# Subgraphs of Large Connectivity and Chromatic Number in Graphs of Large Chromatic Number

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**ABSTRACT**

For each pair $k, m$ of natural numbers there exists a natural number $f(k, m)$ such that every $f(k, m)$-chromatic graph contains a $k$-connected subgraph of chromatic number at least $m$.

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## INTRODUCTION

Mader [1] proved that every graph of minimum degree at least $4k$ contains a $k$-connected subgraph. Thus every $(4k + 1)$-chromatic graph contains a $k$-connected subgraph. In this note we show that a graph of sufficiently large chromatic number contains a subgraph that has both large connectivity and large chromatic number. This result, which is useful for finding general configurations in graphs of large chromatic number (see [3]), was first stated in [2] but the proof given there is in error. We prove the following:

**Theorem.** Every graph $G$ of chromatic number greater than $p = \max(100k^3, 10k^2 + m)$ contains a $(k + 1)$-connected subgraph of chromatic number at least $m$.

## NOTATION AND PROOF OF THE THEOREM

For any vertex set $A$ of $G$ we denote by $G(A)$ the subgraph of $G$ induced by $A$ and $G - A$ denotes $G[V(G)\backslash A]$. As usual $\chi(G)$ denotes the chromatic number of $G$. The number of neighbors in $A$ of a vertex $v$ is denoted $d_A(v)$. The *A-weight* of $v$ is defined as

$$w_A(v) = 2k + 1 - \frac{2k}{p} \min[d_A(v), p]$$

and, for each vertex set $S$ of $G$, we put

$$w_A(S) = \sum w_A(v),$$

where the summation is taken over all $v$ in $S$. Note that $w_A(S) \geq 1$ always. Finally, we put $W = 10k^2$.

We now prove the theorem. Without loss of generality we can assume that $G$ is $(p + 1)$-color-critical and hence all vertices of $G$ have degree at least $p$. If $G$ is $(k + 1)$-connected, there is nothing to prove, so $G$ has a separating vertex set $S$ with at most $k$ vertices. If $A$ is the union of the (vertex sets of) some connected components of $G - S$ then clearly

$$|S| \leq w_A(S) \leq W. \tag{1}$$

Among all pairs $S, A$ where $S$ is a separating vertex set and $A$ the union of some (but not all) vertex sets of connected components of $G - S$ satisfying (1), we choose one such that $|A|$ is minimal. We shall prove that $G(A \cup S)$ has the desired properties.

$$\chi[G(A \cup S)] \geq m. \tag{2}$$

***Proof of (2).*** Since $G$ is $(p+1)$-color-critical, $\chi(G-A) \le p$. If $\chi[G(A)] \le p - |S|$, then any $p$-coloring of $G - A$ can be extended to a $p$-coloring of $G$, which is impossible. So $\chi[G(A)] > p - |S|$ and, by (1)

$$\chi[G(A \cup S)] \ge \chi[G(A)] \ge p - |S| + 1 > 10k^2 + m - W = m,$$

which proves (2).

It remains to be shown that $G(A \cup S)$ is $(k+1)$-connected. We first prove an auxiliary result:

$$\text{For each } v \text{ in } S, d_A(v) \ge k + 1. \tag{3}$$

***Proof of (3)*** (by contradiction). Suppose that $d_A(v) \le k$ for some $v$ in $S$. Let $N$ be the set of neighbors of $v$ in $A$. Since $A$ is nonempty and $G$ has minimum degree at least $p$, it follows that

$$|A| \ge p + 1 - |S| \ge p + 1 - W > k.$$

We put $S' = (S\backslash\{v\}) \cup N$ and $A' = A\backslash N$. Then $0 < |A'| < |A|$ and, for every vertex $u$ in $N$,

$$d_{A'}(u) \ge p - W - k + 1.$$

Hence

$$\sum_{u \in N} w_{A'}(u) < k\left[2k + 1 - \frac{2k}{p}(p - W - k)\right].$$

Also

$$w_{A'}(S) - w_A(S) \le W\frac{2k}{p}k.$$

Combining the last two inequalities we get

$$\begin{aligned} w_{A'}(S') &\le w_A(S) + W\frac{2k}{p}k - w_A(v) + k\left[2k + 1 - \frac{2k}{p}(p - W - k)\right] \\ &\le W + W\frac{2k^2}{p} - 2k - 1 + \frac{2k^2}{p} + k\left(1 + \frac{2kW}{p} + \frac{2k^2}{p}\right) \\ &\le W + \frac{1}{5}k - 2k - 1 + \frac{1}{50k} + k + \frac{1}{5}k + \frac{1}{50} \\ &< W. \end{aligned}$$

Hence the pair $S', A'$ satisfies (1), contradicting the minimality of $|A|$. This proves (3).

$$G(A \cup S) \quad \text{is} \quad (k + 1)\text{-connected}. \tag{4}$$

***Proof of (4)*** (by contradiction). Suppose $S'$ is a separating vertex set of $G(A \cup S)$ such that $|S'| \leq k$. Then the vertex set of $G(A \cup S) - S'$ can be partitioned into two nonempty sets $A_1 \cup S_1$ and $A_2 \cup S_2$ such that there is no edge from $A_1 \cup S_1$ to $A_2 \cup S_2$ and $A_1 \cup A_2 \subseteq A$, $S_1 \cup S_2 \subseteq S$. By (3), each of $A_1$, $A_2$ is nonempty. Then each of $S' \cup S_1$ and $S' \cup S_2$ is a separating vertex set of $G$ and without loss of generality we can assume that

$$w_A(S_1) \leq w_A(S_2) \leq W.$$

In particular, $w_A(S_1) \leq (W/2)$. Now

$$\begin{aligned} w_{A_1}(S' \cup S_1) &= w_{A_1}(S') + [w_{A_1}(S_1) - w_A(S_1)] + w_A(S_1) \\ &\leq k(2k + 1) + \frac{W}{2}\frac{2k}{p} \cdot k + \frac{W}{2} \\ &\leq W. \end{aligned}$$

Hence the pair $S' \cup S_1$, $A_1$ satisfies (1), contradicting the minimality of $|A|$. This proves (4) and the theorem.

The Theorem shows that

$$f(k, m) \leq 100k^3 + m,$$

where $f(k, m)$ is the (smallest) number satisfying the statement of the abstract. We obtain the lower bound

$$f(k, m) \geq k + m - 2$$

as follows: Take $k - 1$ disjoint copies of the complete graph $K_{k-1}$. For each vertex set $S$ containing precisely one vertex of each $K_{k-1}$ we add a $K_{m-2}$ and join it completely to $S$. Then the resulting graph $G_{k,m}$ has chromatic number $k + m - 3$ and no $k$-connected subgraph of $G_{k,m}$ contains vertices of two distinct $K_{m-2} - s$. Hence every $k$-connected subgraph of $G_{k,m}$ is $(m - 1)$-colorable.

## References

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