A Simple Proof of the Upper Bound Theorem

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Let \( c_i(n, d) \) be the number of \( i \)-dimensional faces of a cyclic \( d \)-polytope on \( n \) vertices. We present a simple new proof of the upper bound theorem for convex polytopes, which asserts that the number of \( i \)-dimensional faces of any \( d \)-polytope on \( n \) vertices is at most \( c_i(n, d) \). Our proof applies for arbitrary shellable triangulations of \((d-1)\)-spheres. Our method provides also a simple proof of the upper bound theorem for \( d \)-representable complexes.

1. INTRODUCTION

Let \( c_i(n, d) \) be the number of \( i \)-dimensional faces of a cyclic \( d \)-polytope on \( n \) vertices. In this note we present a simple proof of the upper bound theorem (UBT) for convex polytopes, which asserts that the number of \( i \)-dimensional faces of any \( d \)-polytope on \( n \) vertices is at most \( c_i(n, d) \). We will consider only simplicial polytopes, since it is well known that it suffices to prove the UBT in this case ([8, p. 80], [13, sect. 2.5]).

The UBT was conjectured by Motzkin in 1957 [14], and proved by McMullen in 1970 ([12], [13, chp. 5]). Another proof was given by Bondesan and Brønsted [2]. Stanley [15] proved that the assertion of the UBT holds also for triangulations of \((d-1)\)-spheres (see also [7], [10] and [16]).

McMullen's proof uses a fundamental result of Bruggesser and Mani [3], which asserts that the boundary complex of a convex polytope is shellable. This notion is crucial also here, and in fact our proof applies for arbitrary shellable triangulations of \((d-1)\)-spheres.

Our method supplies also a simple proof of a theorem conjectured by Katchalski and Perles and proved independently by Eckhoff [5] and by the second author [9]. This theorem asserts that if \( \mathcal{K} \) is a family of \( n \) convex sets in \( \mathbb{R}^d \) and \( \mathcal{K} \) has no intersecting subfamily of size \( d+r+1 \), then the number of intersecting \( k \)-subfamilies of \( \mathcal{K} \) for \( d+1 \leq k \leq d+r \) is at most

\[
\sum_{i=0}^{d} \binom{n-r}{i} \cdot \binom{r}{k-i}.
\]

Equality holds, e.g. if \( \mathcal{K} = \{K_1, \ldots, K_n\} \) where \( K_1 = K_2 = \cdots = K_r = \mathbb{R}^d \) and \( K_{r+1}, \ldots, K_n \) are hyperplanes in general position in \( \mathbb{R}^d \).

2. ON THE NUMBER OF ELEMENTARY COLLAPSES

We begin with a combinatorial lemma. Equivalent formulations of it were proved by Frankl [6] and by the second author [9], and a generalization was proved by the first author [1]. Here we present a short proof, following the approach of [1].

**Lemma 2.1.** Suppose \( n > 1 \), \( N = \{1, 2, \ldots, n\} \) and \( 1 \leq s \leq n \leq s \). For \( 1 \leq i \leq n \) let \( A_i \) and \( B_i \) be subsets of \( N \) that satisfy

\[
|A_i| < s \quad \text{and} \quad |B_i| > m, \quad \text{for} \quad 1 \leq i \leq h.
\]

\[
A_i \subseteq B_i, \quad \text{for} \quad 1 \leq i \leq h.
\]

\[
A_i \not\subseteq B_j, \quad \text{for} \quad 1 \leq i < j \leq h.
\]

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Then

\[ h \leq \binom{n - m + s}{s}. \]

**Proof.** Clearly we may assume that \(|A_i| = s\) for \(1 \leq i \leq h\). Let \(V = \mathbb{R}^{n - m + s}\) be the \((n - m + s)\)-dimensional real space and let \(v_1, v_2, \ldots, v_n\) be vectors in general position in \(V\) (i.e., every set of \(\leq n - m + s\) of them is linearly independent). Let \(\wedge V\) denote the exterior algebra over \(V\), equipped with the usual wedge product \(\wedge\) (see [4] or [11] for general information on exterior algebra).

For \(1 \leq i \leq h\) define \(y_i = \Lambda_{j \neq A_i} v_j \in \wedge^i V\) and \(\bar{y}_i = \Lambda_{k \in N \setminus B_i} v_k\). By (2.1), (2.2) and the general position of the \(v_j\),

\[ y_i \wedge \bar{y}_i \neq 0, \quad \text{for } 1 \leq i \leq h. \]

By (2.3)

\[ y_i \wedge \bar{y}_i = 0, \quad \text{for } 1 \leq i < j \leq h. \]

To complete the proof we show that the set \(\{y_i : 1 \leq i \leq h\}\) is linearly independent in \(\Lambda^i V\) and thus \(h \leq \dim(\Lambda^i V) = \binom{n - m + s}{s}\). Indeed, suppose this is false and let

\[ \sum_{i=1}^{h} c_i y_i = 0 \]

be a linear dependence, with \(c_i \neq 0\) for \(i \in I\). Put \(j = \max\{i : i \in I\}\). Combining (2.5) and (2.6) we obtain \(0 = (\sum_{i=1}^{h} c_i y_i) \wedge \bar{y}_j = c_j y_j \wedge \bar{y}_j\), which, together with (2.4), supplies the contradiction \(c_j = 0\).

A face \(S\) of a simplicial complex \(C\) is free if \(S\) is contained in a unique maximal face \(M\) of \(C\). The operation of deleting \(S\) and all faces that contain it is an **elementary-collapse**. If the size of \(S\) is \(s\) and the size of \(M\) is \(m\), it is called an **elementary-\((s, m)\)-collapse**. A **collapse process** on \(C\) is a sequence \(C = C_0 \supseteq C_1 \supseteq \cdots \supseteq C_t\) of simplicial complexes such that for \(1 \leq i \leq t\) \(C_i\) is obtained from \(C_{i-1}\) by an elementary-collapse.

The following lemma plays a crucial role in our proofs.

**Lemma 2.2.** Let \(s, m\) be nonnegative integers, \(s \leq m\), and let \(C\) be a simplicial complex on \(n\) vertices. The number of all elementary-\((s', m')\)-collapses, with \(s' \leq s\) and \(m' \geq m\), in any collapse process on \(C_i\) is at most \(\binom{n - m + s}{s-s'}\).

**Proof.** Let \(C = C_0 \supseteq C_1 \supseteq \cdots \supseteq C_t\) be a collapse process on \(C\). Let \(S_i\) and \(M_i\) be the free face and the maximal face, corresponding to the \(i\)th elementary-collapse, \(1 \leq i \leq t\). Let \((A_i, B_i)_{i=1}^{t}\) be the subsequence of \((S_0, M_0)_{i=1}^{t}\) consisting of those pairs \((S_n, M_i)\) with \(|S_i| \leq s\) and \(|M_i| \geq m\). One can easily check that \((A_i, B_i) (1 \leq i \leq h)\) satisfy the hypotheses of Lemma 2.1. Therefore \(h \leq \binom{n - m + s}{s-s'}\).

### 3. The Upper Bound Theorem for Shellable Spheres

Let \(C\) be a triangulation of a \((d-1)\)-sphere on a set \(N = \{1, 2, \ldots, n\}\) of \(n\) vertices. Let \(f = f_i(c)\) be the number of \(i\)-dimensional faces of \(C\), \(0 \leq i \leq d-1\), and put \(f_{-1} = 1\). The \(h\)-vector \((h_0, \ldots, h_d)\) of \(C\) is defined by the equations

\[ f_j = \sum_{i=0}^{j+1} \binom{d-i}{d-j-1} h_i \quad -1 \leq j \leq d-1. \]  

(3.1)

(See, e.g. [15] or [13, chp. 5] where \(g_k^{(d)}\) is used for \(h_{k+1})\).
As is well known ([8, sect. 9.2], [13, p. 171]) \( C \) satisfies the Dehn–Somerville equations that can be written as
\[
h_i = h_{d-i}, \quad 0 \leq i \leq [d/2]. \tag{3.2}
\]
For \( 0 \leq i \leq d \) define \( \tilde{h}_i = \sum_{j=0}^i h_j \). Thus
\[
h_i = \tilde{h}_i - \tilde{h}_{i-1}. \tag{3.3}
\]
Substituting (3.2) and then (3.3) in (3.1) one can express every \( f_j \) as a linear combination of \( \{\tilde{h}_i : 0 \leq i \leq [d/2]\} \) with nonnegative coefficients as follows:
\[
f_j = \begin{cases} 
\sum_{i=0}^{[d/2]} \left[ \binom{d-i-1}{j-i-1} \left( \binom{d-i}{j-1} - \binom{d-i}{j-2} \right) \right] \tilde{h}_i & \text{for odd } d \\
\sum_{i=0}^{[d/2]} \left[ \binom{d-i-1}{j-i-1} \left( \binom{d-i}{j-1} - \binom{d-i}{j-2} \right) \right] \tilde{h}_i + \left( \binom{d}{j} \binom{d/2}{j-1} \right) \tilde{h}_{d/2} & \text{for even } d.
\end{cases} \tag{3.4}
\]
It is well known (see, e.g. [13, p. 172]) that for the cyclic \( d \)-polytope with \( n \) vertices (or any neighbourly \( d \)-polytope with \( n \) vertices) \( h_i = \binom{n-d+i}{i} \), \( 0 \leq i \leq [d/2] \), and thus \( \tilde{h}_i = \binom{n-d+i}{i} \), \( 0 \leq i \leq [d/2] \). In view of (3.4), in order to prove the UBT it is enough to show that
\[
\tilde{h}_i \leq \binom{n-d+i}{i}. \tag{3.5}
\]
We proceed to show that (3.5) is satisfied by any \((d-1)\)-shellable sphere.

For \( F \subset N \) let \( F \) denote the set of all subsets of \( F \). \( C \) is shellable if its maximal faces are all of dimension \( d-1 \) and can be ordered \( F_1, F_2, \ldots, F_t \), so that for
\[
1 \leq k \leq t-1 \quad \bar{F}_k \cap \left( \bigcup_{i=k+1}^{t} \bar{F}_i \right) = \bigcup_{j=1}^{s_k} \bar{G}_j,
\]
where \( G^*_i \) are \( s_k \) \( d \)-faces of \( C \) of dimension \( d-2 \). In this case define, for \( 0 \leq i \leq t \), \( C_i = \bigcup_{h=i}^{t} \bar{F}_h \) (thus \( C_t = \emptyset \)). For \( 1 \leq i \leq t-1 \) put \( S_i = F_i \cap \bigcup_{j=1}^{i} \bar{G}_j \) and define \( S_0 = \emptyset \). One can easily check that \( S_i \) is a free face of \( C_{i+1} \) and \( C_i \) is obtained from \( C_{i-1} \) by deleting \( S_i \) and all faces that contain it, i.e. by an elementary \((|S_i|, d)\)-collapsible.

Let \( g_i \) denote the number of elementary \((i, d)\)-collapses in the shelling of \( C \) (i.e. \( g_i = |\{ k : s_k = i \} | \)). It is well-known (see [13, p. 175]) that \( g_i = h_i \), \( 0 \leq i \leq d \). (Indeed, the number of \( j \)-faces deleted in an elementary \((i, d)\)-collapse is \( \binom{d-i}{j-1} \).) Since \( C_i = \emptyset \)
\[
f_j = \sum_{i=0}^{j+1} \binom{d-i}{j+1-i} g_i \quad \text{for } -1 \leq j \leq d-1.
\]
Therefore \( g_0, \ldots, g_d \) satisfy the defining equations (3.1) for the \( h_i \)s and hence \( g_i = h_i \) for \( 0 \leq i \leq d \). Since \( h_i = \sum_{i=0}^{j+1} g_i \) is just the number of all elementary \((i', d)\)-collapses with \( i' \leq i \) in the collapse process \( C = C_0 \supset C_1 \supset \cdots \supset C_t \) on \( C \), Lemma 2.2 implies (3.5). This proves the UBT for shellable triangulations of spheres. Bruggesser and Mani [3] proved that the boundary complex of any convex polytope is shellable, and thus the UBT for convex polytopes follows.

4. \( d \)-REPRESENTABLE COMPLEXES

Let \( C \) be a simplicial complex on the vertex set \( N \). \( C \) is \( d \)-representable if there exists a family \( \mathcal{X} = \{ K_1, \ldots, K_n \} \) of convex sets in \( \mathbb{R}^d \) such that \( S \subset C \) if \( \bigcap_{i \in S} K_i \neq \emptyset \). \( C \) is \( d \)-collapsible if there exists a collapse process \( C = C_0 \supset C_1 \supset \cdots \supset C_t \) in which every elementary-collapse is of type \((d, m)\) for some \( m \geq d \) and \( C_i \) has no faces of size \( \geq d \). In
this case, let $h_i$ denote the number of elementary $- (d, d + i) -$ collapses in the process $(i \geq 0)$. Clearly in each such collapse precisely $(d + i - 1)$-dimensional faces of $C$ were deleted for $d - 1 \leq j \leq d + i - 1$. Let $f_j$ denote the number of $j$-dimensional faces of $C$. Suppose $f_{d+r} = 0$ and put $h_i = \sum_{j=i}^{r} h_j (0 \leq i \leq r)$. (Thus $h_{r+1} = 0$.) Clearly for $d \leq j \leq d + r - 1$

$$f_j = \sum_{i=j+1-d}^{r} h_i \binom{j+1-d}{i} = \sum_{i=j+1-d}^{r} (h_i - h_{i+1}) \binom{j+1-d}{i} = \sum_{i=j+1-d}^{r} h_i \binom{j+1-d}{i}.$$

By Lemma 2.2 $h_i \leq \binom{n-i}{d}$ for $i \geq 0$. Therefore we have:

**Theorem 4.1.** For $d \leq j \leq d + r - 1$ let $f_j$ denote the number of faces of dimension $j$ of a $d$-collapsible complex on $n$ vertices. If $f_{d+r} = 0$ then

$$f_j \leq \sum_{i=j+1-d}^{r} \binom{n-i}{d} \binom{i-1}{j-d} = \sum_{i=0}^{d-r} \binom{n-r}{i} \binom{r}{j+1-i}.$$

This theorem was first proved in [9]. By a fundamental result of Wegner [17], every $d$-representable complex is $d$-collapsible, and thus the assertion of Theorem 4.1 holds for $d$-representable complexes.

**References**


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