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NOTE

ON THE DENSITY OF SETS OF VECTORS*

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Answering a question of Erdös, Sauer [4] and independently Perles and Shelah [5] found the maximal cardinality of a collection $\mathcal{F}$ of subsets of a set N of cardinality n such that for every subset $M \subset N$ of cardinality m $|\{C \cap M: C \in \mathcal{F}\}| < 2^m$. Karpovsky and Milman [3] generalised this result. Here we give a short proof of these results and further extensions.

Let ω be the set of nonnegative integers. For fixed positive integers $n, p_1, \ldots, p_n$ put $N = \{1, 2, \ldots, n\}$ and define

$$\mathcal{F} = \mathcal{F}(n, p_1, \ldots, p_n) = \{f: N \to \omega: f(i) < p_i \text{ for all } i \in N\}. \tag{1}$$

For $f \in \mathcal{F}$ and $I \subset N$ let $P^I(f) \equiv f|_I: I \to \omega$ be the restriction of f to I. For $\mathcal{R} \subset \mathcal{F}$ define $P^I(\mathcal{R}) = \{P^I(f): f \in \mathcal{R}\}$. We say that $\mathcal{R}$ is *I-dense* if $P^I(\mathcal{R}) = P^I(\mathcal{F})$. If S is a family of subsets of N, then $\mathcal{R}$ is *S-dense* if $\mathcal{R}$ is I-dense for some $I \in S$. The collection $\mathcal{F}^S$ defined below is clearly *not* S-dense:

$$\mathcal{F}^S = \{f \in \mathcal{F}: (\forall I \in S)(\exists i \in I)[f(i) > 0]\}$$
$$(= \{f \in \mathcal{F}: (\forall I \in S)[P^I(f) \not\equiv 0]\}). \tag{2}$$

As a corollary of the main result of this paper (Theorem 1) we show that if $\mathcal{R}$ is not S-dense, then $|\mathcal{R}| \leq |\mathcal{F}^S|$. This contains as special cases the results of Sauer [4] and Perles and Shelah [5], and of Karpovski and Milman [3].

Before stating Theorem 1, we need one more definition; $\mathcal{R} \subset \mathcal{F}$ is *monotone* if $f \in \mathcal{R}$, $g \in \mathcal{F}$ and $f \leq g$ imply $g \in \mathcal{R}$. Note that $\mathcal{F}^S$ is monotone and that a monotone family $\mathcal{B} \subset \mathcal{F}$ is not S-dense iff $\mathcal{B} \subset \mathcal{F}^S$. (If $P^I(f) \equiv 0$ for some $I \in S$ and $f \in \mathcal{B}$, then the monotonicity of $\mathcal{B}$ implies that $\mathcal{B}$ is I-dense.)

Theorem 1. *For every $\mathcal{R} \subset \mathcal{F}$ there exists a monotone $\mathcal{B} \subset \mathcal{F}$ such that*
(a) $|\mathcal{B}| = |\mathcal{R}|$, and
(b) $|P^I(\mathcal{B})| \leq |P^I(\mathcal{R})|$ *for all $I \subset N$.*

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Proof. Among all sets $\mathscr{B} \subset \mathscr{F}$ that satisfy (a) and (b) let $\mathscr{B}_0$ be one for which the sum

$$M(\mathscr{B}) = \sum_{f \in \mathscr{B}} \sum_{i=1}^{n} f(i) \tag{3}$$

is maximal. To complete the proof we show that $\mathscr{B}_0$ is monotone.

For $1 \leq i \leq n$, $0 \leq j < p_i - 1$ and $f \in \mathscr{F}$ define $\bar{T}_{ij}(f) \in \mathscr{F}$ as follows:

$$(\bar{T}_{ij}(f))(k) = \begin{cases} f(k) & \text{if } k \neq i, \\ f(i) & \text{if } k = i \text{ and } f(i) \neq j, \\ j+1 & \text{if } k = i \text{ and } f(i) = j. \end{cases}$$

For $f \in \mathscr{B}_0$ define

$$T_{ij}(f) = \begin{cases} \bar{T}_{ij}(f) & \text{if } \bar{T}_{ij}(f) \notin \mathscr{B}_0, \\ f & \text{otherwise.} \end{cases}$$

Thus the effect of the operator T_{ij} is to increase $f(i)$ by 1, provided $f(i) = j$ but only if the modified f lies outside $\mathscr{B}_0$. Note that $\mathscr{B}_0$ is not monotone iff $T_{ij}(\mathscr{B}_0) \neq \mathscr{B}_0$ for some $1 \leq i \leq n$ and $0 \leq j \leq p_i - 1$.

We now show that $T_{ij}(\mathscr{B}_0)$ satisfies (a) and (b).

(a) It is easily checked that if $f, g \in \mathscr{B}_0$, then $f \neq g$ implies $T_{ij}(f) \neq T_{ij}(g)$, and thus $|T_{ij}(\mathscr{B}_0)| = |\mathscr{B}_0| = |\mathscr{R}|$.

(b) Suppose $I \subset N$. We shall show that $|P^I(T_{ij}(\mathscr{B}_0))| \leq |P^I(\mathscr{B}_0)|$. Indeed, if $g \in P^I(T_{ij}(\mathscr{B}_0)) \setminus P^I(\mathscr{B}_0)$, it is easily checked that $i \in I$ and $g(i) = j+1$. Define a function $g' : I \to \omega$ by

$$g'(k) = \begin{cases} g(k) & \text{if } k \neq i, \\ j & \text{if } k = i. \end{cases}$$

We claim that $g' \in P^I(\mathscr{B}_0) \setminus P^I(T_{ij}(\mathscr{B}_0))$. Indeed since $g \in P^I(T_{ij}(\mathscr{B}_0))$ there exists an $f \in \mathscr{B}_0$ such that $g = P^I(T_{ij}(f))$. However $g \notin P^I(\mathscr{B}_0)$ and thus $T_{ij}(f) \neq f$. Therefore $f(i) = j$ and $g' = P^I(f) \in P^I(\mathscr{B}_0)$. If $g' = P^I(T_{ij}(f'))$ for some $f' \in \mathscr{B}_0$, then $T_{ij}(f') = f' \in \mathscr{B}_0$ (since $g'(i) = j \neq j+1$), and thus $P^I(f') = g'$ and $\bar{T}_{ij}(f') \notin \mathscr{B}_0$ (since $P^I(\bar{T}_{ij}(f')) = g \notin P^I(\mathscr{B}_0)$). Thus $T_{ij}(f') = \bar{T}_{ij}(f') \neq f'$, a contradiction. This shows that $g' \notin P^I(T_{ij}(\mathscr{B}_0))$.

Since the mapping $g \to g'$ is 1–1, we conclude that $|P^I(T_{ij}(\mathscr{B}_0))| \leq |P^I(\mathscr{B}_0)|$ as claimed, and that $T_{ij}(\mathscr{B}_0)$ satisfied (b).

If $T_{ij}(\mathscr{B}_0) \neq \mathscr{B}_0$, then the sum $M(\mathscr{B}_0)$ defined in (3) is strictly smaller than $M(T_{ij}(\mathscr{B}_0))$, contradicting the choice of $\mathscr{B}_0$. Therefore $T_{ij}(\mathscr{B}_0) = \mathscr{B}_0$ for all $1 \leq i \leq n$ and $0 \leq j < p_i - 1$, and thus $\mathscr{B}_0$ is monotone. This completes the proof. □

For positive integers $p_1, \ldots, p_n$ and for a family S of subsets of N define

$$f(n; p_1, \ldots, p_n; S) = \max\{|\mathscr{R}| : \mathscr{R} \subset \mathscr{F}, \mathscr{R} \text{ is not } S\text{-dense}\}. \tag{4}$$

Theorem 1 implies the following corollary.

Corollary 1. *For every family S of subsets of N*, $f(n; p_1, \ldots, p_n; S) = |\mathcal{F}^S|$.

Proof. Clearly $f(n; p_1, \ldots, p_n; S) \geq |\mathcal{F}^S|$. To see the converse inequality suppose $\mathcal{R} \subset \mathcal{F}$ is not S-dense. By Theorem 1 there exists a monotone $\mathcal{B} \subset \mathcal{F}$ that is not S-dense, with $|\mathcal{B}| = |\mathcal{R}|$. By the remark preceding Theorem 1 $\mathcal{B} \subset \mathcal{F}^S$, and thus $|\mathcal{R}| = |\mathcal{B}| \leq |\mathcal{F}^S|$. □

Remarks. (1) Suppose $n \geq m > 0$. Corollary 1, with $p_1 = p_2 = \cdots = p_n = 2$ and $S = \{I \subset N, |I| = m\}$ gives:

$$f(n; 2, \ldots, 2; S) = |\mathcal{F}^S| = \sum_{i=0}^{m-1} \binom{n}{i}.$$

This is the result of Sauer [4] and Perles and Shelah [5] mentioned in the abstract.

(2) Suppose $n_1 \geq m_1 \geq 1$, $n_2 \geq m_2 \geq 1, \ldots, n_s \geq m_s \geq 1$, $q_1, \ldots, q_s > 1$. For $1 \leq i \leq s$ define

$$J_i = \left\{ \sum_{\nu=1}^{i-1} n_\nu + k \colon 1 \leq k \leq n_i \right\}.$$

Corollary 1, with $n = \sum_{\nu=1}^{s} n_\nu$, $p_j = q_i$ for $j \in J_i$ and $S = \{I \subset N, |I \cap J_i| = m_i$ for $1 \leq i \leq s\}$ gives

$$f(n; q_1, \ldots, q_1, \ldots, q_s, \ldots, q_s; S)$$
$$= |\mathcal{F}^S| = \prod_{i=1}^{s} q_i^{n_i} - \prod_{i=1}^{s} \sum_{j=0}^{n_i - m_i} \binom{n_i}{j} (q_i - 1)^j.$$

This is the result of Karpovsky and Milman [3] mentioned in the abstract.

(3) Theorem 1 contains definitely more than Corollary 1. As an example we state one immediate consequence of it. Suppose $n \geq 3$, and $\mathcal{F} = \mathcal{F}(n, 2, \ldots, 2)$. Define $h = \max |\mathcal{R}|$, where the maximum is taken over all $\mathcal{R} \subset \mathcal{F}$ such that for every $I \subset N$, $|I| = 3$ implies $|P^I(\mathcal{R})| \leq 6$ (i.e., $P^I(\mathcal{R})$ misses at least two different functions $f \in P^I(\mathcal{F})$). Then

$$h = 1 + n + [\tfrac{1}{4}n^2].$$

The proof follows easily from Theorem 1 and Turan's theorem for triangles (see [1, pp. 294–295]). We omit the details.

(4) Suppose $1 \leq m \leq n$ and put $\mathcal{F} = \mathcal{F}(n, 2, \ldots, 2)$. A set $\mathcal{R} \subset \mathcal{F}$ is called *m-doubly-dense* if there exists an $I \subset N$, $|I| = m$, such that for every $g : I \to \{0, 1\}$ there exist $f_1, f_2 \in \mathcal{R}$ that satisfy

$$P^I(f_1) = P^I(f_2) = g \quad \text{and} \quad P^{N-I}(f_1 + f_2) \equiv 1.$$

Combining the method of this paper with the theorem of Hall and König [1, pp. 52–53] and the theorem of Erdös, Ko and Rado [2] we can prove [6] that the maximum cardinality of a set $\mathcal{R} \subset \mathcal{F}$ that is not m-doubly-dense is precisely

$$h(m,n)=\begin{cases}\displaystyle\sum_{i=0}^{(m+n-1)/2}\binom{n}{i} & \text{if } m+n \text{ is odd,}\\[2ex] \displaystyle\binom{n-1}{\frac{1}{2}(n+m)}+\sum_{i=0}^{(m+n-2)/2}\binom{n}{i} & \text{if } m+n \text{ is even.}\end{cases}$$

This result has some applications in functional analysis. Those will appear in [6].

Note added in proof

P. Frankl (On the trace of finite sets, J. Combin. Theory (A) 34 (1983) 41–45) used, independantly, a method similar to ours and proved the assertions of Remarks (1) and (3).

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