COMMUNICATION

A NOTE ON SUBDIGRAPHS OF DIGRAPHS WITH LARGE OUTDEGREES

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Communicated by Daniel J. Kleitman
Received 8 November 1983

In his survey article [3] Nash Williams gives a list of unsolved problems. The last problem is the following.

Let an \((n, \geq q)\)-digraph denote a digraph without loops and parallel directed edges on a set of \(n\) vertices such that the outdegree of every vertex is at least \(q\). If \(D\) is an \((m+n, \geq q+r)\)-digraph, must there be some subdigraph of \(D\) which is an \((m, \geq q)\) or an \((n, \geq r)\) digraph?

The following proposition shows that the answer is "No" even if we allow a somewhat weaker conclusion.

**Proposition.** For every \(k>0\) and every prime \(p \equiv 3 \pmod{4}\) that satisfies \(p > k^2 \cdot 2^{2k-2}\) there exists a \((p, \geq \frac{1}{2}(p-1))\)-digraph that contains neither \((k, \geq \frac{1}{2}k)\)-nor \((p-k, \frac{1}{2}(p-1) - k + 1)\)-subdigraphs.

**Proof.** Let \(T = T_p = (V, E)\) denote the tournament whose vertices are the elements of \(Z_p\), where \((i, i+s)\) is a directed edge iff \(s\) is a quadratic residue \((\neq 0)\) modulo \(p\). Clearly \(T\) is a \((p, \geq \frac{1}{2}(p-1))\)-digraph. Every subdigraph of \(T\) on \(k\) vertices has exactly \(\binom{k}{2}\) edges and thus cannot be a \((k, \geq \frac{1}{2}k)\)-digraph. Consider a subdigraph \(D\) of \(T\) on a set \(W\) of \(p-k\) vertices. By a theorem of Graham and Spencer [1] there exists some \(v \in W\) that dominates all vertices of \(V-W\) and thus in \(D\) the outdegree of \(v\) is \(\frac{1}{2}(p-1) - k < \frac{1}{2}(p-1) - k + 1\). \qee

The assumption \(p > k^2 \cdot 2^{2k-2}\) is not the best possible. Indeed, for \(k = 2\) we can take \(p = 7\). The tournament \(T_7\) is a \((7, \geq 3)\)-digraph with neither \((2, \geq 1)\)-nor \((5, \geq 2)\)-subdigraph.

It is worth noting that using Difference Sets we can construct many examples of \((n+2, \geq q+1)\)-digraphs that are not tournaments and contain neither \((n, \geq q)\)-nor

* Research supported in part by the Weizmann Fellowship for Scientific Research.

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(2, \geq 1)-subdigraphs. As an illustration we use the difference set $A = \{2, 3, 8, 10, 20\}$ in $\mathbb{Z}_{21}$ (see [2]) to construct a $(21, \geq 5)$-digraph with neither $(19, \geq 4)$- nor $(2, \geq 1)$-subdigraphs. Let $D = (V, E)$ be the digraph whose vertices are the elements of $\mathbb{Z}_{21}$, where $(i, i + s)$ is a directed edge iff $s \in A$. Clearly $D$ is a $(21, \geq 5)$-digraph and since $A \cap (-A) = \emptyset$ (in $\mathbb{Z}_{21}$) we conclude that it contains no $(2, \geq 1)$-subdigraph. Consider a subdigraph $H$ of $D$ on a set $W$ of 19 vertices and put $V - W = \{i, j\}$. Since $A$ is a difference set there exist $a, b \in A$ such that $a - b = i - j$, i.e., $i - a = j - b$. Define $k = i - a = j - b$. By the definition of $D$, $(k, i)$ and $(k, j)$ are edges of $D$ and thus the outdegree of $k$ in $H$ is $3 < 4$.

References