}^2$ or $\mathbb{R}^3$, $P^0 = \{P_j^0\}$, is according to,

$$P_{2i}^{k+1} = P_i^k \tag{8}$$

$$P_{2i+1}^{k+1} = \sum_j \alpha_j P_{i-j}^k \tag{9}$$

The limit of this scheme is a curve. Similar to (6), the limit curve can be written in terms of the basic limit function as,

$$S^\infty P^0(t) = \sum_j P_j^0 \phi_S(t-j) \tag{10}$$

It is (6) and (10) which justify the term "basic limit function".

Equation (10) is a parametric representation of the curve. Each component of the curve (2 components in $\mathbb{R}^2$, and 3 in $\mathbb{R}^3$) is a limit function of the subdivision scheme.

Remark 3. Note that if the points given are sampled from a closed curve, or generate a closed control polygon, there is no need for boundary treatment, even if the number of points is finite. The points are then arranged in a periodic way. (The first point is identified as the one after the last, and so on. See the discussion of the periodic case in the solution of Exercise 7 in Section 6).

By (6) and (10) the i smoothness of the basic limit function of a scheme determines the smoothness of the limit functions/curves generated by the scheme.

The basic limit function has an important property to wavelet constructions. This property is formulated in the next theorem.

Theorem 2. *The basic limit function of a convergent scheme S with a mask* $\mathbf{a}$*, satisfies the refinement equation,*

$$\phi_S(t) = \sum_j a_j \phi_S(2t-j).$$

This result follows from (6), from the stationarity of the scheme, which implies that

$$S^\infty(S\delta) = \sum_i (S\delta)_i \phi_S\left(\frac{x - \frac{i}{2}}{\frac{1}{2}}\right),$$

and from the observation that

$$(S\delta)_i = a_i.$$

1.4 Approximation order

The material in this section applies to subdivision schemes in general, and not only to interpolatory schemes.

The limit of a subdivision scheme, for initial data $f_i^0 = F(ih), \quad i \in \mathbb{Z}$ approximates the sampled function F, if F is smooth enough. The quality of the approximation improves with the density of sampling, namely with the reduction in the size of h. A measure of the rate of reduction in the approximation error as a function of h, is the notion of **approximation order**.

Definition 4. A subdivision scheme S, has approximation order n, if for the initial data $f_i^0 = F(ih), \quad i \in \mathbb{Z}$, with F smooth enough

$$|(F - S^\infty f^0)(x)| \leq Ch^n, \quad x \in \mathbb{R},$$

where the constant C may depend on F, n, x, S, but not on h.

The approximation order of a subdivision scheme depends directly on the space of polynomials for which the scheme is exact.

Theorem 3. *The approximation order of a convergent subdivision scheme S, which is exact for π_n, is $n+1$.*

Proof: Consider $G = F - T^n_{F;x}$, where $T^n_{F;x}$ is the Taylor polynomial of F of degree n at the point x. Then

$$G^{(j)}(x) = 0, \quad j = 0, 1, \ldots, n \ , \quad G^{(n+1)} = F^{(n+1)}. \tag{11}$$

Now, since S is exact for π_n, it follows that $T^n_{F;x} = S^\infty(f^0 - g^0)$, where $g_i^0 = G(ih), \quad i \in \mathbb{Z}$. Therefore, F and G have the same error,

$$F - S^\infty f^0 = F - T^n_{F;x} + S^\infty(f^0 - g^0) - S^\infty f^0 = G - S^\infty g^0. \tag{12}$$

In the following we bound the error of G at x.

The stationarity of S implies that

$$(S^\infty g^0)(x) = \sum_i g_i^0 \phi_S(\frac{x - ih}{h}),$$

where ϕ_S is the basic limit function of the scheme. By (7),

$$(G - S^\infty g^0)(x) = \sum_i \phi_S(\frac{x}{h} - i)(G(x) - g_i^0) = \sum_{i \in I_h(x)} (G(x) - G(ih))\phi_S(\frac{x}{h} - i),$$

where $I_h(x) = \{i : \phi_S(\frac{x}{h} - i) \neq 0\}$. Since the support of ϕ_S is finite, the number of elements in $I_h(x)$ is bounded by a constant independent of x, h.

Denote this constant by N_S, and denote the support of ϕ_S by $M_S = [K_1, K_2]$. Let $\|\phi_S\| = \max_{x\in M_S} |\phi_S(x)|$, and $\Omega_{x,h} = [x - hK_1, x + hK_2]$. Then we get

$$|(G-S^\infty g^0)(x)| = \|\phi_S\| \sum_{i\in I_h(x)} |G(x)-G(ih)| \leq N_S\|\phi_S\| \max_{y\in\Omega_{x,h}} |G(x)-G(y)|.$$

Now, (11) implies

$$\max_{y\in\Omega_{x,h}} |G(x) - G(y)| \leq \max_{y\in\Omega_{x,h}} |F^{(n+1)}(y)|[K_2 - K_1]^{n+1}h^{n+1},$$

which together with (12) completes the proof. □

For the spline subdivision schemes the approximation order is 2, because these schemes are exact for linear polynomials only. Yet, it is possible to get approximation order $m + 1$ for splines of degree m, Yet, it is possible to get approximation order $m + 1$ for splines of degree m, if instead of the initial data $F(jh)$ at the point $j \in \mathbb{Z}$, a certain fixed, local, finite linear combination of $F(jh)$ is used. (See the Bibliographical notes).

The scheme in Example 1 has approximation order 2, the scheme in Example 2 has approximation order 4, while the schemes in Example 3 have approximation order 2, except for the scheme with $w = \frac{1}{16}$ which has approximation order 4.

Exercise 6. Determine the approximation order of the $2N$-point Dubuc-Deslauriers scheme.

2 Non-stationary, univariate interpolatory schemes, exact for exponentials

It is non known how to generate limit curves which are circles by stationary subdivision schemes. Yet if we sample points from a circle, and then apply an insertion rule, which is exact for the functions

$$\{\exp(\lambda_j t),\ j = 1, 2, 3\},\quad \lambda_1 = 0,\ \lambda_2 = \sqrt{-1},\ \lambda_3 = -\lambda_2,$$

we get in the limit that circle from which the initial control points where sampled. This observation is based on the following parametric representation of the circle with center (x_0, y_0) and radius r,

$$x(t) = x_0 + r\cos t, \tag{13}$$
$$y(t) = y_0 + r\sin t, \tag{14}$$

and on the relation

$$\text{span}\{\exp(\lambda_j t),\ \ j = 1, 2, 3\} \supseteq \text{span}\{1, \cos t, \sin t\}.$$

The resulting scheme is non-stationary, since the insertion rule depends on the refinement level. Recently a related non stationary scheme, generating circles, was used in the construction of a bivariate scheme, which generates a surface of revolution from a given curve (see the Bibliographical notes).

Example 4. Here we construct a 2-point insertion rule which is exact for all functions in the span of the two exponential functions $\{1, \exp(\lambda t)\}$, $\lambda \in \mathbb{R}$. Note that $1 = \exp(0t)$, for all t. First assume that our data $\{f_j\}$ is given on the grid $h\mathbb{Z}$. Then interpolating the data (jh, f_j), $(jh+h, f_{j+1})$ by a function of the form $a + b\exp(\lambda t)$, we obtain the interpolant

$$p_{h,j}(t) = \frac{f_{j+1} - f_j \exp(\lambda h)}{1 - \exp(\lambda h)} + \frac{f_j - f_{j+1}}{1 - \exp(\lambda h)} \exp(-\lambda jh) \exp(\lambda t).$$

Substituting $t = jh + \frac{h}{2}$ into $p_{h,j}(t)$, we get the insertion rule

$$f_{j+\frac{1}{2}} = \frac{\exp(\lambda \frac{h}{2})}{1 + \exp(\lambda \frac{h}{2})} f_j + \frac{1}{1 + \exp(\lambda \frac{h}{2})} f_{j+1}$$

As can be easily observed, the insertion rule does not depend on j, but depends on h. This leads, by taking $h = 2^{-k}$, to the following non-stationary interpolatory subdivision scheme,

$$f_{2j}^{k+1} = f_j^k \tag{15}$$

$$f_{2j+1}^{k+1} = \frac{\exp(\lambda 2^{-(k+1)})}{1 + \exp(\lambda 2^{-(k+1)})} f_j^k + \frac{1}{1 + \exp(\lambda 2^{-(k+1)})} f_{j+1}^k. \tag{16}$$

As $k \to \infty$ the insertion rule above tends to the insertion rule of Example 1, at the rate $0(2^{-k})$.

The mask of this non-stationary scheme depends on the level of refinement (k), and is given by

$$a_0^k = 1, \quad a_{-1}^k = \frac{1}{1 + \exp(\lambda 2^{-(k+1)})}, \quad a_1^k = \frac{\exp(\lambda 2^{-(k+1)})}{1 + \exp(\lambda 2^{-(k+1)})}.$$

Exercise 7. Construct explicitly the mask of a 4-point insertion rule which interpolates circles exactly.
Hint: Construct an insertion rule by interpolation with a function from the span of the four functions $\{1, t, \cos t, \sin t\}$.

In general, an insertion rule based on $2N$ symmetric points, obtained by interpolating the values at the $2N$ points by a set of $2N$ real or complex exponentials always exists. Yet, in order to get a rule with real coefficients, the set of $2N$ exponentials should be such that a complex exponential is in the set only if its conjugate complex exponential is also in the set. The resulting insertion rule for a given set of $2N$ exponentials depends on the level of refinement, but is the same everywhere on the same level. (See Example 4, and the solution of Exercise 7 in Section 6). Moreover, it can be shown, that for an insertion rule with real coefficients, based on $2N$ symmetric points, which is exact for $2N$ exponentials, the coefficients as functions of the refinement level k, tend at the rate 2^{-k} to the coefficients of the $2N$-point Dubuc-Deslauries rule. Also, the limit functions generated by such a scheme have the same smoothness as the smoothness of the limit functions, generated by the $2N$-point Dubuc-Deslauriers scheme.

3 Tensor-product interpolatory schemes for surfaces

To define a subdivision scheme which generates surfaces, we first have to choose the topology of the control points. In this chapter we deal with two types of regular topologies for surfaces.

The first is that of a quad-mesh. The control points $P = \{P_{i,j} : \quad (i,j) \in \mathbb{Z}^2\}$, are in $\mathbb{R}^3$. Each control point has two indices which reflect the topology of the quad-mesh. The control point $P_{i,j}$ is connected by a topological "edge" to the four control points $P_{i+1,j}, P_{i-1,j}, P_{i,j+1}, P_{i,j-1}$. A set of four points P^1, P^2, P^3, P^4,, constitutes a topological "face", if $P^\ell, P^{\ell+1}$ are connected by a topological "edge", for $\ell = 1,2,3,4$ with $P^5 = P^1$. The quad-mesh is topologically equivalent to the regular square-grid consisting of the vertices $\mathbb{Z}^2$, with the grid lines parallel to the two axes.

In an interpolatory subdivision scheme relative to such a topology, there are two insertion rules, one for the insertion of a control point corresponding to an "edge", and one for the insertion of a "face" control point. The resulting scheme is of the form

$$P^{k+1}_{2i,2j} = P^k_{i,j} \ , \tag{17}$$

$$P^{k+1}_{2i+1,2j} = \sum_{\ell} \gamma_\ell P^k_{i-\ell,j} \ , \tag{18}$$

$$P^{k+1}_{2i,2j+1} = \sum_{\ell} \gamma_\ell P^k_{i,j-\ell} \ , \tag{19}$$

$$P^{k+1}_{2i+1,2j+1} = \sum_{\ell,\nu} \beta_{\ell,\nu} P^k_{i-\ell,j-\nu} \ . \tag{20}$$

The insertions (18) and (19) are of "edge" control points (edge-vertices) and the insertions in (20) are of "face" control points (face vertices).

If the above scheme is convergent, then the limit is a surface with all the control points from all refinement levels on it, namely

$$\cup_{k\in\mathbb{Z}_+} \cup_{(i,j)\in\mathbb{Z}^2} P^k_{i,j} \subset S^\infty P^0 .$$

Exercise 8. Define the notion of a convergent bivariate subdivision scheme on the square-grids with vertices $2^{-k}\mathbb{Z}^2$, in analogy to Definition 1.
Hint: Use bilinear interpolation to the values on the vertices $2^{-k}\mathbb{Z}^2$.

One method for getting the two insertion rules is by taking a tensor-product of a univariate interpolating scheme. The tensor product insertion rules obtained from the insertion rule (1) are as in (17)-(20), with

$$\gamma_i = \alpha_i, \quad i \in \mathbb{Z}, \quad \beta_{i,j} = \alpha_i \alpha_j, \quad (i,j) \in \mathbb{Z}^2. \tag{21}$$

It is not difficult to see that one can perform any bivariate tensor-product subdivision scheme by repeatedly performing the corresponding univariate scheme S, in the following way:

$$P_{\bullet,j}^{k+\frac{1}{2}} = S P_{\bullet,j}^{k}, \quad j \in 2^{-k}\mathbb{Z}, \tag{22}$$

$$P_{i,\bullet}^{k+1} = S P_{i,\bullet}^{k+\frac{1}{2}}, \quad i \in 2^{-(k+1)}\mathbb{Z}. \tag{23}$$

Here we used the convention that the univariate subdivision S is applied to the index, which is denoted by $\bullet$, while the other index is considered as a fixed parameter. Thus after the first stage the following points are defined

$$P^{k+\frac{1}{2}} = \{P_{i,j}^{k+\frac{1}{2}} \ : \ i \in 2^{-(k+1)}\mathbb{Z}, \ j \in 2^{-k}\mathbb{Z}\}.$$

After the second stage the defined points are

$$P^{k+1} = \{P_{i,j}^{k+1} \ : \ (i,j) \in 2^{-(k+1)}\mathbb{Z}^2\}.$$

Exercise 9. Verify that the interpolatory tensor-product scheme, given by (17)-(21), can be performed as a two-stage univariate scheme, regarding one index as a parameter, each stage with a different index as a parameter. Show that the result is independent of the order in which the two indices are chosen to be regarded as parameters.

The basic limit function of a tensor-product scheme $S \times S$, obtained from a convergent univariate scheme S, is related to ϕ_S by

$$\phi_{S\times S}(t_1, t_2) = \phi_S(t_1)\phi_S(t_2).$$

The limit surface generated by $S \times S$ from the initial control points P^0 is

$$(S \times S)^\infty P^0(u,v) = \sum_{(i,j)\in\mathbb{Z}^2} P_{i,j}^0 \phi_S(u-i)\phi_S(v-j). \tag{24}$$

Equation (24) is a parametric representation of the limit surface. Each of its three components

$$x_1(u,v), \quad x_2(u,v), \quad x_3(u,v),$$

is a limit function of $S \times S$, since for $f^0 = \{f_\mu^0 \ : \ \mu \in \mathbb{Z}^2\}$, regarded as function values at the points $\{\mu \ : \ \mu \in \mathbb{Z}^2\}$, the limit function generated by the subdivision scheme $S \times S$, is given by

$$(S \times S)^\infty f^0(u,v) = \sum_{(i,j)\in\mathbb{Z}^2} f_{i,j}^0 \phi_S(u-i)\phi_S(v-j). \tag{25}$$

Remark 4. The representation (24) is not used in applications. It is the collection of control points at a certain refinement level with the corresponding topology that is used. In practice the few first refinement levels (3-5), are sufficient for representing the limit surface.

Exercise 10. Consider the tensor-product scheme $S \times S$, derived from a convergent univariate interpolatory subdivision scheme S, with mask $\mathbf{a} = \{a_i\}$.

1. Determine the mask of $S \times S$ in terms of the coefficients in the mask $\mathbf{a} = \{a_i\}$.
2. If S is exact for π_m (S reproduces π_m), what space of bivariate polynomials is reproduced by $S \times S$?
3. Determine the two tensor-product insertion rules corresponding to the univariate subdivision scheme in Example 2.

4 The butterfly scheme - an interpolatory subdivision scheme on triangulations

The second type of a regular topology of control points that we consider for surfaces, is that of a regular triangulation, where each vertex has valency six, namely is connected to six other vertices in the triangulation.

We recall here, that a triangulation is a topological net defined on a set of vertices and is realizable as a collection of connected planar triangles in $\mathbb{R}^3$. A triangulation consists of a set of vertices, a set of edges, each connecting two vertices, and a set of triangles, each consisting of three vertices and three edges. All the vertices of the triangulation are vertices of triangles in the triangulation. Each edge in the triangulation belongs to exactly two triangles (a triangulation without boundaries). Two triangles with a common edge are termed neighbouring triangles.

A regular triangulation (each vertex has valency six) is topologically equivalent to the "three directional grid", which consists of the vertices of $\mathbb{Z}^2$ with edges connecting the point (i,j) to the points $(i \pm 1, j)$, $(i, j \pm 1)$, $(i+1, j+1)$, $(i-1, j-1)$, for all $(i,j) \in \mathbb{Z}^2$.

An interpolatory scheme on a regular triangulation consists of one insertion rule for new vertices corresponding to the edges of the current triangulation. The vertices of the refined triangulation are the union of the vertices of the current triangulation and those inserted. In the refined triangulation, an inserted vertex P, corresponding to an edge e in the current triangulation, is connected to the two vertices belonging to e in the current triangulation, and to the four inserted vertices corresponding to edges in the current triangulation, which constitute the two triangles sharing e (see Figure 3).

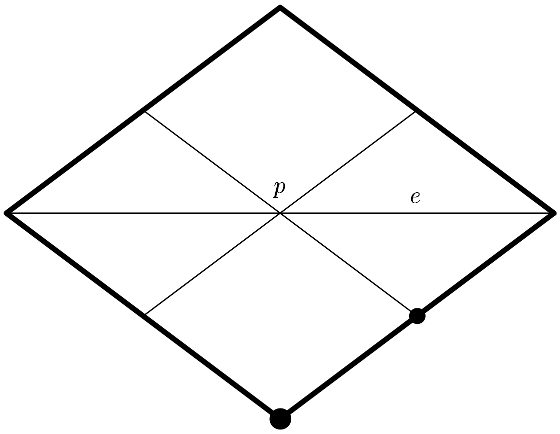

Fig. 3. The connectivity in the refined triangulation

Thus all inserted points have valency six, while the vertices which belong also to the current triangulation retain their valencies. It follows from this observation, that if the initial triangulation is regular, so are all the refined triangulations.

The butterfly scheme is an interpolatory subdivision scheme on triangulations, which in a certain sense extends the univariate four point scheme of Example 3.

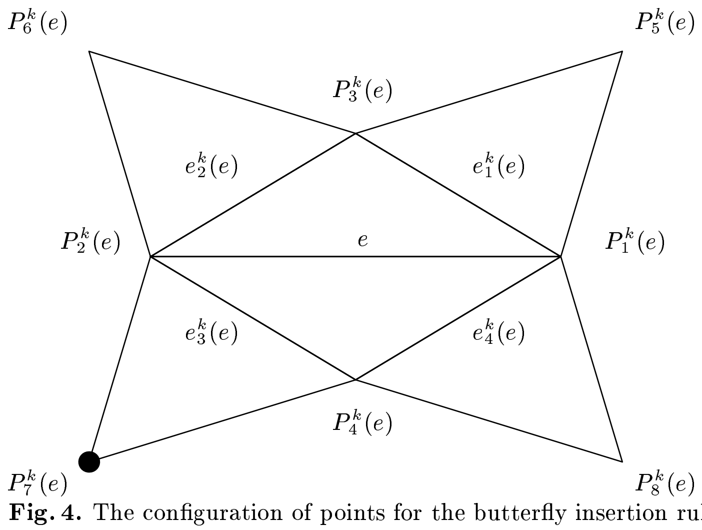

Fig. 4. The configuration of points for the butterfly insertion rule

To present the insertion rule of the butterfly scheme, we introduce the following notations (see Figure 4.). The vertices of an edge e in the triangulation $\mathbf{T^k}$ at refinement level k, are denoted by $P_1^k(e),\ P_2^k(e)$. Further, we denote by $T_1^k(e),\ T_2^k(e)$ the two triangles sharing e, by $P_3^k(e),\ P_4^k(e)$ the vertices of $T_1^k(e),\ T_2^k(e)$, which are not on e, and by $e_j^k(e),\ j=1,2,3,4$, the four edges of these two triangles, which are not e. Finally, $P_j^k(e),\ j=5,6,7,8$, denote the vertices of the neighbouring triangles to $T_1^k(e),\ T_2^k(e)$, which are different from $P_j^k(e),\ j=1,2,3,4$.

The insertion rule for the vertex corresponding to the edge e is

$$P_e^{k+1} = \frac{1}{2}(P_1^k(e) + P_2^k(e)) + 2w(P_3^k(e) + P_4^k(e)) - w\sum_{j=5}^{8} P_j^k(e). \qquad (26)$$

Exercise 11. Show that on the three directional grid, if the values given are constant in one of the three grid directions $(1,0),\ (0,1),\ (1,1)$, namely if $f_{i,j}^0 = f_{0,j}^0$ or $f_{i,j}^0 = f_{i,0}^0$ or $f_{i,j}^0 = f_{0,j-i}^0$, then the butterfly scheme inserts new values which are constant in the same direction as the original data is, and could be computed by the univariate 4-point scheme of Example 3, along the other two grid directions.

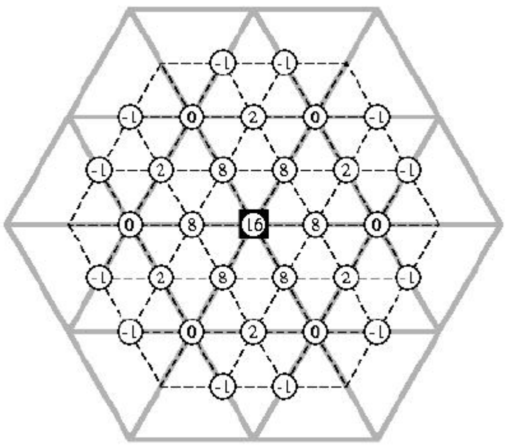

Fig. 5. The mask of the butterfly scheme with $w = \frac{1}{16}$ on a symmetric 3-directional mesh

The mask of the butterfly scheme with $w = \frac{1}{16}$ on a regular grid of equilateral triangles is given in Figure 5. Note that the weights in the figure have to be divided by 16.

The butterfly scheme, when applied to values at the vertices of $\mathbb{Z}^2$, and when the initial triangulation is that of the three directional grid, is exact for the space of all bivariate linear polynomials $\pi_1(\mathbb{R}^2)$. For the particular choice $w = \frac{1}{16}$, it is exact for all bivariate cubic polynomials $\pi_3(\mathbb{R}^2)$.

Exercise 12. Verify the last statement.

Example 5. In this example a coarse triangulation is refined four times by the butterfly insertion rule (26). The initial triangulation and the refined triangulation after 4 iterations are shown in Figures 6,7 respectively.

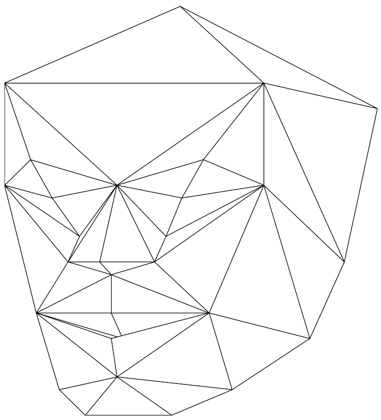

Fig. 6. Initial triangulation

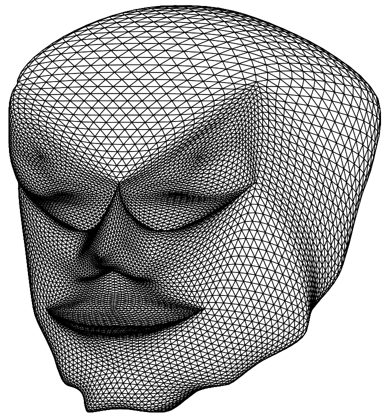

Fig. 7. 4 iterations with the butterfly insertion rule

Facts: The butterfly subdivision scheme on a regular triangulation generates C^1 limit surfaces/functions for $0 < w \leq w^*$, with $w^* > \frac{1}{12}$. The biggest w^* is not known. The butterfly scheme can be applied to triangulations of any topology. Yet the butterfly insertion rule (26) generates C^1 surfaces/functions only near vertices of valency between 4 to 7. There are special insertion rules to be applied near extraordinary vertices of the triangulation (vertices of valency not equal to 6), which together with the butterfly insertion rule (26) applied near vertices of valency 6, result in C^1 surfaces/functions. These special insertion rules togethder with the butterfly insertion rule (constituting together the 'modified butterfly scheme'), generate 'good-looking' surfaces near extraordinary vertices, and improve the quality of the surfaces/functions, generated by the butterfly insertion rule (26), near extraordinary points of valency between 4 to 7 (see the Bibliographical notes).

5 Bibliographical notes

A more detailed tutorial, but less up to-date, on subdivision schemes in Computer Aided Geometric Design is [6]. An extensive mathematical treatment of stationary subdivision schemes, where the basic limit function is investigated, and where the refinement equation of Theorem 2 is derived for the first time is [1]. Also the equivalence discussed in Remark 1 is proved there.

A method for achieving the maximal possible approximation order, by a judgicious choice of the initial data, is developed in [15]. The 4-point schemes of Example 3 and the corresponding tensor-product schemes on regular quad-meshes, are presented and analyzed in [7]. The Dubuc-Deslauriers schemes (DD-schemes) are presented and investigated in [4,5,2]. The DD 4-point scheme is further analyzed in [3]. The 6-point scheme is discussed in [6]. A proof of an extension of Theorem 3 to the multivariate setting can be found in [9]. Non-stationary interpolatory subdivision schemes exact for a finite number of exponentials are studied in [16,11]. Non-stationary subdivision schemes, generating circles are first presented in [10], and extended to schemes generating surfaces of revolution in [17]. The tensor-product of the 4-point DD-scheme is extended to general quad-meshes (quad-meshes with extraordinary control points, where the valency is different from four) in [14]. The butterfly scheme is presented in [8], and shown to generate C^1 limit functions on regular triangulations for $w \in (0, w^*)$, in [12,13]. In the latter it is shown that $w^* > \frac{1}{12}$. The special rules for the 'modified butterfly scheme' (near extraordinary points) are given in [18].

6 Solutions of selected exercises

Exercise 1. Verify that the scheme from Example 2 is exact for cubics.

Solution: The insertion rule in Example 2 is

$$f_{2i+1}^{k+1} = \frac{9}{16}(f_i^k + f_{i+1}^k) - \frac{1}{16}(f_{i-1}^k + f_{i+2}^k), \ \ i = 0, \dots, 2^k N - 2, \ \ k = 0, 1, 2, \dots .$$

To check that it is exact for cubics, it is enough by linearity to check that it is exact for the monomials x^j, $j = 0, 1, 2, 3$. Let $f_i^k = 1$, $i \in \mathbb{Z}$, then since $2(\frac{9}{16} - \frac{1}{16}) = 1$, $f_{2i+1}^{k+1} = 1$, $i \in \mathbb{Z}$, and the scheme is exact for constants. Observe that since spaces of polynomials up to a fixed degree are shift invariant, it is sufficient to check the claim for f_1^{k+1}.

For x^1 let $f_i^k = (i2^k),\ i \in \mathbb{Z}$. Then

$$f_1^{k+1} = \frac{9}{16}2^{-k} - \frac{1}{16}[-2^{-k} + 2(2^{-k})] = \frac{1}{2}2^{-k} = 2^{-(k+1)} .$$

Similarly for x^2, let $f_i^k = (i2^k)^2$. Then

$$f_1^{k+1} = \frac{9}{16}(2^{-k})^2 - \frac{1}{16}[2^{-2k} + 4(2^{-2k})] = \frac{1}{4}2^{-2k} = (2^{-(k+1)})^2 .$$

The verification of exactness for x^3 follows from the exactness for π_2 and from the symmetry of the insertion rule relative to $\frac{1}{2}$.

Exercise 3. Show that the limit functions generated by the 4-point scheme are not in C^2.

Solution: For all values of w except for $w = \frac{1}{16}$, the scheme is exact for linear polynomials, and is not exact for quadratic polynomials. Hence by Theorem 1 the scheme does not generate C^2 functions.

For $w = \frac{1}{16}$, the scheme is exact for cubic polynomials (see Exercise 1), therefore the necessary condition for C^2 limit functions of Theorem 1 holds. We use the local representation of the scheme in the neighbourhood of a diadic point, in terms of a finite dimensional matrix, to show that the scheme with $w = \frac{1}{16}$ generates limit fuctions which do not have a second derivative at diadic points.

Let a diadic point $t = n_0 2^{-m}$ be fixed. Consider the five diadic points closest to t at level $k+m$, namely $t + j2^{-k-m},\ j = -2, -1, 0, 1, 2$, with the corresponding values generated by the scheme at these points

$$F^k = (F_{-2}^k, F_{-1}^k, F_0^k, F_1^k, F_2^k)^T .$$

Then at level $k+m+1$ we have

$$F^{k+1} = AF^k , \tag{27}$$

where the matrix A is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1/16 & 9/16 & 9/16 & -1/16 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1/16 & 9/16 & 9/16 & -1/16 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

implying that $F_0^k = F_0^0$ for all $k \in \mathbb{Z}_+$, as is expected from the fact that the scheme is interpolatory.

The first two largest eigenvalues of A are $\lambda_1 = 1,\ \lambda_2 = \frac{1}{2}$, with corresponding eigenvectors $v^{(1)} = \mathbf{e} = (1,1,1,1,1)^T$ and $v^{(2)} = \ell = (-2,-1,0,1,2)^T$.

The other three eigenvalues are $\lambda_3 = \lambda_4 = \frac{1}{4}$, $\lambda_5 = \frac{1}{8}$, with eigenvectors $v^{(j)}$, $j = 3, 4, 5$. Note that $v^{(4)}$ is a generalized eigenvector satisfying $Av^{(4)} = \frac{1}{4}v^{(4)} + v^{(3)}$. Expanding F^0 in the basis of the eigenvectors of A, $F^0 = \sum_{j=1}^{5} \alpha_j v^{(j)}$, we get

$$F^k = \sum_{j=3}^{5} \alpha_j A^k v^{(j)} + \alpha_2 (\frac{1}{2})^k \ell + F_0^0 \mathbf{e} \ , \tag{28}$$

and $\lim_{k\to\infty} F^k = F_0^0 \mathbf{e}$.

Denote $f = S^\infty f^0$. If $f'(t)$ exists, then

$$f'(t)\mathbf{e}* = \lim_{k\to\infty} 2^{m+k} D^{-1}(F^k - F_0^0\mathbf{e}) \ ,$$

where $\mathbf{e}* = (1, 1, 0, 1, 1)^T$, and where $D = \mathrm{diag}\{-2, -1, 1, 1, 2\}$. But by (28)

$$f'(t)\mathbf{e}* = \lim_{k\to\infty} 2^m D^{-1} \sum_{j=3}^{5} \alpha_j (2A)^k v^{(j)} + 2^m \alpha_2 D^{-1} \ell \ .$$

Thus the limit exists and equals $2^m \alpha_2 \mathbf{e}*$.

Similarly if $f''(t)$ exists, then

$$f''(t)\mathbf{e}* = \lim_{k\to\infty} 4^{m+k} D^{-2}(F^{k-1} - 2F^k + F_0^0\mathbf{e}) \ ,$$

which by (28) would lead to

$$f''(t)\mathbf{e}* = \lim_{k\to\infty} 4^{m+1} D^{-2}(I - 2A) \sum_{j=3}^{5} \alpha_j (4A)^{k-1} v^{(j)} \ .$$

But the limit above does not exist since the eigenvalue $\frac{1}{4}$ has geometric multiplicity 1 and algebraic multiplicity 2.

Exercise 4.

1. Construct the Dubuc-Deslauriers subdivision scheme, based on 6 points.
2. Construct a 6-point insertion rule, by taking a convex combination of the 6-point insertion rule constructed in part 1 of the exercise, and the 4-point insertion rule of Dubuc-Deslauriers constructed in Example 2.
3. What is the maximal possible smoothness of the limit functions generated by the subdivision rule constructed in part 2 of the exercise?

Solution:

1. To construct the 6-point Dubuc-Deslauriers scheme, we solve for the coefficients of the symmetric 6-point insertion rule $\{\alpha_j,\ j=\pm3,\pm2,\pm1\}$, by requiring the rule to be exact for $x^i,\ i=0,1,2,3,4,5$. Since π_5 is invariant under shifts, it is sufficient to consider insrtion at $\frac{1}{2}$ based on the values at the points $-2,-1,0,1,2,3$. Moreover, by the symmetry of the configuration of points relative to the inserted one, we get that $\alpha_{-j}=\alpha_j,\ j=1,2,3$. Thus the system of equations for the unknowns $\alpha_j,\ j=1,2,3$, is

$$2(\alpha_1+\alpha_2+\alpha_3)=1\ ,\quad (\text{exactness for } x^0)$$

$$\alpha_1(1+0)+\alpha_2(4+1)+\alpha_3(9+4)=\frac{1}{4}\ ,\quad (\text{exactness for } x^2)$$

$$\alpha_1(1+0)+\alpha_2(16+1)+\alpha_3(81+16)=\frac{1}{16}\ .\quad (\text{exactness for } x^4)$$

Note that exactness for the monomials x^1, x^3, x^5, follows from the symmetry of the problem together with the above three equations. The solution of this system is $\alpha_1=\frac{75}{128}$, $\alpha_2=-\frac{25}{256}$, $\alpha_3=\frac{3}{256}$, which yields the insertion rule

$$f^{k+1}_{2j+1}=\frac{75}{128}(f^k_j+f^k_{j+1})-\frac{25}{256}(f^k_{j-1}+f^k_{j+2})+\frac{3}{256}(f^k_{j-2}+f^k_{j+3})\ .$$

2. Let $0\le\mu\le1$, then the convex combination of the two insertion rules yields the coefficients

$$\alpha_1=\frac{9}{16}(1-\mu)+\mu\frac{75}{128}=\frac{9}{16}+\frac{3}{128}\mu\ ,$$

$$-\alpha_2=\frac{1}{16}(1-\mu)+\mu\frac{25}{256}=\frac{1}{16}+\frac{9}{256}\mu\ ,$$

$$\alpha_3=\frac{3}{256}\mu\ .$$

Defining $\theta=\frac{3}{256}\mu$, we get the 6-point scheme (see the Bibliographical notes)

$$f^{k+1}_{2j+1}=(\frac{9}{16}+2\theta)(f^k_j+f^k_{j+1})-(\frac{1}{16}+3\theta)(f^k_{j-1}+f^k_{j+2})+\theta(f^k_{j-2}+f^k_{j+3})\ . \tag{29}$$

3. For $\theta\neq\frac{3}{256}$ the insertion rule (29) is exact for cubic polynomials. Therefore by Theorem 1 the corresponding limit functions can be at most C^3. For $\theta=\frac{3}{256}$ the insertion rule (29) is exact for π_5 and therefore by Theorem 1 the functions generated by the scheme can be at most C^5.

Exercise 6. Determine the approximation order of the $2N$-point Dubuc-Deslauriers scheme.

Solution: Since the $2N$-point Dubuc-Deslauriers scheme is exact for π_{2N-1}, the approximation order of the scheme, in view of Theorem 3, is $2N$.

Exercise 7. Construct explicitly the mask of a 4-point insertion rule which interpolates circles exactly.
Hint: Construct an insertion rule by interpolation with a function from the span of the four functions $\{1, t, \cos t, \sin t\}$.

Solution: Since spaces of exponentials are shift invariant, and since the space spanned by $\{1, t, \cos t, \sin t\}$ corresponds to the exponents $\lambda_1 = \lambda_2 = 0, \lambda_3 = i, \lambda_4 = -i$, with $i^2 = -1$, we consider the k-th level insertion rule at the point $\theta 2^{-k-1}$ based on the values at the points $-\theta 2^{-k}, 0, \theta 2^{-k}, 2\theta 2^{-k}$.

The system of equations for the coefficients α_j^k, $j = -1, 0, 1, 2$ in the insertion rule at level k is

$$\begin{aligned}
\alpha_{-1}^k + \alpha_0^k + \alpha_1^k + \alpha_2^k &= 1 \\
-\theta 2^{-k}\alpha_{-1}^k + \theta 2^{-k}\alpha_1^k + 2\theta 2^{-k}\alpha_2^k &= \theta 2^{-k-1} \\
\cos(\theta 2^{-k})\alpha_{-1}^k + \alpha_0^k + \cos(\theta 2^{-k})\alpha_1^k + \cos(2\theta 2^{-k})\alpha_2^k &= \cos(\theta 2^{-k-1}) \\
\sin(\theta 2^{-k})\alpha_{-1}^k + \sin(\theta 2^{-k})\alpha_1^k + \sin(2\theta 2^{-k})\alpha_2^k &= \sin(\theta 2^{-k-1})\ .
\end{aligned}$$

The first two equations correspond to exactness for t^0, t^1. The other two equations correspond to exactness for $\cos t, \sin t$. The solution of this system (obtained with the software Mathematica) leads to the insertion rule

$$\begin{aligned}
f_{2j+1}^{k+1} &= \frac{-1}{16\cos^2(\theta 2^{-k-2})\cos(\theta 2^{-k-1})}(f_{j-1}^k + f_{j+2}^k) \\
&+ \frac{(1 + 2\cos(\theta 2^{-k-1}))^2}{16\cos^2(\theta 2^{-k-2})\cos(\theta 2^{-k-1})}(f_j^k + f_{j+1}^k)\ .
\end{aligned}$$

Note that this insertion rule tends to the 4-point Dubuc-Deslauriers insertion rule of Example 2 as k tends to infinity, at the rate $O(2^{-k})$.

The above insertion rule together with $f_{2j}^{k+1} = f_j^k$, when applied to equidistributed points $\{f_j^0,\ j = 0, \ldots, N\}$ on a circle arc of angle $N\theta$, fills the circle arc through the points f_j^0, $j = 2, \ldots, N-2$. In case the points are equidistributed on the whole circle, then the application of this subdivision scheme, recovers the circle. For this we have to deal with the periodic case. Let $N_k = 2^k(N+1) - 1$, then at level k the control points are f_j^k, $j = 0, \ldots, N_k$. Using the periodic boundary conditions $f_j^k = f_{N_k+1+j}^k$, $j = -1, -2$, and $f_{N_k+j}^k = f_{j-1}^k$, $j = 1, 2$, the points at level $k+1$ are defined with the above rules. Note that with these boundary conditions the new inserted point $f_{N_{k+1}}^{k+1}$ can be obtained also as f_{-1}^{k+1}.

Exercise 11. Show that on the three directional grid, if the values given are constant in one of the three grid directions $(1,0)$, $(0,1)$, $(1,1)$, namely if $f^0_{i,j} = f^0_{0,j}$ or $f^0_{i,j} = f^0_{i,0}$ or $f^0_{i,j} = f^0_{0,j-i}$, then the butterfly scheme inserts new values which are constant in the same direction as the original data is, and could be computed by the univariate 4-point scheme of Example 3, along the other two grid directions.

Solution: By the symmetry of the insertion rule of the butterfly scheme relative to the three directions of the mesh, it is sufficient to consider the case $f^0_{i,j} = f^0_{0,j}$, $i,j \in \mathbb{Z}$. The insertion rule of the butterfly scheme has three different forms on the three directional mesh. For a new vertex corresponding to an edge in the $(1,0)$ direction, the insertion rule is

$$\begin{aligned} f^{k+1}_{2i+1,j} &= \frac{1}{2}(f^k_{i,j} + f^k_{i+1,j}) + 2w(f^k_{i+1,j+1} + f^k_{i,j-1}) \\ &\quad - w(f^k_{i+2,j+1} + f^k_{i,j+1} + f^k_{i-1,j-1} + f^k_{i+1,j-1}) \ . \end{aligned}$$

For a new vertex corresponding to an edge in the $(0,1)$ direction the insertion rule is

$$\begin{aligned} f^{k+1}_{i,2j+1} &= \frac{1}{2}(f^k_{i,j} + f^k_{i,j+1}) + 2w(f^k_{i+1,j+1} + f^k_{i-1,j}) \\ &\quad - w(f^k_{i+1,j+2} + f^k_{i+1,j} + f^k_{i-1,j-1} + f^k_{i-1,j+1}) \ . \end{aligned}$$

Finally, for a new vertex corresponding to an edge in the $(1,1)$ direction the rule is

$$\begin{aligned} f^{k+1}_{2i+1,2j+1} &= \frac{1}{2}(f^k_{i,j} + f^k_{i+1,j+1}) + 2w(f^k_{i+1,j} + f^k_{i,j+1}) \\ &\quad - w(f^k_{i+2,j+1} + f^k_{i+1,j+2} + f^k_{i-1,j} + f^k_{i,j-1}) \ . \end{aligned}$$

Under the assumption $f^k_{i,j} = f^k_{0,j}$, $i,j \in \mathbb{Z}$, namely that at level k the values generated by the scheme are independent of the first index, we get from the first insertion rule

$$\begin{aligned} f^{k+1}_{2i+1,j} &= \frac{1}{2}(f^k_{0,j} + f^k_{0,j}) + 2w(f^k_{0,j+1} + f^k_{0,j-1}) \\ &\quad - w(f^k_{0,j+1} + f^k_{0,j+1} + f^k_{0,j-1} + f^k_{0,j-1}) = f^k_{0,j} \ , \end{aligned}$$

implying that at level $k+1$ the values attached to new points on the old grid lines with fixed second index depend only on the second index. For a new vertex corresponding to the $(0,1)$ direction we get from the second insertion rule

$$\begin{aligned} f^{k+1}_{i,2j+1} &= \frac{1}{2}(f^k_{0,j} + f^k_{0,j+1}) + 2w(f^k_{0,j+1} + f^k_{0,j}) \\ &\quad - w(f^k_{0,j+2} + f^k_{0,j} + f^k_{0,j-1} + f^k_{0,j+1}) \\ &= (\frac{1}{2} + w)(f^k_{0,j} + f^k_{0,j+1}) - w(f^k_{0,j+2} + f^k_{0,j-1}) \ , \end{aligned}$$

which is the insertion rule of the 4-point scheme with respect to the second index. The third insertion rule gives the same rule as above. Thus, new points at level $k+1$ which are on new grid lines with constant second index are computed according to the 4-point scheme.

In conclusion, the grid lines with fixed second index correspond to univariate points, and the values attached to these grid lines are computed by the 4-point scheme. More precisely, let $g_j^k = f_{0,j}^k,\ j \in \mathbb{Z}$, then g^k evolves according to the 4-point scheme, and $f_{i,j}^k = g_j^k,\ i,j \in \mathbb{Z}$.

Acknowledgement: The author thanks Nurit Alkalai, Tom Duchamp and Malcolm Sabin for providing figures to this chapter.

References

1. Cavaretta, A.S., W. Dahmen and C.A. Micchelli, Stationary subdivision, Memoirs of AMS **93**, (1991).
2. Daubechies, I., *Ten Lectures on Wavelets*, CBMS-NSF conference series in Applied Math. SIAM (1992).
3. Daubechies, I. and J. Lagarias, Two scale difference equations II. Local regularity, infinite products of matrices and fractals, SIAM J. Math. Anal., **23** (1992) 1031-1079.
4. Deslauriers, G. and S. Dubuc, Symmetric iterative interpolation, Constr. Approx. **5** (1989), 49-68.
5. Dubuc, S., Interpolation through an iterative scheme, J. Math. Anal. and Appl. **114** (1986), 185-204.
6. Dyn, N., Subdivision schemes in Computer-Aided Geometric Design, in *Advances in Numerical Analysis, Vol. II*, W. Light, (ed.), Wavelets, Subdivision Algorithms and Radial Basis Functions, Clarendon Press, Oxford, (1992) 36–104.
7. Dyn, N., J.A. Gregory and D. Levin, A four-point interpolatory subdivision scheme for curve design, Computer Aided Geometric Design **4** (1987), 257-268.
8. Dyn, N., J.A. Gregory and D. Levin, A butterfly subdivision scheme for surface interpolation with tension control, ACM Trans. on Graphics, **9** (1990), 160-169.
9. Dyn, N. and D. Levin, Interpolatory subdivision schemes for the generation of curves and surfaces, in *Multivariate Approximation and Interpolation*, W. Haussmann and K. Jetter (eds.), Birkauser Verlag, Basel, 91-106 (1990).
10. Dyn, N. and D. Levin, Stationary and non-stationary binary subdivision schemes, in *Mathematical Methods in Computer Aided Geometric Design II*, T. Lyche, and L.L. Schumaker (eds.), Academic press, 209-216 (1992).
11. Dyn, N., D. Levin and A. Luzzatto, Non-stationary interpolatory subdivision schemes reproducing spaces of exponential polynomials, preprint.
12. Dyn, N., D. Levin and C.A. Micchelli, Using parameters to increase smoothness of curves and surfaces generated by subdivision, Computer Aided Geometric Design **7**, 129-140 (1990).
13. Gregory, J.A., An introduction to bivariate uniform subdivision in *Numerical Analysis 1991* Griffiths, D.F. and Watson, G.A., (eds.), Pitman Research Notes in Mathematics, Longman Scientific and Technical, 103-117 (1991).

14. Kobbelt, L., Interpolatory subdivision on open quadrilateral nets with arbitrary topology, In *Proceedings of Eurographics 96*, Computer Graphic Forum, 409-420,(1996).
15. Levin, A., Combined Subdivision Schemes with Applications to Surface Design, Ph.D. Thesis, Tel-Aviv University (1999).
16. Luzzatto, A., Multi-Scale Signal Processing, Based on Non-Stationary Subdivision, Ph.D. thesis, Tel-Aviv University (2000).
17. G. Morin, J. Warren and H. Weimer A subdivision scheme for surfaces of revolution, Computer-Aided Geometric Design, **18**, 483-503 (2001).
18. Zorin, D., P, Schroeder and W. Sweldens, Interpolating subdivision for meshes with arbitrary topology, *Computer Graphics Proceedings (SIGGRAPH 96)*,189-192 (1996).