Blending Based Chaikin type Subdivision Schemes for Nets of Curves

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Abstract. The paper presents a new subdivision scheme, which constructs a surface approximating a given net of 3D-curves. Similar to the well known Chaikin algorithm for points, having a refinement step based on piecewise linear interpolation of the control points followed by evaluation at 1/4 and 3/4 of the local parameter value, the refinement step in the proposed subdivision scheme is based on piecewise Coons patch interpolation followed by evaluation at 1/4 and 3/4 of the local parameters values in both directions, which results in a refined net of curves. We prove the convergence of the scheme to a continuous surface. The proof is based on the "proximity" of the scheme to a new, convergent subdivision scheme for points. Some examples, illustrating the performance of our scheme, are given.

§1. Introduction

Subdivision algorithms for points are efficient iterative means to generate recursively denser and denser sequences of points. At each step of the subdivision recursion a new sequence of points is obtained by weighted average of topologically neighboring points previously determined. The averaging coefficients (used to generate the new points) together with their topological location characterize the subdivision scheme and form the so called mask of the scheme. When the mask and the topological relation between the points are well chosen, the iterative procedure is convergent to at least a continuous surface.

The goal of this paper is to define an efficient procedure for generating recursively denser and denser nets of curves from a given net of curves. The strategy we present is based on to the following steps: we start by using a simple and local transfinite scheme to interpolate all the transfinite information given by the net of curves. This produces a piecewise

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interpolating surface which is locally evaluated at two parameter values for each direction in order to generate, locally, four new curves. Then a new net of curves is defined by patching together the locally defined curves and it is used to restart the procedure. We prove that this iterative scheme converges, and that the limit is a continuous surface. In fact we expect that the limit surface is C^1, as are the limit curves generated by the Chaikin algorithm and as indicated by our simulations (see Section 4), but we do not have a proof. From similar reasons we also expect that our scheme has shape preserving properties. In case our scheme has these two properties, then it is advantageous over algebraic transfinite constructions of C^1 piecewise local patches, in the same way that Chaikin algorithm is advantageous over piecewise cubic Hermite polynomials.

The paper is organized as follows. In Section 2 we first recall shortly the Chaikin subdivision algorithm for points and some simple facts about transfinite interpolation of four curves by Coons patch and about bilinear interpolation of four points. Then we describe the blending based Chaikin type subdivision algorithm for net of curves. Next in Section 3 we discuss its convergence. The latter is based on important consequences of what we call the M-property of curves, which allows us to show the "proximity" of the intersection points of the generated nets of curves and the points generated by a new convergent subdivision scheme for points discussed in the Appendix (see [5] for a detailed investigation of the proximity conditions and their consequences). In the closing Section 4, some figures are given to illustrate the performance of the blending-based Chaikin-type subdivision algorithm on two examples of net of curves.

§2. The Blending-Based Chaikin-type Algorithm for Net of Curves

The Chaikin subdivision algorithm for points -also known as corner cutting algorithm- was introduced by Chaikin in [1] already in the 70s. It was preceded by the work of de Rahm [3], which was the first work about subdivision schemes. The Chaikin algorithm is known to converge to a quadratic spline curve with the initial control points as coefficients of the B-splines. This subdivision algorithm in each iteration (see Figures 1 and 2 below) first constructs the polygonal line through the control points (also termed control polygon) and then every linear piece of the control polygon is evaluated at $\frac{1}{4}$ and $\frac{3}{4}$ of its length. This procedure generates a denser set of control points which is used at the next iteration (see [1] and [4] for a detailed analysis of the algorithm).

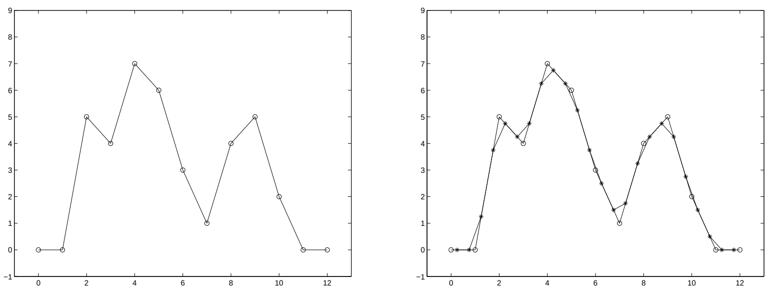

Fig. 1. *The control polygon (left) and its evaluation at $\frac{1}{4}, \frac{3}{4}$ of each edge (right).*

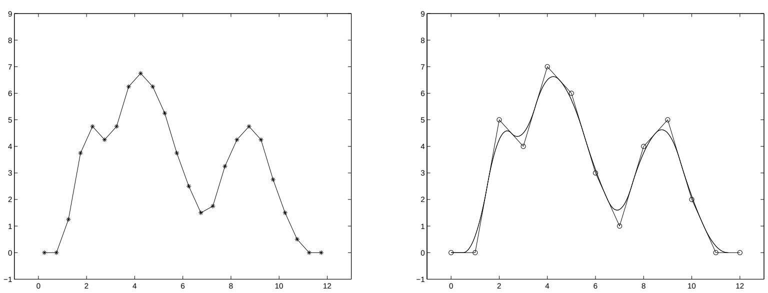

Fig. 2. *The control polygon after* 1*-iteration (left) and the limit quadratic spline with the initial control polygon (right).*

To construct a surface approximating a given net of curves we propose a subdivision procedure based on repeated piecewise Coons transfinite interpolation followed by its evaluation at the values $\frac{1}{4}$ and $\frac{3}{4}$ of the two local parameters.

We begin by recalling the definition of a Coons patch and of a bilinear interpolant ([2], [6]).

Definition 1. *For $\phi_0(s), \phi_1(s)$ with $s \in [0,d]$ and $\psi_0(t), \psi_1(t)$ with $t \in [0,d]$ such that $P_{ji} := \phi_i(jd) = \psi_j(id)$, $i,j = 0,1$, the bilinear surface interpolating the four intersection points of the four curves is*

$$\mathcal{BL}(\{P_{ij}\}_{i,j=0}^{1}; d)(s,t) := (1-\tfrac{s}{d})\big((1-\tfrac{t}{d})P_{00}+\tfrac{t}{d}P_{01}\big)+\tfrac{s}{d}\big((1-\tfrac{t}{d})P_{10}+\tfrac{t}{d}P_{11}\big)\,,$$

and the Coons patch interpolating the four curves is

$$\mathcal{C}(\phi_0,\phi_1,\psi_0,\psi_1;d)(s,t) := (1-\tfrac{s}{d})\psi_0(t)+\tfrac{s}{d}\psi_1(t)+$$

$$(1-\tfrac{t}{d})\phi_0(s)+\tfrac{t}{d}\phi_1(s)-\mathcal{BL}(\{P_{ij}\}_{i,j=0}^{1}; d)(s,t)\,.$$

It is easy to verify the interpolation properties of the Coons patch, *i.e.*

$$\mathcal{C}(\phi_0,\phi_1,\psi_0,\psi_1;d)(0,t) = \psi_0(t), \quad \mathcal{C}(\phi_0,\phi_1,\psi_0,\psi_1;d)(d,t) = \psi_1(t),$$

$$\mathcal{C}(\phi_0,\phi_1,\psi_0,\psi_1;d)(s,0) = \phi_0(s), \quad \mathcal{C}(\phi_0,\phi_1,\psi_0,\psi_1;d)(s,d) = \phi_1(s),$$

and of the bilinear interpolant *i.e.*

$$\mathcal{BL}(\{P_{ij}\}_{i,j=0}^{1}; d)(id, jd) = P_{ij},\ \ i,j = 0,1.$$

Next we continue with the definition of a net of compatible curves N and a piecewise Coons patch surface interpolating a net of compatible curves N.

Definition 2. *A net of continuous curves* $N = N(d, n+1, m+1)$

$$N = \{\phi_0(s), \cdots, \phi_n(s), \psi_0(t), \cdots, \psi_m(t)\}$$

with $\phi_0, \cdots, \phi_n$ *defined on* $[0, md]$ *and* $\psi_0, \cdots, \psi_m$ *defined on* $[0, nd]$ *is said to be compatible if* $\phi_i(jd) = \psi_j(id),\ i = 0, \ldots, n,\ j = 0, \ldots, m.$

Definition 3. *For a compatible net of curves* $N = N(d, n+1, m+1) = \{\phi_0(s), \cdots, \phi_n(s), \psi_0(t), \cdots, \psi_m(t)\}$ *the piecewise Coons patch interpolating the net of curves* N *is locally defined on each sub-domain* $[id, (i+1)d] \times [jd, (j+1)d]$ *as*

$$\begin{aligned}\mathcal{C}(N)(s,t) &= \mathcal{C}(\phi_i, \phi_{i+1}, \psi_j, \psi_{j+1}; d)(s - id, t - jd)\,,\\ &\quad\ i = 0, \ldots, n-1,\ \ j = 0, \ldots, m-1\,.\end{aligned}$$

Before describing the subdivision procedure we introduce some notation.

<u>Notation</u>

- $N = N(d, n, m)$ is a given net of $n \times m$ compatible s-curves and t-curves having common domain of definition $[0, (m-1)d]$ and $[0, (n-1)d]$ respectively;
- $\mathcal{C}(N)$ is the piecewise Coons patch interpolating a net of curves N;
- $BC(\mathcal{C}(N))$ is the net of curves obtained by sampling each Coons patch of $\mathcal{C}(N)$ at $\frac{1}{4}$ and $\frac{3}{4}$ of the two local parameters s and t *i.e.* for $N = N(d, n, m)$

$$\begin{aligned}BC(\mathcal{C}(N)) := \quad &\{\cup_{\ell=1,3} \cup_{j=0}^{m-2} \mathcal{C}(N)(s, (j + \tfrac{\ell}{4})d)\} \bigcup\\ &\{\cup_{\ell=1,3} \cup_{i=0}^{n-2} \mathcal{C}(N)((i + \tfrac{\ell}{4})d, t)\}\end{aligned}$$

  where $s \in [0, (m-1)d]$ and $t \in [0, (n-1)d]$;
- $M(\mathcal{C}(N))$ is the net of curves obtained by sampling each Coons patch of $\mathcal{C}(N)$ at $\frac{1}{2}$ of the two local parameters s and t *i.e.* for $N = N(d, n, m)$

$$M(\mathcal{C}(N)) := \{\cup_{j=0}^{m-2}\mathcal{C}(N)(s, (j + \frac{1}{2})d)\} \bigcup \{\cup_{i=0}^{n-2}\mathcal{C}(N)((i + \frac{1}{2})d, t)\};$$

- $N^{'} = N \cup M(\mathcal{C}(N))$;
- $\mathcal{E}(N)$ is the set of intersection points of the curve network N;
- $\mathcal{BL}(\mathcal{E}(N))$ is the piecewise bilinear surface interpolating the points $\mathcal{E}(N)$;
- the symbol $\| \bullet \|$ stands for $\sup_{t \in I} |f(t)|$, for $f : I \to \mathbb{R}^3$, and where $|\cdot|$ is a fixed norm in $\mathbb{R}^3$.

Note that $N \subset N^{'}$ and $\mathcal{C}(N) = \mathcal{C}(N^{'})$. Note also that for $N = N(d,n,m)$ $BC(\mathcal{C}(N)) = BC(\mathcal{C}(N))(\frac{d}{2}, 2(n-1), 2(m-1))$ and $N^{'} = N^{'}(\frac{d}{2}, 2n-1, 2m-1)$.
We are now in a position to state the

Blending-based Chaikin-type subdivision algorithm (BC-algorithm)

1. Let N_0 be a net of compatible curves
2. For $k = 0, 1, \ldots$
   2.1. $N_{k+1} := BC(\mathcal{C}(N_k))$

§3. Convergence of the Blending-Based Chaikin-Type Subdivision Algorithm

In this section we give conditions on the initial curves determining N_0 which guarantee the convergence to a continuous surface of the BC-algorithm.
First, we introduce the M-property of a curve.

Definition 4. *A curve $\phi(s)$ defined in $[0,d]$ has the M-property over $[0,d]$ if it satisfies the two conditions*

$$1 \ \phi \in C^0[0,d]\,,$$

$$2 \ |[s_1, s_2, s_3]\phi| \ \leq M, \ \textit{for all distinct } s_1, s_2, s_3 \in [0,d]\,,$$

where $[s_1, s_2, s_3]\phi$ is the second order divided difference of ϕ in the points $s_1,\ s_2,\ s_3$, and $|\cdot|$ is a fixed norm in $\mathbb{R}^3$.

Note that the M-property implies that the curve has a first derivative almost everywhere, but it is not necessarily continuous, yet we expect the limit surface to be C^1.
A first trivial consequence of the M-property is,

Lemma 1. *Let $\phi(s)$ with $s \in [0,d]$ have the M-property. Then*

$$|\phi(s) - (1 - \tfrac{s}{d})\phi(0) - \tfrac{s}{d}\phi(d)| \leq \frac{d^2}{4} M\,.$$

Proof: By Newton interpolation formula, we can write

$$\phi(s) = (1-\tfrac{s}{d})\phi(0) + \tfrac{s}{d}\phi(d) + (s-d)s[0,s,d]\phi \tag{1}$$

and thus, due to the M-property of ϕ,

$$|\phi(s) - (1-\tfrac{s}{d})\phi(0) - \tfrac{s}{d}\phi(d)| = |(s-d)s[0,s,d]\phi| \leq \frac{d^2}{4}M\,.$$

□

A second consequence of the M-property is a bound on the distance between a Coons patch interpolating four boundary compatible curves and the bilinear interpolant to the four intersection points of the four curves.

Lemma 2. *Let each of the curves of the compatible net of curves* $N = N(d,2,2) = \{\phi_0(s), \phi_1(s), \psi_0(t), \psi_1(t)\}$ *have the M-property over* $[0,d]$*, and let* $\{P_{ij}\}_{i,j=0}^1$ *be as in Definition 1. Then*

$$\sup_{(s,t)\in[0,d]^2} |\mathcal{C}(\phi_0,\phi_1,\psi_0,\psi_1;d)(s,t) - \mathcal{BL}(\{P_{ij}\}_{i,j=0}^1;d)(s,t)| \leq \frac{d^2}{2}M. \tag{2}$$

Proof: By (1),

$$\begin{aligned}(1-\tfrac{s}{d})\psi_0(t) + \tfrac{s}{d}\psi_1(t) \quad &= \quad (1-\tfrac{s}{d})\left((1-\tfrac{t}{d})\psi_0(0) + \tfrac{t}{d}\psi_0(d) + \right.\\ &\left.(t-d)t[0,t,d]\psi_0\,\right)\\ &+\tfrac{s}{d}\left((1-\tfrac{t}{d})\psi_1(0) + \tfrac{t}{d}\psi_1(d)\right.\\ &\left.+(t-d)t[0,t,d]\psi_1\right)\end{aligned}$$

that is

$$\begin{aligned}(1-\tfrac{s}{d})\psi_0(t) + \tfrac{s}{d}\psi_1(t) \quad &= \quad \mathcal{BL}(\{P_{ij}\}_{i,j=0}^1;d)(s,t) +\\ &(1-\tfrac{s}{d})(t-d)t[0,t,d]\psi_0 +\\ &\tfrac{s}{d}(t-d)t[0,t,d]\psi_1\,,\end{aligned}$$

and, similarly,

$$\begin{aligned}&(1-\tfrac{t}{d})\phi_0(s) + \tfrac{t}{d}\phi_1(s) = \mathcal{BL}(\{P_{ij}\}_{i,j=0}^1;d)(s,t) +\\ &(1-\tfrac{t}{d})(s-d)s[0,s,d]\phi_0 + \tfrac{t}{d}(s-d)s[0,s,d]\phi_1\,.\end{aligned}$$

It follows from Definition 1 that

$$\mathcal{C}(\phi_0,\phi_1,\psi_0,\psi_1;d)(s,t)-\mathcal{BL}(\{P_{ij}\}_{i,j=0}^1;d)(s,t)=$$
$$(1-\tfrac{s}{d})(t-d)t[0,t,d]\psi_0+\tfrac{s}{d}(t-d)t[0,t,d]\psi_1+ \tag{3}$$
$$(1-\tfrac{t}{d})(s-d)s[0,s,d]\phi_0+\tfrac{t}{d}(s-d)s[0,s,d]\phi_1\,,$$

and by the M-property of the four curves that

$$\|\mathcal{C}(\phi_0,\phi_1,\psi_0,\psi_1;d)-\mathcal{BL}(\{P_{ij}\}_{i,j=0}^1;d)\|\leq\frac{d^2}{4}M+\frac{d^2}{4}M=\frac{d^2}{2}M\,.$$

□

An important consequence of the M-property is its "preservation" under Coons patch interpolation and Chaikin-type evaluation.

Theorem 1. *Let each of the curves of the compatible net of curves* $N=N(d,2,2)=\{\phi_0(s),\phi_1(s),\psi_0(t),\psi_1(t)\}$ *have the M-property over* $[0,\frac{d}{2}]$ *and over* $[\frac{d}{2},d]$*. Then the curves*

$$\psi_{\frac{1}{4}}(t):=\mathcal{C}(\phi_0,\phi_1,\psi_0,\psi_1;d)(\tfrac{d}{4},t),\quad \psi_{\frac{3}{4}}(t):=\mathcal{C}(\phi_0,\phi_1,\psi_0,\psi_1;d)(\tfrac{3d}{4},t),$$
$$\phi_{\frac{1}{4}}(s):=\mathcal{C}(\phi_0,\phi_1,\psi_0,\psi_1;d)(s,\tfrac{d}{4}),\quad \phi_{\frac{3}{4}}(t):=\mathcal{C}(\phi_0,\phi_1,\psi_0,\psi_1;d)(s,\tfrac{3d}{4}),$$

have the M-property over $[0,\frac{d}{2}]$ *and over* $[\frac{d}{2},d]$.

Proof: Conditions 1 in Definition 4 simply follows from the Coons patch definition, while for condition 2 we use properties of the second order divided difference operator. For example for $\psi_{\frac{1}{4}}(t)$, by the Coons patch definition we get

$$\begin{aligned}\psi_{\frac{1}{4}}(t)= \quad & \tfrac{3}{4}\psi_0(t)+\tfrac{1}{4}\psi_1(t)+(1-\tfrac{t}{d})\phi_0(\tfrac{d}{4})+\tfrac{t}{d}\phi_1(\tfrac{d}{4})\\ & -\tfrac{3}{4}(1-\tfrac{t}{d})\phi_0(0)-\tfrac{3}{4}\tfrac{t}{d}\phi_1(0)\\ & -\tfrac{1}{4}(1-\tfrac{t}{d})\phi_0(d)-\tfrac{1}{4}\tfrac{t}{d}\phi_1(d)\,.\end{aligned}$$

Since the second divided differences of linear functions vanishes, it follows that for any three values of the parameter in $[0,\frac{d}{2}]$ or in $[\frac{d}{2},d]$, say t_1,t_2,t_3, we can write

$$|[t_1,t_2,t_3]\psi_{\frac{1}{4}}|\;\leq\frac{3}{4}|[t_1,t_2,t_3]\psi_0|\;+|\frac{1}{4}[t_1,t_2,t_3]\psi_1|\;\leq M.$$

An analogous proof can be given for the three other curves $\psi_{\frac{3}{4}}(t)$, $\phi_{\frac{1}{4}}(s)$ and $\phi_{\frac{3}{4}}(s)$. $\square$

With the help of the previous results we are able to show that the BC-algorithm preserves the M-property of the initial net of curves through the iterations.

Definition 5. *A net of compatible curves as in Definition 2 has the M-property if each curve in the net has the M-property over intervals of the form $[\nu\frac{d}{2},(\nu+1)\frac{d}{2}]$, $\nu=0,1,\ldots,2m-1$ for the $\{\phi_i\}$ curves and $\nu=0,1,\ldots,2n-1$ for the $\{\psi_j\}$ curves.*

Theorem 2. *Let N_0 be a net of compatible curves satisfying the M_0-property. Then at each iteration of the BC-algorithm the generated net of curves N_k is a net of compatible curves which satisfies the M_0-property.*

Proof: N_k is a net of compatible curves by construction. Also by construction each piece of curve between two intersection points is taken from at most two patches of $\mathcal{C}(N_{k-1})$. Thus, by Theorem 1 N_k has the M_0-property if N_{k-1} has it with the same constant. $\square$

We finally arrive at a convergence result. Before we prove the convergence of the BC-algorithm we introduce a new bivariate subdivision scheme for points denoted by $S_{\mathbf{a}}$. The mask of $S_{\mathbf{a}}$ is given in the Appendix together with a convergence proof of $S_{\mathbf{a}}$.

Theorem 3. *Let N_0 be a net of compatible curves satisfying the M property. Then the BC-algorithm is convergent to a continuous surface.*

Proof: To prove that the BC-algorithm is convergent to a continuous surface it is sufficient to show that $\{\mathcal{C}(N_k), k\in\mathbb{Z}_+\}$ is a Cauchy sequence. This is the case, since $\{\mathcal{C}(N_k), k\in\mathbb{Z}_+\}$ is a sequence of continuous vector valued functions defined over a fixed domain in the parameter plan. Being a Cauchy sequence, the sequence has a continuous limit, which is the limit of the BC-algorithm.
With $d_k=\frac{1}{2^k}d$, we have by Lemma 2 and by the M-property of all nets of curves generated by the BC-algorithm, as guaranteed by Theorem 2, that

$$\|\mathcal{C}(N_k^{'})-\mathcal{BL}(\mathcal{E}(N_k^{'}))\|\leq\frac{d_{k+1}^2}{2}M\,. \tag{4}$$

Applying the proximity condition (4) to

$$\begin{aligned}\mathcal{C}(N_{k+1})-\mathcal{C}(N_k) &= \mathcal{C}(N_{k+1}^{'})-\mathcal{C}(N_k^{'})=\mathcal{C}(N_{k+1}^{'})-\mathcal{BL}(\mathcal{E}(N_{k+1}^{'})+\\ &\quad \mathcal{BL}(\mathcal{E}(N_k^{'}))-\mathcal{C}(N_k^{'})+\mathcal{BL}(\mathcal{E}(N_{k+1}^{'}))-\mathcal{BL}(\mathcal{E}(N_k^{'})),\end{aligned}$$

we obtain

$$\|\mathcal{C}(N_{k+1})-\mathcal{C}(N_k)\| \leq \frac{d_{k+2}^2}{2}M+\frac{d_{k+1}^2}{2}M+\left\|\mathcal{BL}(\mathcal{E}(N_{k+1}^{'}))-\mathcal{BL}(\mathcal{E}(N_k^{'}))\right\|. \tag{5}$$

It remains to bound the last term above. This is done with the aid of the subdivision scheme $S_{\mathbf{a}}$. Now,

$$\begin{aligned}\|\mathcal{BL}(\mathcal{E}(N_{k+1}^{'}))-\mathcal{BL}(\mathcal{E}(N_k^{'}))\| \leq\ & \|\mathcal{BL}(\mathcal{E}(N_{k+1}^{'}))-\mathcal{BL}(S_{\mathbf{a}}(\mathcal{E}(N_k^{'})))\|+ \\ & \|\mathcal{BL}(S_{\mathbf{a}}(\mathcal{E}(N_k^{'})))-\mathcal{BL}(\mathcal{E}(N_k^{'}))\|.\end{aligned} \tag{6}$$

Since each point of the bilinear patch is a convex combination of the four corners of the patch,

$$\|\mathcal{BL}(\mathcal{E}(N_{k+1}^{'}))-\mathcal{BL}(S_{\mathbf{a}}(\mathcal{E}(N_k^{'})))\| \leq \|\mathcal{E}(N_{k+1}^{'})-S_{\mathbf{a}}(\mathcal{E}(N_k^{'}))\|, \tag{7}$$

where the difference sequence between two sequences defined over the same grid in the (s,t)-plane, is the sequence of differences between corresponding elements in the two sequences. Without loss of generality we assume that the points of $\mathcal{E}(N_k^{'})$ correspond to the grid $d_{k+1}\mathbb{Z}^2$ in the (s,t)-plane. In the following we bound $\|\mathcal{E}(N_{k+1}^{'})-S_{\mathbf{a}}(\mathcal{E}(N_k^{'}))\|$ with the help of two lemmas.

Lemma 3. *For* $q \in d_{k+2}\mathbb{Z}^2 \setminus d_{k+1}\mathbb{Z}^2$,

$$|\mathcal{E}(N_{k+1}^{'})(q)-S_{\mathbf{a}}(\mathcal{E}(N_k^{'}))(q)| \leq \frac{d_{k+1}^2 M}{2}.$$

Proof: By (19), for $q \in d_{k+2}\mathbb{Z}^2 \setminus d_{k+1}\mathbb{Z}^2$ (see Figure 3),

$$S_{\mathbf{a}}(\mathcal{E}(N_k^{'}))(q)=\mathcal{BL}(\mathcal{E}(N_k^{'}))(q).$$

On the other hand by the BC-algorithm and the definition of $N_{k+1}^{'}$

$$\mathcal{E}(N_{k+1}^{'})(q)=BC(\mathcal{C}(N_k))(q).$$

The claim of the Lemma follows now from Lemma 2. $\square$

The more involved result concerns points corresponding to the grid $d_{k+1}\mathbb{Z}^2$.

Lemma 4. *For* $q \in d_{k+1}\mathbb{Z}^2$,

$$|\mathcal{E}(N_{k+1}^{'})(q)-S_{\mathbf{a}}(\mathcal{E}(N_k^{'}))(q)| \leq \frac{3}{2}d_{k+1}^2 M.$$

Proof: For $q \in d_{k+1}\mathbb{Z}^2$, $\mathcal{E}(N^{'}_{k+1})(q)$ is the mid point of a Coons patch of $\mathcal{C}(N_{k+1}) = \mathcal{C}(N^{'}_{k+1})$. By Definition 1

$$\mathcal{E}(N^{'}_{k+1})(q) = \frac{1}{2}\sum_{e\in F}\mathcal{E}(N^{'}_{k+1})(q+(d_{k+2})e) - \frac{1}{4}\sum_{e\in E}\mathcal{E}(N^{'}_{k+1})(q+(d_{k+2})e) \tag{8}$$

where $F = \{(\epsilon,0),(0,\epsilon) : \epsilon \in \{-1,1\}\}$ and $E = \{(\epsilon_1,\epsilon_2) : \epsilon_1,\epsilon_2 \in \{-1,1\}\}$. Since for $e \in E\cup F$, $q+(d_{k+2})e \in d_{k+2}\mathbb{Z}^2 \setminus d_{k+1}\mathbb{Z}^2$ (see Figure 4) we get from (8) and the previous lemma

$$\begin{aligned} |\mathcal{E}(N^{'}_{k+1})(q) - & \tfrac{1}{2}\sum_{e\in F} S_{\mathbf{a}}(\mathcal{E}(N^{'}_{k}))(q+(d_{k+2})e) + \\ & \tfrac{1}{4}\sum_{e\in E} S_{\mathbf{a}}(\mathcal{E}(N^{'}_{k}))(q+(d_{k+2})e)| \le \frac{3}{2}d^2_{k+1}M. \end{aligned} \tag{9}$$

Now, for $e \in E\cup F$, as noticed in the proof of Lemma 3,

$$S_{\mathbf{a}}(\mathcal{E}(N^{'}_{k}))(q+(d_{k+2})e) = \mathcal{BL}(\mathcal{E}(N^{'}_{k}))(q+(d_{k+2})e). \tag{10}$$

Since $\mathcal{BL}(\mathcal{E}(N^{'}_{k}))(q+(d_{k+2})e),\ e\in E$ is a center of its bilinear patch

$$\mathcal{BL}(\mathcal{E}(N^{'}_{k}))(q+(d_{k+2})e) = \frac{1}{4}\sum_{\tilde{e}\in E}\mathcal{BL}(\mathcal{E}(N^{'}_{k}))(q+(d_{k+2})(e+\tilde{e})),$$

and thus,

$$\begin{aligned} \tfrac{1}{4}\sum_{e\in E} S_{\mathbf{a}}(\mathcal{E}(N^{'}_{k}))(q+(d_{k+2})e) &= \tfrac{1}{16}\sum_{\tilde{e},\ e\in E}\mathcal{E}(N^{'}_{k})(q+(d_{k+2})(e+\tilde{e})) \\ &= \tfrac{1}{16}\Bigg(\sum_{e\in E}\mathcal{E}(N^{'}_{k})(q+(d_{k+1})e)) + \\ & \quad 2\sum_{e\in F}\mathcal{E}(N^{'}_{k})(q+(d_{k+1})e) + \\ & \quad 4\mathcal{E}(N^{'}_{k})(q)\Bigg). \end{aligned} \tag{11}$$

Since $\mathcal{BL}(\mathcal{E}(N^{'}_{k}))(q+(d_{k+2})e),\ e\in F$ is a center of a boundary edge of its bilinear patch

$$\mathcal{BL}(\mathcal{E}(N^{'}_{k}))(q+(d_{k+2})e) = \frac{1}{2}\left(\mathcal{E}(N^{'}_{k})(q+(d_{k+1})e) + \mathcal{E}(N^{'}_{k})(q)\right),\ e\in F,$$

and thus in view of (10)

$$\begin{aligned}\frac{1}{2}\sum_{e\in F} S_{\mathbf{a}}(\mathcal{E}(N_k^{'}))(q+(d_{k+2})e) &= \frac{1}{2}\sum_{e\in F} \mathcal{BL}(\mathcal{E}(N_k^{'}))(q+(d_{k+2})e)\\ &= \mathcal{E}(N_k^{'})(q) + \frac{1}{4}\sum_{e\in F}\mathcal{E}(N_k^{'})(q+(d_{k+1})e).\end{aligned} \tag{12}$$

Note that $\{q+(d_{k+1})e,\ e\in F\}$, $\{q+(d_{k+2})(e+\tilde{e}),\ e,\tilde{e}\in E\} = \{q+(d_{k+1})e,\ e\in E\cup F\}$ and q are points of $d_{k+1}\mathbb{Z}^2$ (see Figure 4). Combining (11) with (12) and comparing with the last rule in (19), we conclude that

$$\begin{aligned}\tfrac{1}{2}\textstyle\sum_{e\in F} S_{\mathbf{a}}(\mathcal{E}(N_k^{'}))(q+(d_{k+2})e) &- \tfrac{1}{4}\textstyle\sum_{e\in E} S_{\mathbf{a}}(\mathcal{E}(N_k^{'}))(q+(d_{k+2})e)\\ &= S_{\mathbf{a}}(\mathcal{E}(N_k^{'}))(q).\end{aligned} \tag{13}$$

Substituting (13) into (9) we obtain the claim of the lemma.

We summarize the results of the last two lemmas in a corollary

Corollary 1. *Under the conditions of Theorem 3*

$$\|\mathcal{E}(N_{k+1}^{'}) - S_{\mathbf{a}}(\mathcal{E}(N_k^{'}))\| \le \frac{3}{2}d_{k+1}^2 M. \tag{14}$$

Now, returning to the proof of Theorem 3, we get from (5), (6), (7) and Corollary 1

$$\|\mathcal{C}(N_{k+1})-\mathcal{C}(N_k)\| \le \frac{d_{k+2}^2}{2}M + 2d_{k+1}^2 M + \|\mathcal{BL}(S_{\mathbf{a}}(\mathcal{E}(N_k^{'})) - \mathcal{BL}(\mathcal{E}(N_k^{'}))\|. \tag{15}$$

Thus, it remains to bound $\|\mathcal{BL}(S_{\mathbf{a}}(\mathcal{E}(N_k^{'})) - \mathcal{BL}(\mathcal{E}(N_k^{'}))\|$. By Lemma 5 in the Appendix

$$\|\mathcal{BL}(S_{\mathbf{a}}(\mathcal{E}(N_k^{'})) - \mathcal{BL}(\mathcal{E}(N_k^{'}))\| \le 2\ \|\Delta_1(\mathcal{E}(N_k^{'}))\|, \tag{16}$$

with Δ_1 defined there. To bound $\|\Delta_1(\mathcal{E}(N_k^{'}))\|$, denoted hereafter by τ_k, we obtain from (14)

$$\|\Delta_1(\mathcal{E}(N_{k+1}^{'}))\| \le \|\Delta_1 S_{\mathbf{a}}(\mathcal{E}(N_k^{'}))\| + 3d_{k+1}^2 M \tag{17}$$

using the observation that if two sequence of points in $\mathbb{R}^3$ $\mathbf{P}$, $\mathbf{Q}$ satisfy $\|\mathbf{P}-\mathbf{Q}\| \le \epsilon$ then $\|\Delta_1\mathbf{P}\| \le \|\Delta_1\mathbf{Q}\| + 2\epsilon$. Now, as is shown in the Appendix

$$\Delta_1 S_{\mathbf{a}}(\mathcal{E}(N_k^{'})) = S_{\mathbf{b}_1}\Delta_1(\mathcal{E}(N_k^{'}))$$

and we get from (17) and the fact that $\|S_{\mathbf{b}_1}\| = \frac{3}{4}$ (see Appendix) that

$$\tau_{k+1} \le \frac{3}{4}\tau_k + 3Md_{k+1}^2 = \frac{3}{4}\tau_k + 3Md^2\ 4^{-(k+1)}.$$

Iterating this inequality we arrive at

$$\tau_k \le (\frac{3}{4})^k \tau_0 + 3Md^2\ 4^{-k} \sum_{i=0}^{k-1} 3^i \le (\frac{3}{4})^k \left(\tau_0 + \frac{3}{2}Md^2\right). \tag{18}$$

Recalling that $\tau_k := \|\Delta_1(\mathcal{E}(N_k^{'}))\|$, we obtain from (16) and (18)

$$\|\mathcal{BL}(S_{\mathbf{a}}(\mathcal{E}(N_k))) - \mathcal{BL}(\mathcal{E}(N_k))\| \le 2(\tfrac{3}{4})^k \left(\Delta_1(\mathcal{E}(N_0^{'})) + \tfrac{3}{2}Md^2\right),$$

which, together with (15), proves that $\{\mathcal{C}(N_k), k \in \mathbb{Z}_+\}$ is a Cauchy sequence. $\square$

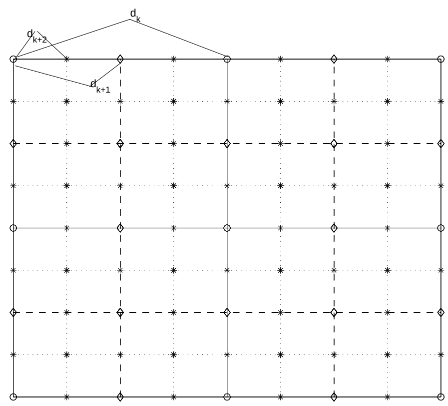

Fig. 3. *Grid in the (s,t)-plane:solid lines for $d_k\mathbb{Z}^2$ (N_k)*
solid lines plus dashed lines for $d_{k+1}\mathbb{Z}^2$ $(N_k^{'})$
dotted lines for $d_{k+2}\mathbb{Z}^2 \setminus d_{k+1}\mathbb{Z}^2$ (N_{k+1})
all line for $d_{k+2}\mathbb{Z}^2$ (corresponding to $N_{k+1}^{'}$)

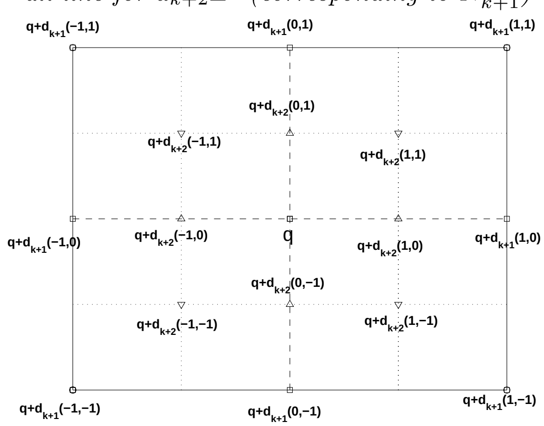

Fig. 4. *Points involved in the proof of Lemma 4:*
$\nabla-$ points of type $q + (d_{k+2})e$; $\triangle-$ points of type $q + d_{k+2}F$;
$\square-$ points of type $q + d_{k+1}(E \cup F)$.

§4. Examples

We conclude the paper by illustrating the application of the BC-algorithm to two finite nets of curves. Note that since no special rules for the "boundary" curves has been defined so far, we only present internal curves. This is why the domain of definition of the generated nets shrinks at each refinement step. The first three figures refer to a net of curves taken from the function f in $[-1, 3] \times [-1, 3]$ defined by

$$f(x, y) := \begin{cases} -\frac{e^{y-7}}{(x-2)} x y^3 & \text{for } x < 0, \\ x^3 y^3 & \text{otherwise.} \end{cases}$$

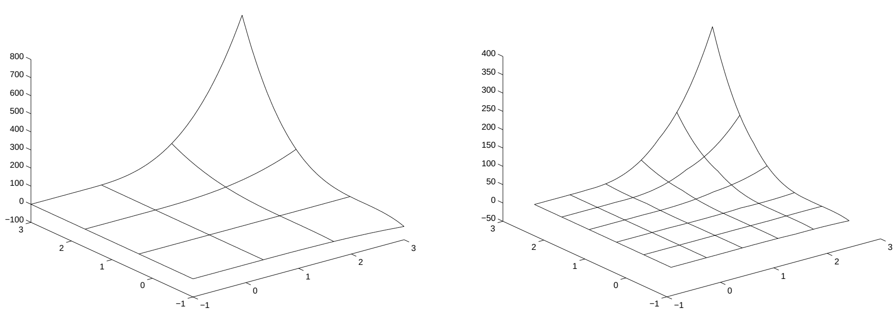

Fig. 5. *Original network (left) and* 1*-iteration network (right)*

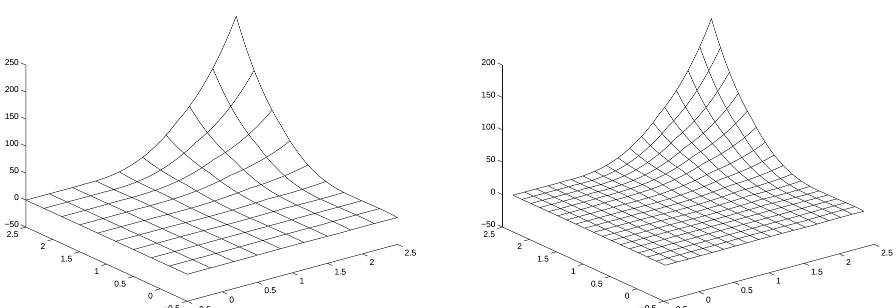

Fig. 6. 2*-iterations network (left)* 3*-iterations network (right)*

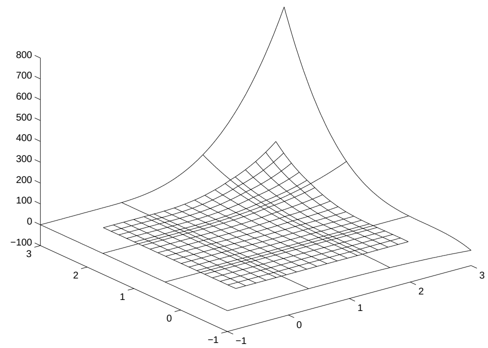

Fig. 7. 3-*iterations network together with the original network*

In the last three figures the net is taken from the Franke function in the square $[0, 1] \times [0, 1]$.

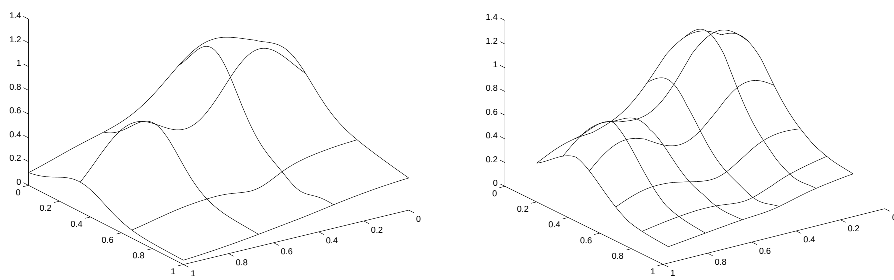

Fig. 8. *Original network (left) and* 1-*iteration network (right)*

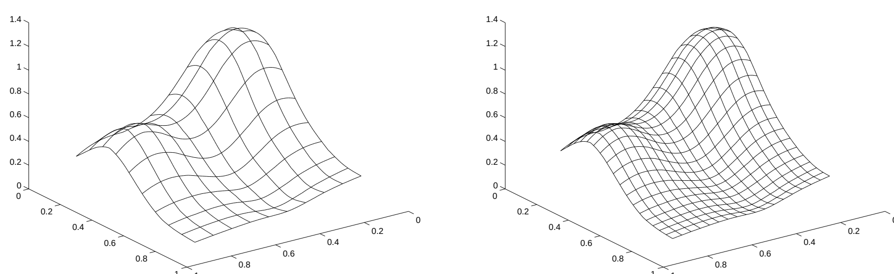

Fig. 9. 2-*iterations network (left)* 3-*iterations network (right)*

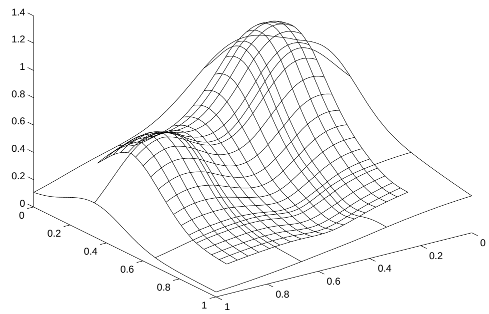

Fig. 10. 3-*iterations network together with the original network*

These simulations and many others, with different type of initial nets of curves, obtained by our Matlab implementation, confirm our conjecture about the C^1-smoothness of the surface generated by the algorithm, and about their shape preservation properties.

§5. Appendix

Consider the subdivision scheme $S_{\mathbf{a}}$ with rules

$$\begin{aligned}
P^{k+1}_{2i,2j+1} &= \tfrac{1}{2}\left(P^k_{i,j} + P^k_{i,j+1}\right)\\
P^{k+1}_{2i+1,2j} &= \tfrac{1}{2}\left(P^k_{i,j} + P^k_{i+1,j}\right)\\
P^{k+1}_{2i+1,2j+1} &= \tfrac{1}{4}\left(P^k_{i,j} + P^k_{i+1,j} + P^k_{i,j+1} + P^k_{i+1,j+1}\right)\\
P^{k+1}_{2i,2j} &= \tfrac{3}{4}P^k_{ij} + \tfrac{1}{8}\left(P^k_{i+1,j} + P^k_{i-1,j} + P^k_{i,j+1} + P^k_{i,j-1}\right)\\
&\quad -\tfrac{1}{16}\left(P^k_{i+1,j+1} + P^k_{i+1,j-1} + P^k_{i-1,j+1} + P^k_{i-1,j-1}\right).
\end{aligned} \tag{19}$$

It is easy to see that the mask of $S_{\mathbf{a}}$, $\mathbf{a} = \{a_\alpha\}_{\alpha\in\mathbb{Z}^2}$, is the sum of two masks one of which is the mask of tensor product linear spline (bilinear spline). In fact,

$$16\mathbf{a} = \begin{matrix} 0 & 0 & 0 & 0 & 0\\ 0 & 4 & 8 & 4 & 0\\ 0 & 8 & \mathbf{16} & 8 & 0\\ 0 & 4 & 8 & 4 & 0\\ 0 & 0 & 0 & 0 & 0 \end{matrix} \; + \; \begin{matrix} -1 & 0 & 2 & 0 & -1\\ 0 & 0 & 0 & 0 & 0\\ 2 & 0 & -\mathbf{4} & 0 & 2\\ 0 & 0 & 0 & 0 & 0\\ -1 & 0 & 2 & 0 & -1 \end{matrix}\;. \tag{20}$$

Note that in this presentation of the mask the bold face entry corresponds to the index $(0,0)$. It follows that the symbol of $S_{\mathbf{a}}$ is

$$\begin{aligned} a(z_1,z_2) &= \tfrac{1}{4}z_1^{-1}z_2^{-1}(1+z_1)^2(1+z_2)^2 - \tfrac{1}{4} + \tfrac{1}{8}\left(z_1^{-2}+z_1^2+z_2^{-2}+z_2^2\right) \\ &-\tfrac{1}{16}\left(z_1^2z_2^2+z_1^{-2}z_2^2+z_1^2z_2^{-2}+z_1^{-2}z_2^{-2}\right) \\ &= \tfrac{1}{4}z_1^{-1}z_2^{-1}(1+z_1)^2(1+z_2)^2- \\ &\tfrac{1}{16}\left(z_1z_2+z_1^{-1}z_2^{-1}-z_1z_2^{-1}-z_1^{-1}z_2\right)^2 \\ &= \tfrac{1}{4}z_1^{-1}z_2^{-1}(1+z_1)^2(1+z_2)^2 - \tfrac{1}{16}z_1^{-2}z_2^{-2}(1-z_1^2)^2(1-z_2^2)^2 \\ &= \tfrac{1}{4}z_1^{-1}z_2^{-1}(1+z_1)^2(1+z_2)^2\,(1- \\ &\tfrac{1}{4}z_1^{-1}z_2^{-1}(1-z_1)^2(1-z_2)^2\Big)\,. \end{aligned} \tag{21}$$

The symbol of the difference scheme in the z_1 direction is

$$\begin{aligned} b_1(z_1,z_2) = \quad & \tfrac{1}{4}z_1^{-1}z_2^{-1}(1+z_1)(1+z_2)^2\,(1- \\ & \tfrac{1}{4}z_1^{-1}z_2^{-1}(1-z_1)^2(1-z_2)^2\Big) \end{aligned} \tag{22}$$

with mask $\mathbf{b}_1$ such that

$$16\mathbf{b}_1 := \begin{matrix} -1 & 1 & 1 & -1 \\ 0 & 4 & 4 & 0 \\ 2 & 6 & \mathbf{6} & 2 \\ 0 & 4 & 4 & 0 \\ -1 & 1 & 1 & -1 \end{matrix} \quad . \tag{23}$$

It is easy to verify from the mask given by (23), that

$$\|S_{\mathbf{b}_1}\|_\infty := \max_{e\in\{0,1\}^2}\{\sum_{\alpha\in\mathbb{Z}^2}|(b_1)_{2\alpha+e}|\} = \max\{\frac{3}{4},\ \frac{3}{4},\ \frac{1}{2},\ \frac{1}{2}\}.$$

Therefore, the norm of the operator $S_{\mathbf{b}_1}$ is $\frac{3}{4}$. By symmetry the norm of the subdivision operator corresponding to the differences in the z_2 direction is also $\frac{3}{4}$. Thus, $S_{\mathbf{a}}$ is a convergent scheme [4].

We prove an important property of the subdivision operator $S_{\mathbf{a}}$.

Lemma 5. *Let* $\mathbf{P}$ *be a set of control points defined on* $\mathbb{Z}^2$*. Then*

$$\|\mathcal{BL}(S_{\mathbf{a}}(\mathbf{P})) - \mathcal{BL}(\mathbf{P})\| \le 2\|\Delta_1\mathbf{P}\|, \tag{24}$$

with $(\Delta_1\mathbf{P})_{i,j} = P_{i+1,j} - P_{i,j}$*.*

Proof: Since $S_{\mathbf{a}}(\mathbf{P})$ is defined on $\frac{1}{2}\mathbb{Z}^2$ we present $\mathcal{BL}(\mathbf{P})$ in terms of control points defined on $\frac{1}{2}\mathbb{Z}^2$ as $\mathcal{BL}(\mathbf{P}) = \mathcal{BL}(BL(\mathbf{P}))$ with $BL(\mathbf{P})$ the points obtained from $\mathbf{P}$ by one refinement step of the bilinear subdivision scheme with symbol $\frac{(1+z_1)^2(1+z_2)^2}{4z_1z_2}$. Since $BL(\mathbf{P})$ are control points defined on $\frac{1}{2}\mathbb{Z}^2$,

$$\|\mathcal{BL}(S_{\mathbf{a}}(\mathbf{P})) - \mathcal{BL}(\mathbf{P})\| = \|\mathcal{BL}(S_{\mathbf{a}} - BL)(\mathbf{P})\| \leq \|(S_{\mathbf{a}} - BL)(\mathbf{P})\|. \quad (25)$$

By (21), the symbol of $S_{\mathbf{a}} - BL$ is of the form

$$\begin{aligned} c(z_1, z_2) &= -\tfrac{1}{16} z_1^{-2} z_2^{-2}(1+z_1)^2(1+z_2)^2(1-z_1)^2(1-z_2)^2 \\ &= d(z_1, z_2)\tfrac{(1+z_1)(1+z_2)^2}{4z_1z_2}(1+z_1) \end{aligned}$$

with $d(z_1, z_2) = -\frac{1}{4z_1z_2}(1-z_1)^2(1-z_2)^2$. Thus,

$$(S_{\mathbf{a}} - BL)(\mathbf{P}) = C_d \; S_{\Delta_1}\Delta_1(\mathbf{P}) \quad (26)$$

where S_{Δ_1} is a subdivision scheme with symbol $\frac{(1+z_1)(1+z_2)^2}{4z_1z_2}$ and C_d is a convolution operator corresponding to the symbol $d(z_1, z_2)$. It is easy to verify that $\|S_{\Delta_1}\| = \frac{1}{2}$ and $\|C_d\| = 4$ so that (24) follows from (25) and (26).

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