PLANAR SEPARATORS

Noga Alon*
Dept. of Mathematics, Sackler Faculty of Exact Sciences
Tel Aviv University
Ramat Aviv, Tel Aviv 69978, Israel

Paul Seymour
Bellcore
445 South St.
Morristown, New Jersey 07960, USA

and

Robin Thomas†
School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332, USA

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† Supported by DIMACS Center, Rutgers University, New Brunswick, New Jersey 08903, USA

ABSTRACT

We give a short proof of a theorem of Lipton and Tarjan, that for every planar graph with $n > 0$ vertices, there is a partition (A, B, C) of its vertex set such that $|A|, |B| < \frac{2}{3}n$, $|C| \leq 2(2n)^{1/2}$, and no vertex in A is adjacent to any vertex in B. Secondly, we apply the same technique more carefully, to deduce that in fact such a partition (A, B, C) exists with $|A|, |B| < \frac{2}{3}n$ and $|C| \leq \frac{3}{2}(2n)^{1/2}$; and this improves the best previously known result. An analogous result holds when the vertices or edges are weighted.

1. THE LIPTON-TARJAN THEOREM

Our first objective is to give a short proof of the following theorem of Lipton and Tarjan [3]. ($V(G)$ denotes the vertex set of the graph G.)

(1.1) *Let G be a planar graph with $n > 0$ vertices. Then there is a partition (A, B, C) of $V(G)$ such that $|A|, |B| < \frac{2}{3}n$, $|C| \leq 2\sqrt{2}\sqrt{n}$, and no vertex in A is adjacent to any in B.*

Proof. We may assume that G has no loops or multiple edges, that $n \geq 3$ and (by adding new edges to G) that G is drawn in the plane in such a way that every region is bounded by a circuit of three edges. (Circuits have no "repeated" vertices.) Let $k = \lfloor \sqrt{2n} \rfloor$. For any circuit C of G we denote by $A(C)$ and $B(C)$ the sets of vertices drawn inside C and outside C, respectively; thus $(A(C), B(C), V(C))$ is a partition of $V(G)$, and no vertex in $A(C)$ is adjacent to any in $B(C)$. Choose a circuit C of G such that

(i) $|V(C)| \leq 2k$

(ii) $|B(C)| < \frac{2}{3}n$

(iii) subject to (i) and (ii), $|A(C)| - |B(C)|$ is minimum.

(This is possible, because the circuit bounding the infinite region satisfies (i) and (ii).)

We suppose, for a contradiction, that $|A(C)| \geq \frac{2}{3}n$. Let D be the subgraph of G drawn in the closed disc bounded by C. For $u, v \in V(C)$, let $c(u,v)$ (respectively, $d(u,v)$) be the number of edges in the shortest path of C (respectively, D) between u and v.

(1) *$c(u,v) = d(u,v)$ for all $u, v \in V(C)$.*

For certainly $d(u,v) \leq c(u,v)$ since C is a subgraph of D. If possible, choose a pair $u, v \in V(C)$ with $d(u,v)$ minimum such that $d(u,v) < c(u,v)$. Let P be a path of D between u and v, with $d(u,v)$ edges. Suppose that some internal vertex w of P belongs to $V(C)$. Then

$$d(u,w) + d(w,v) = d(u,v) < c(u,v) \leq c(u,w) + c(w,v)$$

and so either $d(u,w) < c(u,w)$ or $d(w,v) < c(w,v)$, in either case contrary to the choice of u, v. Thus there is no such w. Let C, C_1, C_2 be the three circuits of $C \cup P$ where $|A(C_1)| \geq |A(C_2)|$. Now $|B(C_1)| < \frac{2}{3}n$, since

$$n - |B(C_1)| = |A(C_1)| + |V(C_1)| > \frac{1}{2}(|A(C_1)| + |A(C_2)| + |V(P)| - 2) = \frac{1}{2}|A(C)| \geq \frac{1}{3}n.$$

But $|V(C_1)| \leq |V(C)|$ since $|E(P)| \leq c(u,v)$, and so C_1 satisfies (i) and (ii). By (iii), $B(C_1) = B(C)$, and in particular $c(u,v) \leq 1$, which is impossible since $d(u,v) < c(u,v)$. This proves (1).

(2) $|V(C)| = 2k$.

For suppose that $|V(C)| < 2k$. Choose $e \in E(C)$, and let P be the two-edge path of D such that the union of P and e forms a circuit bounding a region inside of C. Let v be the middle vertex of P, and let P' be the path $C \backslash e$. Now $P \neq P'$ since $A(C) \neq \emptyset$, and so $v \notin V(C)$ by (1). Hence $P \cup P'$ is a circuit satisfying (i) and (ii), contrary to (iii). This proves (2).

Let the vertices of C be $v_0, v_1, \ldots, v_{2k-1}, v_{2k} = v_0$, in order.

(3) *There are $k+1$ vertex-disjoint paths of D between* $\{v_0, v_1, \ldots, v_k\}$ *and* $\{v_k, v_{k+1}, \ldots, v_{2k}\}$.

For otherwise, by a well-known form of Menger's theorem for planar triangulations, there is a path of D between v_0 and v_k with $\leq k$ vertices, contrary to (1).

Let the paths of (3) be $P_0, P_1, \ldots, P_k$, where P_i has ends v_i, v_{2k-i} $(0 \leq i \leq k)$. By (1),

$$|V(P_i)| \geq \min(2i+1, 2(k-i)+1)$$

and so

$$n = |V(G)| \geq \sum_{0 \leq i \leq k} \min(2i+1, 2(k-i)+1) \geq \frac{1}{2}(k+1)^2.$$

Yet $k+1 > \sqrt{2n}$ by the definition of k, a contradiction. Thus our assumption that $|A(C)| \geq \frac{2}{3}n$ was false, and so $|A(C)| < \frac{2}{3}n$ and $(A(C), B(C), V(C))$ is a partition satisfying the theorem. ■

2. SHIELDS

In the remainder of the paper, we use the same technique more carefully, to improve (1.1) numerically. A *separator* in a graph G is a partition (A, B, C) of $V(G)$ such that $|A|, |B| \leq \frac{2}{3}|V(G)|$ and no vertex in A is adjacent to any vertex in B; and its *order* is $|C|$. (1.1) therefore implies that any planar graph with n vertices has a separator of order $\leq 8^{1/2} n^{1/2}$, and one might ask, what is the smallest constant λ such that every planar graph with n

vertices has a separator of order $\le \lambda n^{1/2}$? The Lipton-Tarjan result (1.1) asserts that $\lambda \le 8^{1/2} \simeq 2.828$, and this was improved by Gazit [2], who showed that $\lambda \le \frac{7}{3} \simeq 2.333$. We shall give a further improvement, showing that $\lambda \le \frac{3}{2} \cdot 2^{1/2} \simeq 2.121$. Incidentally, the best lower bound known appears to be that of Djidjev [1], who showed that

$$\lambda \ge \frac{1}{3} \sqrt{4\pi\sqrt{3}} \simeq 1.555 .$$

Actually we shall prove a slight strengthening, the following (and indeed, we shall prove an extension of (2.1) when the vertices or edges have weights).

(2.1) *Let G be a loopless graph with n vertices, drawn in a sphere Σ. Then there is a simple closed curve F in Σ, meeting the drawing only in vertices, such that* $n_1 + \frac{1}{2} n_3, n_2 + \frac{1}{2} n_3 \le 2n/3$ *and* $n_3 \le \frac{3}{2}(2n)^{1/2}$, *where F passes through* n_3 *vertices and the two open discs bounded by F contain* n_1 *and* n_2 *vertices respectively.*

We shall be concerned with graphs drawn in a disc or sphere Σ, and to simplify notation we shall usually not distinguish between a vertex of the graph and the point of Σ used in the drawing to represent the vertex, or between an edge and the open line segment representing it. A subset of Σ homeomorphic to the closed interval [0, 1] is called an *I-arc*. If G is drawn in Σ, a subset of Σ meeting the drawing only in vertices is *G-normal*.

The proof of (2.1) relies on the notion of a "k-shield". Let $k \ge 0$. A *k-shield* (in Δ) is a loopless graph G drawn in a closed disc Δ, such that

(i) $|V(G) \cap bd(\Delta)| = k$ ($bd(\Delta)$ denotes the boundary of Δ)

(ii) $bd(\Delta)$ is G-normal, and

(iii) for every G-normal I-arc $F \subseteq \Delta$ with ends $x, y \in bd(\Delta)$, there is an I-arc $F' \subseteq bd(\Delta)$ with ends x, y such that $|V(G) \cap F'| \le |V(G) \cap F|$.

One can view the proof of (1.1) as consisting of two parts (omitting the reduction to G being a planar triangulation, which is included only for convenience and can easily be avoided): roughly, we show that for any k, every planar graph either has a separator of order $\le k$, or has a subgraph which is a k-shield; and secondly, we show that any k-shield has at least about $\frac{1}{8} k^2$ vertices. Consequently, any planar graph with no separator of order $\le k$ has at least about $\frac{1}{8} k^2$ vertices, and (1.1) follows.

We shall improve this as follows. First, $\frac{1}{8}$ is the wrong constant; we shall see that any k-shield has at least $\frac{1}{6}k^2$ vertices. ($\frac{1}{6}$ might not be the right constant either.) Secondly, with a little care we can confine ourselves to k-shields in G which contain at most three-quarters of the vertices of G.

In this section we prove that any k-shield has at least $\frac{1}{6}k^2$ vertices, and some related lemmas; and these are applied to prove (2.1) in the next section.

The proof of the next result is due to A. Schrijver (private communication); our original proof was an application of a currently unpublished theorem of Randby about graphs drawn in the projective plane [5], but Schrijver's proof is simpler.

(2.2) *If G is a k-shield then* $|E(G)| \geq \frac{1}{2}k(k-1)$.

Proof. We may assume that $k \geq 3$, that G has no multiple edges, and that G is 2-connected, as is easily seen. It follows that there is a circuit C of G, bounding a closed disc in Δ which includes all the drawing of G. Let the vertices of G in $bd(\Delta)$ be $v_1, \ldots, v_k$, and for $1 \leq i \leq k$ let l_i be the open line segment between v_i and v_{i+1} which is an arc-wise connected component of $bd(\Delta) - \{v_1, \ldots, v_k\}$ (where v_{k+1} means v_1). For $1 \leq i \leq k$, let r_i be the region of G in Δ including l_i. Then for $1 \leq i \leq k$, the boundary of r_i consists of l_i together with a path from C, while every other region of G in Δ is an open disc, and is bounded by a circuit of G. Let us say a *corner* of G is a pair (v, r), where $v \in V(G)$ and r is a region of G in Δ incident with v. For any corner (v, r) there are precisely two edges of G incident with both v and r, unless $r = r_i$ and $v = v_i$ or v_{i+1} for some i, when there is only one such edge. We call any such edge an *arm* of the corner.

We wish to define a new graph G' drawn in Δ. For each $e \in E(G)$, let x_e be a point of the open line segment representing e in the drawing of G. For $1 \leq i \leq k$, let a_i, b_i be distinct points of l_i, so that v_i, a_i, b_i, v_{i+1} occur in order. The vertex set of G' will be

$$\{a_1, b_1, a_2, b_2, \ldots, a_k, b_k\} \cup \{x_e : e \in E(G)\}.$$

The edges of G' correspond to the corners of G. For each corner (v, r) with two arms e, f there is an edge of G' with ends x_e, x_f, drawn within r. For each corner (v, r) with one arm e, let $r = r_i$; then if $v = v_i$ the corresponding edge of G' has ends a_i, x_e, while if $v = v_{i+1}$ it has ends $b_i\ x_e$, and in either case it is drawn within r. This defines

G', and its drawing. We see that every vertex of G' has valency 4 except for $a_1, b_1, \ldots, a_k, b_k$, which all have valency 1. Moreover, each region of G' in Δ either includes a (unique) vertex of G, or is a subset of a region of G in Δ; and every edge of G' is incident with one region of each type.

(1) *Let $F' \subseteq \Delta$ be an I-arc with ends $s, t \in bd(\Delta)$, not passing through any vertex of G'; and let F_1, F_2 be the two I-arcs in $bd(\Delta)$ with ends s, t. Then the number of edges of G' crossed by F' is at least $\min(|F_1 \cap V(G')|, |F_2 \cap V(G')|)$.*

For we may assume (by rerouting F') that $F' \cap r$ is an open line segment or null, for every region r of G' in Δ. As we traverse F' from s to t the regions of G' we pass through correspond alternately to vertices and regions of G and there is a G-normal I-arc F in Δ, passing through the same sequence of vertices and regions. Moreover, we may assume that F and F' have the same ends. Hence F passes through at least $\min(|F_1 \cap V(G)|, |F_2 \cap V(G)|)$ vertices of G, since G is a k-shield; say $|F \cap V(G)| \geq |F_1 \cap V(G)|$. If both ends of F are in $V(G)$, then

$$|F' \cap E(G')| \geq 2|F \cap V(G)| - 2 \geq 2|F_1 \cap V(G)| - 2 = |F_1 \cap V(G')|$$

and a similar computation applies if one or neither end of F is in $V(G)$. This proves (1).

Let us renumber $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ as $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ respectively. From (1) and the result of [4] it follows that

(2) *There are k mutually edge-disjoint paths $P_1, \ldots, P_k$ of G' joining s_i and t_i $(1 \leq i \leq k)$ respectively.*

Since for $1 \leq i < j \leq k, P_i$ and P_j have a common vertex (because they must cross somewhere) and this vertex belongs to no other of the k paths, we deduce that G' has at least $\frac{1}{2} k(k-1)$ vertices of valency 4. Consequently, $|E(G)| \geq \frac{1}{2} k(k-1)$, as required. ■

A k-shield G in Δ is *stable* if for every I-arc $L \subseteq bd(\Delta)$ with ends x, y and with $L \cap V(G) = \{x, y\}$, there is no edge e of G with ends x, y such that $L \cup e$ bounds a region of G in Δ.

(2.3) *If $k \geq 3$ and G is a stable k-shield then $|V(G)| \geq \frac{1}{6} k^2 + \frac{1}{2} k + 1$.*

Proof. Let G be drawn in Δ. If some region of G in Δ is bounded by a two-edge circuit, we may delete one of these two edges. By continuing this process, we may assume there is no such region.

Let the vertices of G drawn in $bd(\Delta)$ be $v_1, ..., v_k$ in order. Add to G a new vertex v_0, edges with ends v_0, v_i $(1 \leq i \leq k)$ and edges with ends v_i, v_{i+1} $(1 \leq i \leq k)$ where v_{k+1} means v_1. We obtain a new planar graph G', with $|V(G')| = |V(G)| + 1$ and $|E(G')| = |E(G)| + 2k$. Moreover, G' can be drawn in a sphere so that no region has boundary consisting of a one- or two-edge circuit. Since $|V(G')| \geq 3$, it follows that $|E(G')| \leq 3|V(G')| - 6$, and hence

$$|E(G)| + 2k \leq 3(|V(G)| + 1) - 6.$$

But by (2.2), $|E(G)| \geq \frac{1}{2} k(k-1)$, and the result follows. ■

Similarly, for $k \geq 2$ any k-shield has $\geq \frac{1}{6} k^2 + \frac{1}{6} k + 1$ vertices. We do not know if the term $\frac{1}{6} k^2$ here is best possible.

(2.4) *Let G be a graph drawn in a closed disc* Δ, *such that* $|V(G) \cap bd(\Delta)| = k$ *and* $bd(\Delta)$ *is G-normal. Suppose that for every G-normal I-arc* $F \subseteq \Delta$ *with ends* $x, y \in bd(\Delta)$ *and* $F \cap bd(\Delta) = \{x, y\}$, *there is an I-arc* $F' \subseteq bd(\Delta)$ *with ends x, y such that* $|F' \cap V(G)| \leq |F \cap V(G)|$. *Then G is a k-shield.*

Proof. For distinct $x, y \in bd(\Delta)$, let

$$d(x,y) = min(|F_1 \cap V(G)|, |F_2 \cap V(G)|)$$

where F_1, F_2 are the two I-arcs in $bd(\Delta)$ with ends x, y. We must show that $|F \cap V(G)| \geq d(x,y)$ for every G-normal I-arc $F \subseteq \Delta$ with ends $x, y \in bd(\Delta)$. We may assume that $F \cap bd(\Delta) \subseteq V(G) \cup \{x, y\}$, and we proceed by induction on $|F \cap bd(\Delta) - \{x,y\}|$. If this quantity is zero the result follows from the hypothesis. Otherwise, there exists $z \in (F - \{x,y\}) \cap bd(\Delta)$, and $z \in V(G)$. Let $F_1, F_2 \subseteq F$ be the I-arcs with ends x, z and z, y respectively. From the inductive hypothesis, $|F_1 \cap V(G)| \geq d(x,z)$ and $|F_2 \cap V(G)| \geq d(z,y)$. But

$$|F \cap V(G)| = |F_1 \cap V(G)| + |F_2 \cap V(G)| - 1$$

and $d(x,y) \leq d(x,z) + d(z,y) - 1$ (since $z \in V(G)$). The result follows. ■

Let us say a *strong* k-shield is a graph G drawn in a closed disc Δ with $|V(G) \cap bd(\Delta)| = k$ and with $bd(\Delta)$ G-normal, such that for every G-normal I-arc $F \subseteq \Delta$ with ends $x, y \in bd(\Delta)$ and with $F \cap bd(\Delta) = \{x, y\}$, either

(i) there is an I-arc $F' \subseteq bd(\Delta)$ with ends x, y such that $|F' \cap V(G)| < |F \cap V(G)|$, or

(ii) one of the two closed discs into which F divides Δ includes all of the drawing of G.

From (2.4) we see that every strong k-shield is a k-shield.

(2.5) *For $k \geq 3$, if G is a strong k-shield then* $|V(G)| \geq \frac{1}{6}k^2 + \frac{5}{6}k + \frac{2}{3}$.

Proof. We may assume that G has no multiple edges. Since $k \geq 3$, it follows that no two vertices in $bd(\Delta)$ are adjacent (for if an edge has both ends in $bd(\Delta)$ then we may choose F to violate conditions (i) and (ii) in the definition of strong k-shield, with the same ends as e and otherwise disjoint but "next to" e). Let the vertices of G in $bd(\Delta)$ be $v_1, \ldots, v_k$ in order. Let r be the region incident with v_1 and v_k (it is unique). Since G is a k-shield, r is not incident with any of $v_2, \ldots, v_{k-1}$. Let $u \neq v_1, v_k$ be incident with r (this exists since v_1, v_k are not adjacent). Add a new vertex v_{k+1} to G and an edge e with ends u, v_{k+1}, forming G'. Draw v_{k+1} in $r \cap bd(\Delta)$, and draw e within $r - bd(\Delta)$.

(1) *G' is a $(k+1)$-shield.*

For let $F \subseteq \Delta$ be a G'-normal I-arc with ends $x, y \in bd(\Delta)$, and with $F \cap bd(\Delta) = \{x, y\}$. Let F_1, F_2 be the two I-arcs in $bd(\Delta)$ with ends x, y. We claim that

$$|F \cap V(G')| \geq min(|F_1 \cap V(G')|, |F_2 \cap V(G')|).$$

If $|F \cap V(G)| > |F_1 \cap V(G)|$, then our claim holds since

$$|F \cap V(G')| \geq |F \cap V(G)| \geq |F_1 \cap V(G)| + 1 \geq |F_1 \cap V(G')|.$$

We assume then that $|F \cap V(G)| \leq |F_i \cap V(G)|$ for $i = 1, 2$. Since G is a strong k-shield, we may assume that the closed disc in Δ bounded by $F \cup F_2$ includes the drawing of G. Hence $F_1 \cap V(G) \subseteq F \cap V(G)$. If $v_{k+1} \notin F_1$, then

$$|F_1 \cap V(G')| = |F_1 \cap V(G)| \leq |F \cap V(G)| \leq |F \cap V(G')|$$

as required. If $v_{k+1} \in F_1$, then either v_{k+1} or u belongs to F (since u belongs to the closed disc bounded by $F \cup F_2$). In the first case $v_{k+1} = x$ or y, and so

$$|F_1 \cap V(G')| = |F_1 \cap V(G)| + 1 \leq |F \cap V(G)| + 1 \leq |F \cap V(G')|.$$

In the second case $u \in F \cap V(G)$ and $u \notin F_1 \cap V(G)$, and so

$$|F_1 \cap V(G')| = |F_1 \cap V(G)| + 1 \leq |F \cap V(G)| \leq |F \cap V(G')| .$$

This proves our claim that

$$|F \cap V(G')| \geq min(|F_1 \cap V(G')| , |F_2 \cap V(G')|).$$

Consequently, G' is a $(k+1)$-shield. This proves (1).

Certainly G' is a stable $(k+1)$-shield, and so from (2.3) we deduce that $|V(G')| \geq \frac{1}{6}(k+1)^2 + \frac{1}{2}(k+1) + 1$. Since $|V(G')| = |V(G)| + 1$, the result follows. ■

3. THE MAIN ARGUMENT

In section 1 we were concerned with the problem of finding a small cutset, defined by a simple closed curve, so that both sides of it contain about the same number of vertices. One can also give the vertices or edges weights, and ask for a small cutset, defined by a simple closed curve, so that both sides contain about the same total weight. This is a little more complicated; for instance, although the analogue of (1.1) holds (that is, $2(2n)^{1/2}$), Gazit's proof of $\frac{7}{3} n^{1/2}$ does not extend, and up to the present $2(2n)^{1/2}$ was the best known. However, we shall show in (3.9) that, for any planar G and for any constant $\lambda \geq 2$, if $\lambda n^{1/2}$ works for the unweighted case then it also works for the weighted case. In particular, our result of $\frac{3}{2}(2n)^{1/2}$ works for the weighted case.

A convenient common generalization of the different ways to assign weights to a planar graph is via "majorities". Let G be a graph drawn in a sphere Σ. A *noose* is a G-normal, simple closed curve $F \subseteq \Sigma$, and its *length* is $|F \cap V(G)|$. A *majority of order k*, where $k \geq 0$ is an integer, is a function *big* which assigns to every noose F of length $\leq k$ a closed disc $big(F) \subseteq \Sigma$ bounded by F, satisfying the following two axioms:

Axiom 1. If $x, y \in \Sigma$ are distinct, and F_1, F_2, F_3 are G-normal I-arcs each between x and y and otherwise disjoint, and $F_1 \cup F_2, F_1 \cup F_3, F_2 \cup F_3$ all have length $\leq k$, and $big(F_1 \cup F_2)$ includes F_3, then $big(F_1 \cup F_2)$ includes one of $big(F_1 \cup F_3)$, $big(F_2 \cup F_3)$.

Axiom 2. If F is a noose with length $\leq min(2, k)$, then either $big(F) - F$ contains a vertex of G, or $big(F)$ includes at least two edges of G.

This is connected with the weighted separator problem via the next two results. $\mathbf{R}_+$ denotes the set of nonnegative real numbers; and if $w : X \to \mathbf{R}_+$ is a function and $Y \subseteq X$, we denote $\Sigma(w(x) : x \in Y)$ by $w(Y)$.

(3.1) *Let G be a graph drawn in a sphere, let* $w : V(G) \to \mathbf{R}_+$ *be a function, and let* $k \geq 0$ *be an integer. Suppose that there is no noose F of length* $\leq k$ *such that*

$$w((D - F) \cap V(G)) + \frac{1}{2} w(F \cap V(G)) \leq \frac{2}{3} w(V(G))$$

for both closed discs D bounded by F. Then G has a majority of order k.

Proof. For each noose F of length $\leq k$ let $big(F)$ be the (unique) closed disc D bounded by F such that

$$w((D - F) \cap V(G)) + \frac{1}{2} w(F \cap V(G)) > \frac{2}{3} w(V(G)) .$$

The axioms may easily be verified. ■

If G is drawn in Σ and $D \subseteq \Sigma$ is a closed disc with $bd(D)$ G-normal, we denote the subgraph of G drawn in D by $G \cap D$.

(3.2) *Let G be a graph in a sphere, let* $w : E(G) \to \mathbf{R}_+$ *be a function, and let* $k \geq 0$ *be an integer. Suppose that*

(i) $w(f) \leq \frac{2}{3} w(E(G))$ *for each* $f \in E(G)$, *and*

(ii) *there is no noose F of length* $\leq k$ *such that* $w(E(G \cap D)) \leq \frac{2}{3} w(E(G))$ *for both closed discs D bounded by F.*

Then G has a majority of order k.

Proof. For each noose F of length $\leq k$, let $big(F)$ be the unique closed disc D bounded by F with $w(E(G \cap D)) > \frac{2}{3} w(E(G))$. Again, the axioms may easily be verified. ■

Let *big* be a majority of order k in G. A noose F is *optimal* if

(i) it has length $\leq k$

(ii) subject to (i), $G \cap big(F)$ is minimal, and

(iii) subject to (i) and (ii), $|F \cap V(G)|$ is maximum.

(3.3) *Let G be a loopless graph drawn in a sphere* Σ, *let big be a majority of order* $k \geq 0$, *and let F be an optimal noose. Then* $G \cap big(F)$ *is a strong stable k-shield in* $big(F)$.

Proof. Let $|F \cap V(G)| = k'$. We claim first that $G \cap big(F)$ is a strong k'-shield in $big(F)$. For let $big(F) = \Delta$, let $F_3 \subseteq \Delta$ be a G-normal I-arc with ends $x, y \in F$ and with $F_3 \cap F = \{x, y\}$, and let F_1, F_2 be the two arcs between x, y in F. Suppose that

$$|F_3 \cap V(G)| \leq |F_1 \cap V(G)|, |F_2 \cap V(G)| \ .$$

Let $\Delta_i \subseteq \Delta$ be the closed disc bounded by $F_i \cup F_3$ $(i = 1, 2)$. We must show that one of Δ_1, Δ_2 includes $G \cap \Delta$. For $i = 1, 2$, $F_i \cup F_3$ is a G-normal noose with length $\leq k$, since

$$|(F_i \cup F_3) \cap V(G)| \leq |F \cap V(G)| \leq k \ .$$

From axiom 1, we may assume that $\Delta_1 = big(F_1 \cup F_3)$. Since F is optimal, it follows that $G \cap \Delta_1 = G \cap \Delta$, that is, Δ_1 includes $G \cap \Delta$, as required. Thus, $G \cap big(F)$ is a strong k'-shield.

We claim that $G \cap big(F)$ is a stable k'-shield. This is clear if $k' \geq 3$ because every strong k'-shield with $k' \geq 3$ is stable, but needs proof if $k' \leq 2$. Suppose that e is an edge of $G \cap \Delta$ with ends $x, y \in bd(\Delta)$, and that $L \subseteq F$ is an I-arc with ends x, y, such that $e \cup L$ bounds a region of G in Δ. Let $F_3 \subseteq \Delta$ be a G-normal I-arc with ends x, y, just on the other side of e from L, in the natural sense. From axiom 2, $big(F_3 \cup L) \not\subseteq \Delta$, and so from axiom 1, $big(F_3 \cup (F - L)) \subseteq \Delta$, contrary to the optimality of F. Thus $G \cap \Delta$ is a stable k'-shield in Δ.

Finally, we claim that $k' = k$. For suppose that $k' < k$ and let r be a region of $G \cap \Delta$ in Δ with $F \cap r \neq \emptyset$. Suppose that $v \in V(G \cap \Delta)$ is incident with r, and $v \notin F$. Choose distinct $x, y \in r \cap F$, and let F_3 be an I-arc with ends x, y and $F_3 \cap F = \{x, y\}$ and $F_3 \subseteq r \cup \{v\}$, passing through v. Since $k > k' \geq 0$ it follows that $|(F_1 \cup F_3) \cap V(G)| \leq 1 \leq k$, where $F_1 \subseteq r \cap F$ is an I-arc between x, y, and $|(F_2 \cup F_3) \cap V(G)| \leq k' + 1 \leq k$, where $F_2 \subseteq F$ is the other I-arc between x, y. By axiom 2, $big(F_1 \cup F_3) \not\subseteq \Delta$, and so by axiom 1, $big(F_2 \cup F_3) \subseteq \Delta$ contrary to the optimality of F.

Hence, every $v \in V(G \cap \Delta)$ incident with r belongs to F. Since $G \cap \Delta$ is a k'-shield it follows that $r \cap F$ is connected. If $F \subseteq r$ then r is incident with no vertex of $G \cap \Delta$ and so $G \cap \Delta$ is null, contrary to the second axiom. Hence $r \cap F$ is an open line segment, with ends $x, y \in V(G)$. Since $G \cap \Delta$ is a k'-shield it follows that r is incident with no vertex of $G \cap \Delta$ except x and y. In particular, $x \neq y$ since G is loopless, and there is an edge of G

with ends x and y, incident with r. But this is impossible since $G \cap \Delta$ is a stable k'-shield and r is incident with no vertex except x and y. We deduce that $k' = k$, as required. ■

Consequently, we have

(3.4) *Let G be a graph in a sphere* Σ, *and let big be a majority of order* $k \geq 0$. *For any noose* $F \subseteq \Sigma$ *of length* $\leq k$, $|V(G) \cap big(F)| \geq \frac{1}{6}k^2 + \frac{5}{6}k + \frac{2}{3}$.

Proof. From the definition of optimal noose, there is an optimal noose F' with $G \cap big(F') \subseteq G \cap big(F)$. We claim that $|V(G \cap big(F'))| \geq \frac{1}{6}k^2 + \frac{5}{6}k + \frac{2}{3}$. If $k \leq 2$ this follows from the second axiom (together with the first if $k = 2$), while for $k \geq 3$ it follows from (2.5), since $G \cap big(F')$ is a strong k-shield by (3.3). Since $|V(G \cap big(F))| \geq |V(G \cap big(F'))|$, the result follows. ■

Let us say a noose in G has *discrepancy* $|n_1 - n_2|$, where it bounds closed discs Δ_1, Δ_2 and $n_i = |V(G) \cap \Delta_i|$ $(i = 1, 2)$. We have immediately from (3.4) that

(3.5) *Let G be a graph in a sphere* Σ, *with n vertices. There is a noose of length* $\leq 6^{1/2}n^{1/2}$ *with discrepancy* $\leq \frac{1}{3}n$.

Proof. Let $k = \lfloor 6^{1/2}n^{1/2} \rfloor$. If G has a majority of order k then by (3.4),

$$|V(G)| \geq \frac{1}{6}k^2 + \frac{5}{6}k + \frac{2}{3} \geq \frac{1}{6}(k+1)^2 > n ,$$

a contradiction. Thus, by (3.1) (with $w(v) = 1$ for all v), there is a noose F of length $\leq k$ such that the discs D_1, D_2 bounded by F satisfy

$$|(D_i - F) \cap V(G)| + \frac{1}{2}|F \cap V(G)| \leq \frac{2}{3}|V(G)| \quad (i = 1, 2) ,$$

or, equivalently, that F has discrepancy $\leq \frac{1}{3}n$. ■

Actually, here the $6^{1/2}$ is irrelevant; all we need from (3.5) is that some noose has discrepancy $\leq \frac{1}{2}n$.

(3.6) *Let G be a graph in a sphere* Σ, *with n vertices. There is a noose of length* $\leq \frac{3}{2}(2n)^{1/2}$ *with discrepancy* $\leq \frac{1}{2}n$.

Proof. We assume $n > 0$. By (3.5) there is a noose with discrepancy $\leq \frac{1}{2}n$. Let us choose such a noose F of minimum order, k say. Let F bound closed discs Δ, Δ' with $|V(G) \cap \Delta| \geq |V(G) \cap \Delta'|$.

(1) *$G \cap \Delta$ is a k-shield in Δ.*

For let F_3 be a G-normal I-arc with ends $x, y \in F$, and let F_1, F_2 be the two I-arcs in F with ends x, y. Suppose that

$$|F \cap V(G)| < |F_1 \cap V(G)|, |F_2 \cap V(G)| .$$

Since

$$|V(G) \cap (\Delta - F)| + \frac{1}{2} |V(G) \cap F| \geq \frac{1}{2} n$$

because $|V(G) \cap \Delta| \geq |V(G) \cap \Delta'|$, we may assume that

$$|V(G) \cap (\Delta_1 - (F_1 \cup F_3))| + \frac{1}{2} |V(G) \cap (F_1 \cup F_3)| \geq \frac{1}{4} n$$

without loss of generality, where Δ_1 is the closed disc in Δ bounded by $F_1 \cup F_3$. But then $F_1 \cup F_3$ has discrepancy $\leq \frac{1}{2} n$, and has order $< k$, contrary to the choice of F. This proves (1).

Now let us choose such F, Δ with $E(G \cap \Delta)$ minimal. It follows that $G \cap \Delta$ is a stable k-shield, and so by (2.3),

$$|V(G \cap \Delta)| \geq \frac{1}{6} k^2 + \frac{1}{2} k + 1 .$$

(for we may assume that $k > \frac{3}{2} (2n)^{1/2}$ since otherwise F satisfies the theorem, and $\frac{3}{2} (2n)^{1/2} \geq 2$ since $n \geq 1$, and so $k \geq 3$). Hence

$$|V(G \cap \Delta')| \geq \frac{1}{6} k^2 + \frac{1}{2} k + 1 - \frac{1}{2} n$$

since F has discrepancy $\leq \frac{1}{2} n$; and so

$$n + k = |V(G)| + |V(G) \cap F| = |V(G \cap \Delta)| + |V(G \cap \Delta')| \geq 2(\frac{1}{6} k^2 + \frac{1}{2} k + 1) - \frac{1}{2} n .$$

It follows that $\frac{3}{2} n \geq \frac{1}{3} k^2 + 2$, and so $k < \frac{3}{2}(2n)^{1/2}$, as required. ■

We deduce

(3.7) *Let G be a graph in a sphere* Σ, *with n vertices, and with a majority of order k. Then* $k \le \frac{3}{2}(2n)^{1/2} - 1$.

Proof. Let *big* be a majority of order k, and suppose that $k \ge \lfloor \frac{3}{2}(2n)^{1/2} \rfloor$. By (3.6), there is a noose F of length $\le k$ with discrepancy $\le \frac{1}{2}n$. By (3.4),

$$|V(G) \cap big(F)| \ge \frac{1}{6}k^2 + \frac{5}{6}k + \frac{2}{3};$$

but

$$|V(G)| + |V(G) \cap F| \ge 2\,|V(G) \cap big(F)| - \frac{1}{2}n$$

since F has discrepancy $\le \frac{1}{2}n$, and so

$$\frac{3}{2}n + k \ge \frac{1}{3}k^2 + \frac{5}{3}k + \frac{4}{3},$$

since $|V(G) \cap F| \le k$. Hence $\frac{1}{3}(k+1)^2 + 1 \le \frac{3}{2}n$, and so $k + 1 < \frac{3}{2}(2n)^{1/2}$, a contradiction. Thus $k < \lfloor \frac{3}{2}(2n)^{1/2} \rfloor$, as required. ∎

From (3.7) and (3.1) we deduce our main result, the following weighted version of (2.1).

(3.8) *Let G be a graph in a sphere with n vertices, and for each vertex v let* $w(v) \ge 0$ *be a real number. There is a noose F with* $|F \cap V(G)| \le \frac{3}{2}(2n)^{1/2}$ *such that*

$$w((D - F) \cap V(G)) + \frac{1}{2}w(F \cap V(G)) \le \frac{2}{3}w(V(G))$$

for both closed discs D bounded by F.

Proof. Let $k = \lfloor \frac{3}{2}(2n)^{1/2} \rfloor$. By (3.7), G has no majority of order k, and the result follows from (3.1). ∎

Similarly one can use (3.7) and (3.2) to deduce a $\frac{3}{2}(2n)^{1/2}$-separator result when the weights are on the edges.

Finally, let us show the following curiosity, which indicates that for finding "separating" nooses of length $\le \lambda n^{1/2}$ where $\lambda \ge 2$, in some sense the unweighted case is the hardest.

(3.9) *Let G be a graph in a sphere, with n vertices, let* $k \ge 2n^{1/2} - 1$ *be a integer, and suppose that there is a noose* F^* *of length* $\le k$ *such that*

$$|(D-F^*)\cap V(G)| + \frac{1}{2}|F^*\cap V(G)| \le \frac{2}{3}|V(G)|$$

for both closed discs D bounded by F^*. Then

(i) *G has no majority of order k*

(ii) *for any function $w : V(G) \to \mathbf{R}_+$ there is a noose F of length $\le k$ such that*

$$w((D-F)\cap V(G)) + \frac{1}{2}w(F\cap V(G)) \le \frac{2}{3}w(V(G))$$

for both closed discs D bounded by F

(iii) *for any function $w : E(G) \to \mathbf{R}_+$ such that $w(f) \le \frac{2}{3}w(E(G))$ for every $f \in E(G)$, there is a noose F of length $\le k$ such that $w(E(G\cap D)) \le \frac{2}{3}w(E(G))$ for both closed discs D bounded by F.*

Proof. Suppose that *big* is a majority of order k. By (3.4),

$$|V(G)\cap big(F^*)| \ge \frac{1}{6}k^2 + \frac{5}{6}k + \frac{2}{3}.$$

But by the hypothesis,

$$|V(G)\cap big(F^*)| - \frac{1}{2}|V(G)\cap F^*| \le \frac{2}{3}n.$$

Moreover, $|V(G)\cap F^*| \le k$, and so

$$\frac{1}{6}k^2 + \frac{5}{6}k + \frac{2}{3} - \frac{1}{2}k \le \frac{2}{3}n,$$

that is, $(k+1)^2 + 3 \le 4n$. But $k+1 \ge 2n^{1/2}$, a contradiction. This proves (i), and (ii) and (iii) follow from (3.1) and (3.2) respectively. ■

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