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COVERING MULTIGRAPHS BY SIMPLE CIRCUITS*

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Abstract. Answering a question raised in [SIAM J. Comput., 10 (1981), pp. 746–750], we show that every bridgeless multigraph with v vertices and e edges can be covered by simple circuits whose total length is at most $\min(\frac{5}{3}e, e+\frac{7}{3}v-\frac{7}{3})$. Our proof supplies an efficient algorithm for finding such a cover.

Key words. bridgeless multigraphs, Eulerian subgraphs, graph algorithms, nowhere-zero flow

AMS (MOS) subject classification. 05

1. The main results. Let $G=(V, E)$ be an undirected bridgeless multigraph (i.e., a multigraph with no isthmus) and put $v=|V|$, $e=|E|$. A family $C_1, \cdots, C_m$ of simple circuits (=cycles) in G is a *cover* of G if every edge of G is in at least one of the circuits (2-cycles are allowed if they contain different edges of G). The *size* of such a cover is the sum of the lengths of the circuits $C_1, \cdots, C_m$. We are interested in the problem of finding covers of minimum size.

Itai, Lipton, Papadimitriou and Rodeh considered this problem in [ILPR]. Their main result is that every bridgeless multigraph G with $v \geqq 2$ vertices and $e \geqq 4$ edges has a cover of size at most

$$\min(3e-6, e+6v-7),$$

and that such a cover can be found in $O(e+v^2)$ time. (Note that since G is a multigraph, it is possible that $e \gg v^2$.) This improves a result of Itai and Rodeh in [IR].

The authors of [ILPR] ask if the multiplicative constants in their bound can be improved. In § 5 we settle this question in the affirmative by proving the following.

THEOREM 5.1. *Every bridgeless multigraph G with v vertices and e edges has a cover of size at most*

$$\min(\tfrac{5}{3}e, e+\tfrac{7}{3}v-\tfrac{7}{3}).$$

Such a cover can be found in polynomial time.

For planar multigraphs we have a better (and, in a sense, best possible) result:

THEOREM 4.2. *Every bridgeless planar multigraph with v vertices and e edges has a cover of size at most*

$$\min(\tfrac{4}{3}e, e+\tfrac{5}{3}v-\tfrac{5}{3}).$$

For a bridgeless multigraph G, let $s(G)$ denote the minimum size of a cover of G. One can easily show that if G is cubic then $s(G) \geqq \frac{4}{3}e$. Therefore, Theorem 4.2 gives the best possible upper bound for every cubic planar multigraph. In fact, $s(G)=\frac{4}{3}e$ for every cubic planar multigraph G.

One can also show (see [ILPR]) that if P is the Petersen graph (with 15 edges), then $s(G)=21$. This implies that if G is a graph obtained by substituting a path of length k for every edge of P, then $s(G)/e(G)=7/5$, where $e(G)=15k$ is the number of edges of G. Therefore, the coefficient $\frac{5}{3}$ in Theorem 5.1 cannot be replaced by any constant smaller than $\frac{7}{5}$.

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In order to prove our results we use some known results about nowhere-zero flows in multigraphs. In the next section we state these results. In § 3 we develop a general method of constructing covers of small size from covers by Eulerian subgraphs. In § 4 we combine this method with the fact that every bridgeless multigraph has a nowhere-zero 8-flow and obtain a slightly weaker version of Theorem 5.1. We also prove Theorem 4.2 in this section. In § 5 we finally use nowhere-zero 6-flow to prove Theorem 5.1.

During the completion of this manuscript we were notified that our main result (Theorem 5.1) was recently proved independently by Bermond, Jackson and Jaeger [BJJ], with a different method.

2. Nowhere-zero flows. If $G=(V,E)$ is a directed multigraph and $v\in V$, then $A^+(v)$ is the set of nonloop edges with tail v and $A^-(v)$ the set with head v. If K is any Abelian group (with additive notation), a *K-flow* in G is a function $f\colon E\to K$ such that for every $v\in V$,

$$\sum\{f(e)\colon e\in A^+(v)\}=\sum\{f(e)\colon e\in A^-(v)\}.$$

If $f(e)\neq 0$ for all $e\in E$, f is called a *nowhere-zero K-flow*. For $k>1$, f is called a *nowhere-zero k-flow* in G if f is a nowhere-zero Z-flow in G such that $-k<f(e)<k$ for all $e\in E$. (Here Z denotes the set of all integers.)

It is easy to see that if G has a nowhere-zero k-flow (K-flow) under some orientation of its edges, then it has one under every orientation, and thus the existence of such a flow depends only on the underlying undirected multigraph.

Tutte [Tu] conjectured that every bridgeless multigraph has a nowhere-zero 5-flow. Jaeger [J1], [J2] proved:

PROPOSITION 2.1 (Jaeger). *Every bridgeless multigraph has a nowhere-zero 8-flow.*

Seymour [Se] improved this result by showing:

PROPOSITION 2.2 (Seymour). *Every bridgeless multigraph has a nowhere-zero 6-flow.*

An *Eulerian multigraph* is a multigraph (not necessarily connected) in which every vertex has an even degree. Equivalently, as is well known, an Eulerian multigraph is an edge disjoint union of cycles. Thus the problem of covering a multigraph by a family of cycles of minimum total size is equivalent to that of covering the multigraph by a family of Eulerian subgraphs of minimum total size. The existence of nowhere-zero flows in a multigraph is closely related to the minimum number of Eulerian subgraphs that cover it. This is shown in the following known results.

PROPOSITION 2.3 (Jaeger [J2]). *Let G be a bridgeless multigraph. The following conditions are equivalent for every $k\geqq 2$:*

(i) *There exists a nowhere-zero Z_k-flow in G.*

(ii) *For every Abelien group K of order k there exists a nowhere-zero K-flow in G.*

(iii) *There exists a nowhere-zero k-flow in G.*

PROPOSITION 2.4 (Mathews [Ma]). *Let G be a bridgeless multigraph. For every $k\geqq 1$, G can be covered by k Eulerian subgraphs iff it has a nowhere-zero Z_2^k-flow.*

In §§ 4, 5, we combine Propositions 2.1–2.4 in order to obtain for every bridgeless multigraph G a cover by Eulerian subgraphs. From this cover we obtain a cover of small size of G using the method we develop in § 3.

Our results showing the connection between nowhere-zero flows and short cycle covers are summarized in Table 1.

3. Generating covers of small size from covers by Eulerian subgraphs. Our main result in this section is the following:

PROPOSITION 3.1. *Let $G=(V,E)$ be a bridgeless multigraph, and let $C=(C_1,C_2,\cdots,C_k)$ be a given cover of G by k Eulerian subgraphs. Then there exists a*

TABLE 1

If $G=(V,E)$ has a nowhere-zero k-flow for $k=$	then G has a cycle cover of length at most:
2	$\|E\|$
4	$\frac{4}{3}\|E\|$
6	$\frac{5}{3}\|E\|$
8	$\frac{12}{7}\|E\|$

cover of G of size at most

$$s=\frac{k\cdot 2^{k-1}\cdot|E|}{2^k-1}. \tag{3.1}$$

Such a cover can be found in $O(2^{k^2}|E|)$ *time.*

Proof. Identify each C_i with the corresponding element of the cycle space of G, i.e., with the characteristic function of C_i, regarded as a function from E to $GF(2)$. For every binary vector $u=(u_1,u_2,\cdots,u_k)$ define $C(u)=\bigoplus_{i=1}^k u_iC_i$. (Here $\bigoplus$ denotes the sum over $GF(2)$.) Obviously $C(u)$ is an Eulerian subgraph of G. For every edge $f\in E$, let $v(f)=(v_1,\cdots,v_k)$ be a binary vector in which $v_i=1$ iff $f\in C_i$. One can easily check that for every vector $u=(u_1,\cdots u_k)$, $f\in C(u)$ iff $\langle v(f),u\rangle=\bigoplus_{i=1}^k v_iu_i=1$. This implies the following:

Fact 1. For every edge $f\in E$, the number of vectors u such that $f\in C(u)$ is precisely 2^{k-1}.

Let $u^{(1)}, u^{(2)},\cdots,u^{(k)}$ be a basis of $(GF(2))^k$. If $f\in E$ then $v(f)\neq 0$, and thus $\langle v(f),u^{(i)}\rangle\neq 0$ (i.e., $f\in C(u^{(i)})$), for at least one index $1\leqq i\leqq k$. Therefore:

Fact 2. For every basis $u^{(1)}, u^{(2)},\cdots,u^{(k)}$ of $(GF(2))^k$, $C(u^{(1)}),\cdots,C(u^{(k)})$ is a cover of G by Eulerian subgraphs.

Let B be the set of all bases of $(GF(2))^k$, and put $b=|B|$. By Fact 2, every element of B induces a cover of G. We now compute the sum of the sizes of these b covers. By symmetry, every nonzero vector $u\in(GF(2))^k$ belongs to exactly $b\cdot k/(2^k-1)$ bases. Combining this with Fact 1, we conclude that every edge $f\in E$ is covered precisely $(b\cdot k/(2^k-1))\cdot 2^{k-1}$ times by the collection of all the b covers associated with the elements of B. Therefore the sum of sizes of these covers is $b\cdot k\cdot 2^{k-1}\cdot|E|/(2^k-1)$, and the average size is just the number s given in (3.1). Thus, there exists a cover of G corresponding to an element of B of size at most s. This establishes the first part of Proposition 3.1. The time bound follows from the fact that

$$b=\frac{1}{k!}\prod_{i=0}^{k-1}(2^k-2^i)\leqq\frac{1}{k!}2^{k^2}. \qquad \square$$

Next we prove the following proposition, which is an improvement of [ILPR, Cor. 1]. (There is a misprint in this corollary. $|E|+s(|V|,2|V|-2)$ should read $|E|+2|V|-2+s(|V|,2|V|-2)$.)

PROPOSITION 3.2. *Suppose that one can find in time* $O(e^2)$ *a cover of size* $\leqq d\cdot e(d>1)$ *for any bridgeless multigraph G with v vertices and e edges. Then we can find a cover of size* $\leqq\min\{de, e+(2d-1)(v-1)\}$ *in time* $O(e+v^2)$.

In order to prove Proposition 3.2 we need the following result of [ILPR]:

LEMMA 3.3. (i) *Let* $T=(V,E_T)$ *be a spanning tree of a multigraph* $G=(V,E)$. *Then there exists an Eulerian subgraph* $C=(V,E_c)$ *of G with* $E_c\supseteq E-E_T$. *C can be found in* $O(|E|)$ *time.*

(ii) *Let $T=(V, E_T)$ be a spanning tree of a bridgeless multigraph G. Then there exists a bridgeless subgraph $H=(V, E_H)$ of G such that $E_H \supseteq E_T$ and $|E_H| \leqq 2|V|-2$. Such T, H can be found in $O(|E|)$ time.*

Proof of Proposition 3.2. Let $G=(V, E)$ be a bridgeless multigraph. Put $v=|V|$ and $e=|E|$. If $de \leqq e+(2d-1)(v-1)$, then $e=O(v)$ and there is nothing to show. Otherwise we argue as follows. Clearly we may assume that G is connected; otherwise apply the theorem to each of its connected components. Let $T=(V, E_T)$ be a spanning *DFS* tree of G. By Lemma 3.3(ii), there exists a bridgeless subgraph $H=(V, E_H)$ of G such that $E_H \supseteq E_T$ and $|E_H| \leqq 2v-2$. Define $\bar{G}=(V, \bar{E})$ where $\bar{E}=E-(E_H-E_T)$. Clearly T is a spanning tree of $\bar{G}$. By Lemma 3.3(i), there exists an Eulerian subgraph $C=(V, E_c)$ of $\bar{G}$ with $E_c \supseteq \bar{E}-E_T$. By assumption there exists a cover of H of size at most $d|E_H|$. This cover together with C forms a cover of G of size at most

$$\begin{aligned} d|E_H|+|E_C| &\leqq d|E_H|+|\bar{E}|=d|E_H|+|E|-|E_H|+|E_T| \\ &= e+(d-1)|E_H|+v-1 \leqq e+(d-1)(2v-2)+v-1=e+(2d-1)(v-1). \end{aligned}$$

T, H and C can be found in $O(e)$ time. The cover of H can be found in $O(|E_H|^2)=O(v^2)$ time. Therefore, the total time bound is $O(e+v^2)$. This completes the proof of Proposition 3.2. □

4. Consequences of nowhere-zero 8-flow and nowhere-zero 4-flow. Combining Proposition 2.1 with Propositions 2.3 and 2.4, one can easily deduce the following result of Jaeger [J2]. His result appears also in [Ma]. The proof in [ILPR] supplies the algorithms and the time bound.

LEMMA 4.1. *Every bridgeless multigraph with e edges can be covered by three Eulerian subgraphs. These subgraphs can be found in $O(e^2)$ time.*

Combining this lemma with Proposition 3.1 (with $k=3$) and Proposition 3.2, we obtain the following weaker version of Theorem 5.1:

Every bridgeless multigraph G with v vertices and e edges has a cover of size at most

$$\min\left(\tfrac{12}{7}e,\ e+\tfrac{17}{7}v-\tfrac{17}{7}\right).$$

Such a cover can be found in $O(e+v^2)$ time.

The following result [Ma] (see also J2]) is equivalent to the four color theorem: Every planar bridgeless multigraph can be covered by two Eulerian subgraphs.

Combining this with Proposition 3.1 with $k=2$ and Proposition 3.2, we obtain:

THEOREM 4.2. *Every bridgeless planar multigraph with v vertices and e edges has a cover of size at most*

$$\min\left(\tfrac{4}{3}e,\ e+\tfrac{5}{3}v-\tfrac{5}{3}\right).$$

Similarly, Jaeger's result [J2] that every 4-edge-connected multigraph has a nowhere-zero 4-flow implies:

THEOREM 4.3. *Every 4-edge-connected multigraph with v vertices and e edges has a cover of size at most*

$$\min\left(\tfrac{4}{3}e,\ e+\tfrac{5}{3}v-\tfrac{5}{3}\right).$$

Such a cover can be found in $O(e+v^2)$ time.

5. A consequence of nowhere-zero 6-flow. In this section we prove our main result.

THEOREM 5.1. *Every bridgeless multigraph G with v vertices and e edges has a cover* *of size at most*

$$\min\left(\tfrac{5}{3}e,\ e+\tfrac{7}{3}v-\tfrac{7}{3}\right).$$

Such a cover can be found in polynomial time.

Proof. By Proposition 3.2 it is enough to show that G has a cover of size $\leqq \frac{5}{3}e$. Let $G_1=(V,E)$ be an orientation of G. By Propositions 2.2 and 2.3, G_1 has a nowhere-zero $Z_2\times Z_3$-flow f_1, i.e., for every $e\in E$, $f_1(e)\in\{(1,0),(1,1),(1,2),(0,1),(0,2)\}$. For any K-flow f and $g\in K$ define

$$E(f,g)=\{e\in E: f(e)=g\}.$$

Put

$$E_1=E(f_1,(1,0))\cup E(f_1,(1,1))\cup E(f_1,(1,2)).$$

Clearly E_1 is an Eulerian subgraph of G. Let G_2 be an orientation of G in which E_1 is a directed Eulerian circuit, and let f_2 be the $Z_2\times Z_3$-flow obtained from f_1 by defining $f_2(e)=f_1(e)$ if the directions of e in G_1 and G_2 coincide, and $f_2(e)=-f_1(e)$ otherwise. Clearly there exists an i, $0\leqq i\leqq 2$, such that

$$|E(f_2,(1,i))|\geqq \tfrac{1}{3}|E_1|=\tfrac{1}{3}(|E(f_2,(1,0))|+|E(f_2,(1,1))|+|E(f_2,(1,2))|).$$

Let f_3 be the flow obtained from f_2 by letting $f_3(e)=f_2(e)$ if $e\notin E_1$ and $f_3(e)=f_2(e)-(0,i)$ if $e\in E_1$. Obviously

$$|E(f_3,(1,0))|=|E(f_2,(1,i))|\geqq \tfrac{1}{3}|E_1|. \tag{5.1}$$

Put $E_3=E(f_3,(1,0))$, $E_2=E\setminus E_3$. The second coordinate of f_3 is a nowhere-zero Z_3-flow in E_2. By Proposition 2.3 there exists a nowhere-zero 3-flow in E_2, which is, of course, also a 4-flow. By Proposition 2.3, E_2 has a Z_4-flow, and by Proposition 2.4, E_2 can be covered by two Eulerian subgraphs C_2 and C_3. By Proposition 3.1 with $k=2$, E_2 has a cover C of size at most $\frac{4}{3}|E_2|=\frac{4}{3}(|E|-|E_3|)$. In order to obtain a cover of G, we add to C an Eulerian subgraph D of G that contains E_3. There are four possibilities to such a subgraph: E_1, $E_1\oplus C_2$, $E_1\oplus C_3$ and $E_1\oplus C_2\oplus C_3$. Let D be that of smallest size. One can easily check that

$$|E_1|+|E_1\oplus C_2|+|E_1\oplus C_3|+|E_1\oplus C_2\oplus C_3|=4|E_3|+2(|E|-|E_3|)=2(|E|+|E_3|).$$

Therefore

$$|D|\leqq\frac{|E|+|E_3|}{2}.$$

Since $|D|\leqq|E_1|$, (5.1) implies

$$|E_3|\geqq\tfrac{1}{3}|D|.$$

C together with D is a cover of G of size at most

$$\begin{aligned}|D|+\tfrac{4}{3}|E|-\tfrac{4}{3}|E_3|&=\tfrac{1}{3}|D|-|E_3|+\tfrac{2}{3}|D|-\tfrac{1}{3}|E_3|+\tfrac{4}{3}|E|\\&\leqq\tfrac{2}{3}|D|-\tfrac{1}{3}|E_3|+\tfrac{4}{3}|E|\leqq\tfrac{1}{3}|E|+\tfrac{1}{3}|E_3|-\tfrac{1}{3}|E_3|+\tfrac{4}{3}|E|=\tfrac{5}{3}|E|.\end{aligned}$$

This establishes the existence of the desired cover.

We now briefly sketch an evaluation of the complexity of the construction. The constructions which are explicitly described in the proof can clearly be executed in $O(e^2)$ time.

However, the proof uses the following two existence theorems:

1. the existence of a nowhere-zero $Z_2 \times Z_3$ flow for every bridgeless multigraph;

2. the fact that one can obtain a $(Z_2)^2$ nowhere-zero flow from a given Z_3 nowhere-zero flow.

In [Yo] Younger shows that the needed $Z_2 \times Z_3$ flow can be formed in $O(v \cdot e)$ time. Statement 2 can be settled by means of maximal matching algorithms, and thus the time complexity certainly does not exceed $O(e^2)$. Thus the total time bound is at most $O(e^2)$, which can be reduced, by Proposition 3.2, to $O(e+v^2)$. □

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