

Splitting Necklaces

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Let N be an opened necklace with ka_i beads of color i , $1 \leq i \leq t$. We show that it is possible to cut N in $(k-1) \cdot t$ places and partition the resulting intervals into k collections, each containing precisely a_i beads of color i , $1 \leq i \leq t$. This result is best possible and solves a problem of Goldberg and West. Its proof is topological and uses a generalization, due to Bárány, Shlosman and Szücs, of the Borsuk-Ulam theorem. By similar methods we obtain a generalization of a theorem of Hobby and Rice on L_1 -approximation. © 1987 Academic Press, Inc.

1. INTRODUCTION

Suppose a necklace opened at the clasp has $k \cdot n$ beads, chosen from t different colors. Suppose there are $k \cdot a_i$ beads of color i , $1 \leq i \leq t$. A k -splitting of the necklace is a partition of the necklace into k parts, each consisting of a finite number of nonoverlapping intervals of beads whose union captures precisely a_i beads of color i , $1 \leq i \leq t$. The size of the k -splitting is the number of cuts that form the intervals of the splitting, which is one less than the total number of these intervals. If the beads of each color appear contiguously on the opened necklace, then any k -splitting of the necklace must contain at least $k-1$ cuts between the beads of each color, and hence its size is at least $(k-1) \cdot t$. The following theorem shows that this number of cuts always suffices to form a k -splitting.

THEOREM 1.1. *Every necklace with ka_i beads of color i , $1 \leq i \leq t$, has a k -splitting of size at most $(k-1) \cdot t$. The number $(k-1) \cdot t$ is best possible.*

As mentioned in [AW], the problem of finding k -splittings of small size arises naturally when k mathematically oriented thieves steal a necklace with $k \cdot a_i$ jewels of type i , and wish to divide it fairly between them, wasting as little as possible of the metal in the links between jewels. As shown in [BL, BLe] this problem also has some applications to VLSI circuit design.

Theorem 1.1 solves a problem raised in [GW], and settles the conjecture raised in [AW] (see also [O]). Its proof is topological and uses a generalization, due to Bárány, Shlosman, and Szücs [BSS] of the well-known Borsuk–Ulam theorem [Bo] (see also [D]). Similar methods were used in [AFL] to solve the general Kneser problem.

The case $k=2$ of the theorem was first proved in [GW]. In [AW] there is a short proof of this case, using the Borsuk–Ulam theorem. In both papers an appropriate continuous generalization of the discrete problem is solved. This continuous problem is closely related to a theorem of Hobby and Rice [HR] on L_1 approximation. Using our methods we can generalize the Hobby–Rice theorem and prove

THEOREM 1.2. *Let $\mu_1, \mu_2, \dots, \mu_t$ be t continuous probability measures on the unit interval. Then it is possible to cut the interval in $(k-1) \cdot t$ places and partition the $(k-1) \cdot t + 1$ resulting intervals into k families F_1, F_2, \dots, F_k such that $\mu_i(\cup F_j) = 1/k$ for all $1 \leq i \leq t, 1 \leq j \leq k$. The number $(k-1) \cdot t$ is best possible.*

The case $k=2$ of the last Theorem is the Hobby–Rice theorem [HR].

Our paper is organized as follows. In Section 2 we formulate the continuous version of the k -splitting problem and prove that it generalizes the discrete one. In sections 3 and 4 we apply the topological results of [BSS] to solve the continuous problem. In Section 5 we modify this solution to prove Theorem 1.2.

2. CONTINUOUS SPLITTING

Let $I = [0, 1]$ be the unit interval. An *interval t -coloring* is a coloring of the points of I by t colors, such that for each $i, 1 \leq i \leq t$, the set of points colored i is (Lebesgue) measurable. Given such a coloring, a *k -splitting of size r* is a sequence of numbers $0 = y_0 \leq y_1 \leq y_2 \leq \dots \leq y_r \leq y_{r+1} = 1$ and a partition of the family of $r+1$ intervals $F = \{[y_i, y_{i+1}]: 0 \leq i \leq r\}$ into k pairwise disjoint subfamilies F_1, \dots, F_k whose union is F , such that for each $1 \leq j \leq k$ the union of the intervals in F_j captures precisely $1/k$ of the total measure of each of the t colors. Clearly, if each color appears contiguously and colors occupy disjoint intervals, the size of each k -splitting is at least $(k-1) \cdot t$. Therefore, the next theorem, which we prove in Sections 3, 4, is best possible.

THEOREM 2.1. *Every interval t -coloring has a k -splitting of size $(k-1) \cdot t$.*

It is not difficult to check that the theorem implies Theorem 1.1, which is its discrete version. Indeed given an opened necklace of $k \cdot n$ beads as in

Theorem 1.1, convert it into an interval coloring by partitioning $I = [0, 1]$ into $k \cdot n$ equal segments and coloring the j -th segment by the color of the j -th bead of the necklace. By Theorem 2.1 there is a k -splitting into the families of intervals F_1, \dots, F_k , with $(k-1) \cdot t$ cuts, but these cuts need not occur at the endpoints of the $k \cdot n$ segments. We show, by induction on the number r of the "bad" cuts, that this splitting can be modified to form a k -splitting of the same size with no bad cuts, i.e., a splitting of the discrete necklace. If $r=0$ there is nothing to prove. Assuming the result for all $r' < r$, suppose the number of bad cuts is $r > 0$. Then there is some i , $1 \leq i \leq t$, such that there is a positive number of bad cuts in the interior of segments belonging to color i . Construct a multigraph G on the set of vertices F_1, F_2, \dots, F_k with an edge $\{F_j, F_k\}$ corresponding to each bad cut in color i between an interval belonging to F_j and an interval belonging to F_k . Since the measure of color i captured by each F_j is an integral multiple of $1/kn$, if a vertex F_j of G has a positive degree, its degree is at least 2. Thus G contains a cycle. We can now slide all the cuts corresponding to the edges of this cycle by the same amount, without changing the measure of any color captured by the F_j 's, until one of the cuts reaches the boundary of its small segment. This decreases the number of the bad cuts by at least one and completes the proof of the induction step. Hence Theorem 2.1 implies Theorem 1.1.

3. THE PROOF OF THEOREM 2.1

Theorem 2.1 follows from the following two assertions.

PROPOSITION 3.1. *Theorem 2.1 holds for every odd prime k .*

PROPOSITION 3.2. *The validity of Theorem 2.1 for (t, k) and for (t, l) implies its validity for $(t, k \cdot l)$.*

Theorem 2.1 for $k=2$ was proved in [GW] (see also [AW] for a short proof). Hence, by Propositions 3.1 and 3.2 it holds for all t, k .

The proof of Proposition 3.1 is topological and is given in the next section. We concluded this section with the (easy) proof of Proposition 3.2.

Proof of Proposition 3.2. Given an interval t -coloring c , we obtain a $k \cdot l$ -splitting of size $(k \cdot l - 1) \cdot t$ as follows. Begin by using $(k-1) \cdot t$ cuts to form k families of intervals each capturing $1/k$ of the measure of each color. For each of these families, consider the interval coloring formed by placing its intervals next to each other and rescaling to total length 1. Using $(l-1) \cdot t$ cuts, we obtain an l -splitting of this coloring. Transforming back

to the original interval coloring, this adds together to $k \cdot (l-1) \cdot t$ more cuts, so altogether we have $(k-1) \cdot t + k(l-1) \cdot t = (kl-1) \cdot t$ cuts which form the desired $k \cdot l$ -splitting.

4. THE PROOF OF PROPOSITION 3.1

We begin by stating two results of Bárány, Shlosman, and Szücs [BSS], which are crucial for our proof. Let k be an odd prime, and suppose $m \geq 1$. Let $X = X_{m,k}$ denote the CW -complex consisting of k disjoint copies of the $m \cdot (k-1)$ dimensional ball with an identified boundary $S^{m(k-1)-1}$. We define a free action of the cyclic group Z_k on X by defining w , the action of its generator, as follows, (see [Bou, Chapter 13], for the definition of a free group action on a topological space). Represent $S^{m(k-1)-1}$ as the set of all m by k real matrices (a_{ij}) satisfying $\sum_{j=1}^k a_{ij} = 0$ for all $1 \leq i \leq m$ and $\sum_{i,j} a_{ij}^2 = 1$. Define now

$$w(a_{ij}) = (a_{ij+1}),$$

where $j+1$ is reduced modulo k . Thus w just cyclically shifts the columns of a matrix representing a point of $S^{m(k-1)-1}$. Trivially, this action is free, i.e., $w(x) \neq x$ for all $x \in S^{m(k-1)-1}$. The map w is extended from $S^{m(k-1)-1}$ to $X_{m,k}$ as follows. Let (y, r, q) denote a point of X from the q -th ball with radius r and $S^{m(k-1)-1}$ -coordinate y . Then

$$w(y, r, q) = (wy, r, q+1),$$

where $q+1$ is reduced modulo k . Since k is prime, w defines a free Z_k action on $X = X_{m,k}$.

LEMMA 4.1 [BSS]. *For any continuous map $h: X \rightarrow R^m$ there exists an $x \in X$ such that $h(x) = h(wx) = \dots = h(w^{k-1}x)$.*

Put $N = (k-1) \cdot (m+1)$ and let Δ^N denote the N -dimensional simplex, i.e., $\Delta^N = \{(x_0, x_1, \dots, x_N) \in R^{N+1}, x_i \geq 0 \text{ and } \sum_{i=0}^N x_i = 1\}$. The *support* of a point $x \in \Delta^N$ is the minimal face of Δ^N that contains x . Let $y = y_{N,k}$ denote the following CW -complex;

$$y_{N,k} = \{(y_1, y_2, \dots, y_k): y_1, \dots, y_k \in \Delta^N$$

and the supports of the y_i 's are pairwise disjoint\}.

There is an obvious free Z_k action on $Y_{N,k}$; its generator γ maps (y_1, \dots, y_k) into (y_2, \dots, y_k, y_1) .

Let T and R be two topological spaces and suppose that Z_k acts freely

on both. Let α and β denote the actions of the generator of Z_k on T and R , respectively. We say that a continuous mapping $f: T \rightarrow R$ is Z_k -equivariant if $f \circ \alpha = \beta \circ f$, (cf. [Bou, Chapter 13]).

Recall that for $s \geq 0$, a topological space T is s -connected if for all $0 \leq l \leq s$, every continuous mapping of the l dimensional sphere S^l into T can be extended to a continuous mapping of the $l+1$ dimensional ball B^{l+1} with boundary S^l into T .

LEMMA 4.2 [BSS]. *Suppose k is an odd prime, $m \geq 1$, $N = (k-1)(m+1)$ and let $X = X_{m,k}$, $Y = Y_{N,k}$, w and γ be as in the preceding paragraphs. Then Y is $N-k = \dim X - 1$ connected and thus there is a Z_k -equivariant map $f: X \rightarrow Y$.*

We can now prove Proposition 3.1. Let k be an odd prime and let c be an interval t -coloring. Put $X = X_{t-1,k}$, $Y = Y_{(k-1) \cdot t, k}$ and define a continuous function $g: Y \rightarrow R^{t-1}$ as follows. Let $y = (y_1, y_2, \dots, y_k)$ be a point of Y . Recall that each y_i is a point of Δ^N , i.e., is an $(N+1)$ -dimensional vector with nonnegative coordinates whose sum is 1, and that the supports of the y_i 's are pairwise disjoint. Put $x = (x_0, x_1, \dots, x_N) = 1/k (y_1 + y_2 + \dots + y_k)$, and define a partition of $[0, 1]$ into $N+1$ intervals I_0, I_1, \dots, I_N , where $I_0 = [0, x_0]$, $I_j = [\sum_{i=0}^{j-1} x_i, \sum_{i=0}^j x_i]$, $1 \leq j \leq N$. Notice that since the supports of the y_i 's are pairwise disjoint, if $x_j > 0$ (i.e., the interval I_j has positive length), then there is a unique l , $1 \leq l \leq k$ such that the j -th coordinate of y_l is positive. For $1 \leq l \leq k$, let F_l be the family of all those I_j 's such that the j -th coordinate of y_l is positive. Note that the sum of lengths of these I_j 's is precisely $1/k$. For $1 \leq i \leq t-1$, define $g_i(y)$ to be the measure of the i -th color in $\cup F_l$. Finally, put $g(y) = (g_1(y), g_2(y), \dots, g_{t-1}(y))$. One can easily check that $g: Y \rightarrow R^{t-1}$ is continuous. Moreover, for $1 \leq l \leq k$ and $1 \leq i \leq t-1$, $g_i(\gamma^l y)$ is the measure of the i -th color in $\cup F_l$. By Lemma 4.2 there exists a Z_k -equivariant map $f: X_{t-1,k} \rightarrow Y_{(t-1) \cdot k, k}$. Define $h = g \circ f: X \rightarrow R^{t-1}$. By Lemma 4.1 there is some $x \in X$ such that $h(x) = h(wx) = \dots = h(w^{k-1}x)$. By the equivariance of f , $y = f(x)$ satisfies $g(y) = g(\gamma y) = \dots = g(\gamma^{k-1}y)$. But this means that each of the families of intervals F_1, F_2, \dots, F_k corresponding to y captures precisely $1/k$ of the measure of each of the first $t-1$ colors. Since the total measure of each F_j is $1/k$, each F_j captures precisely $1/k$ of the measure of the last color, as well. Dividing the length 0 intervals arbitrarily between the F_j 's we conclude that there is a k -splitting of size $N = (k-1) \cdot t$, as desired. This completes the proof of Proposition 3.1.

5. GENERALIZING THE HOBBY-RICE THEOREM

Given t measures μ_1, \dots, μ_t on the unit interval, a k -splitting of size r is a sequence of numbers $0 = y_0 \leq y_1 \leq \dots \leq y_r \leq y_{r+1} = 1$ and a partition of the family of $r+1$ intervals $F = \{[y_i, y_{i+1}]: 0 \leq i \leq r\}$ into k pairwise disjoint subfamilies F_1, \dots, F_k whose union is F , such that for each $1 \leq j \leq k$ and $1 \leq i \leq t$,

$$\mu_i(\cup F_j) = \frac{1}{k} \mu_i([0, 1]).$$

A close look at the proof of Theorem 2.1 given above will convince the reader that we made no use of the fact that the t measures considered there come from an interval coloring. The only requirement is that these are continuous measures and that the sum of the t measures of any interval is its length. We thus have:

LEMMA 5.1. *Let m_1, m_2, \dots, m_t be t continuous measures on the unit interval and suppose*

$$m_1([0, \alpha]) + \dots + m_t([0, \alpha]) = \alpha$$

for all $0 \leq \alpha \leq 1$. Then, for all $k > 1$, there exists a k -splitting of size $(k-1) \cdot t$.

Theorem 1.2 is derived from this lemma using compactness arguments. Indeed, let $\mu_1, \mu_2, \dots, \mu_t$ be t continuous probability measures on the unit interval I . Suppose $\varepsilon > 0$. Define the following t measures m_1, \dots, m_t on I . For $1 \leq j < t$ put $m_j = \mu_j/k$ and define $m_t = (\mu_t + \varepsilon \cdot m_L)/(1 + \varepsilon) \cdot k$, where m_L is the usual Lebesgue measure. Put $m = m_1 + \dots + m_t$ and define $f: [0, 1] \rightarrow [0, 1]$ by $f(x) = m([0, x])$. The function f is continuous, onto and strictly increasing and thus its inverse f^{-1} is also continuous and strictly increasing. For $1 \leq j \leq t$ let m'_j be the measure given by $m'_j(S) = m_j(f^{-1}(S))$. Clearly, for every $0 \leq \alpha \leq 1$, $m'_1([0, \alpha]) + \dots + m'_t([0, \alpha]) = \sum_{j=1}^t m_j(f^{-1}[0, \alpha]) = m([0, f^{-1}(\alpha)]) = \alpha$. Therefore, by Lemma 5.1, there is a k -splitting of size $(k-1) \cdot t$ for the measures m'_1, m'_2, \dots, m'_t . The function f^{-1} will carry this k -splitting into a k -splitting of the same size for the measures m_1, m_2, \dots, m_t . Let F_1, F_2, \dots, F_k be the k collections of intervals that form this splitting. By the definition of the m'_j 's, these collections almost form a k -splitting for the original measures $\mu_1, \mu_2, \dots, \mu_t$. Indeed

$$\mu_i(\cup F_j) = \frac{1}{k} \mu_i([0, 1]) \quad \text{for } 1 \leq i < t, 1 \leq j \leq k \quad (5.1)$$

and $\mu_t(\cup F_j) + \varepsilon m_L(\cup F_j) = (1 + \varepsilon)/k$ for $1 \leq j \leq k$, i.e.,

$$\frac{1}{k} - \frac{k-1}{k} \varepsilon \leq \mu_t(\cup F_j) \leq \frac{1+\varepsilon}{k} \quad \text{for } 1 \leq j \leq k. \quad (5.2)$$

By choosing a sequence $\varepsilon_t \rightarrow 0$ and obtaining a convergent subsequence of the sequence of k -splittings of size $(k-1) \cdot t$ satisfying (5.1) and (5.2) for these ε_t 's, we obtain a k -splitting of size $(k-1) \cdot t$ for the measures μ_1, \dots, μ_t . This completes the proof of Theorem 1.2.

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