The Longest Cycle of a Graph with a Large Minimal Degree

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ABSTRACT

We show that every graph G on n vertices with minimal degree at least n/k contains a cycle of length at least $[n/(k-1)]$. This verifies a conjecture of Katchalski. When $k = 2$ our result reduces to the classical theorem of Dirac that asserts that if all degrees are at least $\frac{1}{2}n$ then G is Hamiltonian.

1. INTRODUCTION

For a graph $G = \langle V(G), E(G)\rangle$ let $\delta(G)$ denote the minimal degree of vertex of G and let $c(G)$ denote the circumference, i.e., the size of a longest cycle of G. In this note we prove

Theorem 1.1. Suppose $n > k \geq 2$ are integers and that G is a graph on n vertices with $\delta(G) \geq n/k$. Then $c(G) \geq [n/(k-1)]$.

This result was conjectured by M. Katchalski [3], who also proved it for $k \leq 4$. For $k = 2$ it reduces to the classical theorem of Dirac ([2], see also [1, p. 135]) and asserts that G is Hamiltonian provided $\delta(G) \geq \frac{1}{2}n$.

The following example shows that Theorem 1.1 is almost sharp: Suppose $n > k \geq 2$ and put $r = [(n-1)/k]$. Then $n = kr + s$, where $1 \leq s \leq k$ and r is the largest integer $<n/k$. Let G be the graph consisting of k complete graphs on $r + 1$ vertices, $k - s + 1$ of which share a common vertex. Then G has n vertices, $\delta(G) = r$, and G contains no cycle of size $>r + 1$. It is worth noting, though, that there is a possible (slight) strengthening of Theorem 1.1, namely, $c(G) \geq n/(k-1)$ (without the integer part), provided $\delta(G) \geq n/k$. At the moment we can prove this strengthened version only for $n \gg k$.

Journal of Graph Theory, Vol. 10 (1986) 123–127
 CCC 0364-9024/86/010123-05$04.00

To prove Theorem 1.1 we need several known results and several new lemmas. These are presented in Section 2 and are applied in Section 3 to deduce the theorem.

2. LEMMAS

Lemma 2.1 (Dirac [2], see also [4, problems 10.21, 10.27, pp. 67–68]). Let G be a graph on n vertices.

(a) If $\delta(G) > 1$ then $c(G) \geq \delta(G) + 1$.
(b) If $\delta(G) \geq \frac{1}{2}n > 1$ then G is Hamiltonian.
(c) If G is 2-connected then

$$c(G) \geq \min(n, 2\delta(G)).$$

Lemma 2.2 (Lovász [4, problem 10.19, p. 67]). Let x and y be two distinct vertices of a 2-connected graph G and suppose that each vertex of G other than x and y has degree at least d. Then there is a path of length $\geq d$ between x and y. ∎

Lemma 2.3. Let z be a vertex of a 2-connected graph G and suppose that each vertex of G other than z has degree $\geq d$. Then either G contains a cycle of size $\geq 2d$ including z or any two distinct vertices of G are connected by a path of length $\geq d - 1$.

Proof. If $G - z$ is 2-connected the second possibility holds, by Lemma 2.2. Otherwise, let B and C be two endblocks of $G - z$ and let $b \in V(B)$ and $c \in V(C)$ be the corresponding cutvertices (possibly $b = c$). Clearly z has a neighbor $b' \in V(B) - b$ and a neighbor $c' \in V(C) - c$. By Lemma 2.2, B contains a (b', b)-path of length $\geq d - 1$ and C contains a (c', c)-path of length $\geq d - 1$. These paths, the edges zb', zc', and a (b, c)-path in $G - [z + (B - b) + (C - c)]$ form a cycle of length $\geq 2d$ containing z. ∎

Remark. As noted by one of the referees, one can prove that, in fact, under the hypotheses of Lemma 2.3, there is a path of length at least d between any two vertices of G. For our purposes, however, Lemma 2.3 suffices.

To prove Theorem 1.1, we have to study the block structure of the graph G. This is done in the rest of this section.

Suppose $n > k(k - 1)$, where $k \geq 2$, and let G be a connected graph on n vertices with $\delta(G) \geq n/k$. Call a block of G *large* if it has at least $n/k + 1$ vertices, otherwise call it *small*. Let $\mathcal{B}$ denote the set of all large blocks of G, and $\mathcal{C}$ the set of all cutvertices of G that belong to at least two large blocks. Finally, let H denote the bipartite graph with classes of vertices $\mathcal{B}$ and $\mathcal{C}$, in which $B \in \mathcal{B}$ is joined to $c \in C$ iff $c \in V(B)$. Note that H is a subgraph of the block-cutvertex tree of G induced by $\mathcal{B} \cup \mathcal{C}$, and is therefore a forest.

Lemma 2.4

(a) $|\mathcal{B}| \le k - 1$.
(b) $|\mathcal{C}| \le k - 2$.
(c) Every vertex of G belongs to at least one large block of G.
(d) If y is a cutvertex of G that belongs to a unique large block, then y is joined to at least $n/k - k + 2 + |\mathcal{C}|$ vertices of this block.

Proof. Note that every endblock of G is a large block. Indeed, if B is an endblock it contains a vertex v which is not a cutvertex of G and all the $\ge n/k$ neighbors of v belong to $V(B) - v$. Hence B is large. For $c \in \mathcal{C}$ let $d(c)$ denote the degree of c in H, i.e., the number of large blocks of G that contain c. Let $\gamma = \gamma(H)$ denote the number of connected components of H. Since H is a forest with classes of vertices $\mathcal{B}$ and $\mathcal{C}$;

$$|E(H)| = \sum\{d(c)\colon c \in \mathcal{C}\} = |\mathcal{B}| + |\mathcal{C}| - \gamma\,. \tag{2.1}$$

Thus

$$\begin{aligned} n &\ge |\cup\{V(B) B \in \mathcal{B}\}| = \sum\{|V(B)| B \in \mathcal{B}\} - \sum\{[d(c) - 1]\colon c \in \mathcal{C}\} \\ &\ge |\mathcal{B}|(n/k + 1) + |\mathcal{C}| - \sum\{d(c)\colon c \in \mathcal{C}\} = |\mathcal{B}|(n/k) + \gamma \\ &> |\mathcal{B}| n/k\,. \end{aligned}$$

This verifies (a).

Equality (2.1) and (a) imply

$$|\mathcal{C}| \le \sum\{[d(c) - 1]\colon c \in \mathcal{C}\} = |\mathcal{B}| - \gamma \le k - 2\,,$$

which proves (b).

To prove (c) assume it is false and let $v \in V(G)$ be a counterexample. Note that every vertex that belongs to a small block of G is a cutvertex, since its degree is at least n/k. Thus v is a cutvertex. Let $B_1, B_2, \ldots, B_r$ be the blocks of G that contain v. Clearly $\Sigma_{i=1}^{r} |V(B_i) - v| \ge n/k$ and every vertex of $\cup_{i=1}^{r} V(B_i)$ is a cutvertex. Thus there are at least $n/k > k - 1$ vertices of H within a distance 2 (in H) from v. Therefore H has more than $k - 1$ endvertices, each of which is a large block of G, contradicting (a). This contradiction proves (c).

To prove (d), let y be a cutvertex of G that belongs to a unique large block of G. Consider the subgraph F obtained from G by deleting all edges that do not belong to large blocks. By (a) and (c) F has at most $k - 1 - r$ connected components, where $r = |\mathcal{C}|$ and y belongs to one of them. Clearly y has at most one neighbor in G in any other component, whereas in its own component it is joined in G only to vertices of its own large block. Thus the degree of y in this block is at least $n/k - (k - 2 - r) = n/k - k + 2 + r$. This completes the proof of the lemma. ∎

3. PROOF OF THEOREM 1.1

Suppose $n > k \geq 2$ and let G be a graph on n vertices with $\delta(G) \geq n/k$. We must show that

$$c(G) \geq [n/(k - 1)]. \tag{3.1}$$

If $[n/(k - 1)] \leq \lceil n/k \rceil + 1$ this follows from Lemma 2.1(a). Thus we may assume

$$n/(k - 1) \geq [n/(k - 1)] \geq \lceil n/k \rceil + 2 \geq n/k + 2,$$

i.e.,

$$n \geq 2k(k - 1). \tag{3.2}$$

Clearly we may also assume that G is connected; otherwise add bridges to make it connected. By Lemma 2.4 G has at least $n - k + 2$ vertices that belong to exactly one large block and it has at most $k - 1$ large blocks. Thus there is a block B containing $m \geq (n - k + 2)/(k - 1) > n/(k - 1) - 1$ such vertices. Let D be the induced subgraph of B on these m $(\geq [n/(k - 1)])$ vertices. Since D is obtained from B by deleting vertices that belong to at least two large blocks of G, Lemma 2.4(b), (d) implies that the degree of every vertex of D is at least $n/k - k + 2$. If $2(n/k - k + 2) \geq m$, D is Hamiltonian, by Lemma 2.1(b), and since $m \geq [n/(k - 1)]$ the desired result follows. Thus we may assume that $m > 2(n/k - k + 2)$. Similarly, if D is 2-connected the desired result follows from Lemma 2.1(c) and inequality (3.2), which guarantees that $2(n/k - k + 2) \geq n/(k - 1)$. Thus we may assume that D is not 2-connected. We consider three possible cases, according to the block structure of D.

Case 1. Each block of D is a connected component of it. Let A and C be two distinct blocks of D. Since B is 2-connected there are two vertex-disjoint paths in B connecting $a_i \in V(A)$ to $c_i \in V(C)\,(i = 1, 2)$ and using no edges of A or C. By Lemma 2.2, A contains an (a_1, a_2)-path of length at least $n/k - k + 2$ and C contains a (c_1, c_2)-path of length at least $n/k - k + 2$. Altogether we have a cycle of size at least $2(n/k - k + 2) + 2 > n/(k - 1)$, as needed.

Case 2. D has two nonadjacent endblocks A and C in the same connected component. As in the previous case, the 2-connectivity of B implies the existence of two vertex-disjoint paths in B connecting $a_i \in V(A)$ to $c_i \in V(C)\,(i = 1, 2)$

and using no edges of A or C. If A or C contains a cycle of size at least $2(n/k - k + 2) \geq n/(k - 1)$, (3.1) follows. Otherwise, by Lemma 2.3, A contains an (a_1, a_2)-path of length at least $n/k - k + 1$ and C contains a (c_1, c_2)-path of length at least $n/k - k + 1$. This gives a cycle of size at least $2 + 2(n/k - k + 1) \geq n/(k - 1)$, implying (3.1).

Case 3. D contains two adjacent endblocks A, C. Let x be the unique common vertex of A and C. By the 2-connectivity of B, B contains a path between some $a \in A, a \neq x$, and some $c \in C, c \neq x$, using no edges of A or C. By Lemma 2.2 A contains an (a, x)-path of length at least $n/k - k + 2$ and C contains a (c, x)-path of length at least $n/k - k + 2$. These paths, together with the previous (a, c)-path, form a cycle of size at least $2(n/k - k + 2) + 1 > n/(k - 1)$. This completes the proof of the theorem. ∎

ACKNOWLEDGMENT

This research was supported in part by a Weizmann Fellowship for Scientific Research.

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