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A NOTE ON THE DECOMPOSITION OF GRAPHS INTO ISOMORPHIC MATCHINGS

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All graphs considered are finite, undirected, with no loops and no multiple edges. A graph H is said to have a G-decomposition if it is the union of pairwise edge-disjoint subgraphs each isomorphic to G. We denote this situation by $G|H$.

Many results are known about G-decomposition, for references see e.g. [1] and [6]. In this paper we establish some necessary and sufficient conditions for a graph H to have a tK_2-decomposition, where tK_2 is the graph consisting of t independent edges. Our result implies, as a very special case, the main result of Bialostocki and Roditty [3], that states that if G is a graph with e edges and maximum degree Δ, then, with a finite number of exceptions, $3K_2|G$ iff $3|e$ and $\Delta \leqq e/3$.

For every graph G, $E(G)$ is the set of edges of G and $e(G)=|E(G)|$. $\Delta(G)$ is the maximum degree of G and $\chi'(G)$ is the chromatic index (=edge-chromatic number) of G.

We begin with the following simple lemma, which is proved in [2]:

LEMMA 1. *Let G be a graph and let $M, N \subset E(G)$ be disjoint matchings of G with $|M|>|N|$. Then there are disjoint matchings M' and N' of G such that $|M'|=|M|-1$, $|N'|=|N|+1$ and $M' \cup N' = M \cup N$.* □

As an easy consequence we obtain

LEMMA 2. *For every graph G and every $t>1$, $tK_2|G$ iff*

$$t|e(G) \quad \text{and} \quad \chi'(G) \leqq e(G)/t. \tag{1}$$

PROOF. If $tK_2|G$ then obviously (1) holds. Conversely, if (1) holds put $r=e(G)/t$. Since $\chi'(G) \leqq r$, there are r disjoint matchings $F_1, \ldots, F_r$ of G that cover $E(G)$. By repeated application of Lemma 1 to pairs of these r matchings that differ in size by two or more we obtain r disjoint matchings $E_1, \ldots, E_r$ of G that cover $E(G)$ and $|E_i|=t$ for all $1 \leqq i \leqq r$. □

REMARK 1. König's Theorem (for proof see [4, p. 105]), asserts that for every bipartite graph G, $\chi'(G)=\Delta(G)$. This and Lemma 2 imply that for every bipartite graph G $tK_2|G$ iff

$$t|e(G) \quad \text{and} \quad \Delta(G) \leqq e(G)/t. \tag{2}$$

This result is stated as Lemma 3.2 of [5]. In Theorem 1 below we prove that the same result holds for every graph G, with a finite number of exceptions for every value of t.

REMARK 2. Vizing's Theorem (for proof see [4, pp. 107—108]) asserts that for every graph G, $\chi'(G) \leqq \Delta(G)+1$. This and Lemma 2 imply that if $t|e(G)$ and $\Delta(G)<e(G)/t$, then $tK_2|G$.

The following lemma is proved in [4, p. 119]:

LEMMA 3. *If G is a graph and $\chi'(G)=\Delta(G)+1$ then*

$$e(G) \geqq \frac{1}{8}(3(\Delta(G))^2+6\cdot\Delta(G)-1). \quad \square$$

Now we are ready to prove our main result:

THEOREM 1. *For every $t>1$ and for every graph G that satisfies*

$$e(G)>(8/3)t^2-2t, \tag{3}$$

the following two conditions are equivalent:

$$tK_2|G. \tag{4}$$

$$t|e(G) \quad \text{and} \quad \Delta(G) \leqq e(G)/t. \tag{5}$$

PROOF. Clearly (4) implies (5) (even if G does not satisfy (3)). Conversely, assuming G satisfies (3) and (5) let us prove (4). Put $\Delta=\Delta(G)$ and $e=e(G)$. If $\Delta<e/t$ then Remark 2 implies (4), and if $\chi'(G)=\Delta$ then Lemma 2 implies (4). Thus we are left with the case that $\chi'(G)=\Delta+1$ and $\Delta=e/t$. We shall show that this case contradicts (3). By Lemma 3

$$8e \geqq 3\Delta^2+6\Delta-1 = 3(e/t)^2+6(e/t)-1. \tag{6}$$

Since the left side of (6) is divisible by e/t, and (3) implies that $e/t>1$, (6) implies

$$8e \geqq 3(e/t)^2+6(e/t).$$

The last inequality implies $e \leqq (8/3)t^2-2t$, which contradicts (3). Thus $tK_2|G$ and the theorem is established. $\square$

REMARK 3. For $t>1$ let G_t be the disjoint union of K_{2t-1} (= the complete graph on $2t-1$ vertices), and any graph H with $t-1$ edges. Clearly $e(G_t)=2t^2-2t$, $\Delta(G_t)=2t-2$ and $\chi'(G_t)=\chi'(K_{2t-1})=2t-1$. Thus G_t satisfies (5), and by Lemma 2, G_t does not satisfy (4). This shows that the lower bound for $e(G)$ in condition (3) is not very far from being best possible.

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