The CW-Inequalities for Vectors in $l_1$

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Let $l_1$ denote the metric space the elements of which are all sequences $x = (x_1, x_2, \ldots)$ of real numbers satisfying $\sum_{i=1}^{\infty} |x_i| < \infty$. The distance $d(x, y)$ between $x$ and $y = (y_1, y_2, \ldots)$ is $d(x, y) = \sum_{i=1}^{\infty} |x_i - y_i|$. Let $v_0, v_1, \ldots, v_m, u_0, u_1, \ldots, u_k$ be a sequence of $n = m + k$ not necessarily distinct vectors in $l_1$, where $k > 0$, $m = k + 2u + 1$ and $u \geq 0$. We show that

$$\sum_{0 \leq i < j < m \atop (i-j) \equiv (u, v_0, u_1, \ldots, v_{m-1}, u_0, u_1, \ldots, u_{k-1}} d(v_i, v_j) + \sum_{0 \leq i < j < m \atop 0 \leq u \leq k} d(v_i, u_j) \\ \geq \sum_{0 \leq i < j < m \atop (i-j) \equiv (u, v_0, u_1, \ldots, v_{m-1}, u_0, u_1, \ldots, u_{k-1}} d(u_i, u_j).$$

This solves a conjecture of Deza and Laurent and implies, as special cases, inequalities of Deza, Laurent, Kelly, Barahona and Mahjoub.

1. THE CW-INEQUALITIES

Let $l_1$ denote, as usual, the metric space the elements of which are all real sequences $x = (x_1, x_2, \ldots)$ satisfying $\sum_{i=1}^{\infty} |x_i| < \infty$. The distance $d(x, y)$ between the vector $x$ above and $y = (y_1, y_2, \ldots)$ is $d(x, y) = \sum_{i=1}^{\infty} |x_i - y_i|$. In this note we prove the following class of inequalities, conjectured by Deza and Laurent [4].

**Theorem 1.1 (the CW-inequalities).** Let $k > 0$ and $u \geq 0$ be two integers. Put $m = k + 2u + 1$ and let $v_0, v_1, \ldots, v_{m-1}, u_0, u_1, \ldots, u_{k-1}$ be a sequence of $m + k$ not necessarily distinct vectors in $l_1$. Then, for $U = \{ \pm 1, \pm 2, \ldots, \pm u \}$,

$$\sum_{0 \leq i < j < m \atop (i-j) \equiv (u, v_0, u_1, \ldots, v_{m-1}, u_0, u_1, \ldots, u_{k-1}} d(v_i, v_j) + \sum_{0 \leq i < j < m \atop 0 \leq u \leq k} d(v_i, u_j) \geq \sum_{0 \leq i < j < m \atop (i-j) \equiv (u, v_0, u_1, \ldots, v_{m-1}, u_0, u_1, \ldots, u_{k-1}} d(u_i, u_j). \tag{1.1}$$

In order to state (1.1) in a more compact form, it is convenient to introduce the following notation. For an undirected finite graph $G = (V, E)$, the set of vertices $V$ of which is a set of vectors in $l_1$, put $d_G(u, v) = \sum_{u \in E} d(u, v)$. Let $W = W(v_0, v_1, \ldots, v_{m-1}, u_0, u_1, \ldots, u_{k-1})$ denote the graph the vertices of which are $v_0, v_1, \ldots, v_{m-1}$, in which $v_i$ and $v_j$ are an edge if there is an $s$, $-u \leq s \leq u$ such that $i - j \equiv s \pmod{m}$. Thus $W$ is simply the Cayley graph of the group $Z_m$ with respect to the generating set $1, 2, \ldots, u$. Let $B = B(v_0, v_1, \ldots, v_{m-1}, u_0, \ldots, u_{k-1})$ denote the complete bipartite graph on the classes of vertices $\{v_0, \ldots, v_{m-1}\}$ and $\{u_0, \ldots, u_{k-1}\}$, and let $K_n$ and $K_u$ denote the complete graphs on the sets $\{v_0, \ldots, v_{m-1}\}$ and $\{u_0, \ldots, u_{k-1}\}$, respectively. In this notation, inequality (1.1) is simply the assertion $d(W) + d(B) \geq d(K_m) + d(K_u)$. As the graphs $K_n$ and $K_u$ are cliques, and the graph $W$ is sometimes called an anti-web, the authors of [4, 5] suggested that the inequalities (1.1) should be called the Clique-Web Inequalities or, for short, the CW-Inequalities.

Several special cases of the CW-Inequalities have been proved by various authors. The case $u = 0$ is the class of hypermetric inequalities, first proved by Deza in 1960 [3], and later independently, by Kelly [6]. (Note that if $u = 0$ and $k = 1$ the obtained inequality is just the triangle inequality.) The case $k = 2$ was proved by Barahona and Mahjoub [1]. Deza and Laurent [4, 5] established the cases $u = 1$ and $u = 2$ as well as the case $k \geq (u - 1)(u^2 + u - 2) - 2u - 1$.

The CW-inequality, when applied to distinct points, defines a facet of the complete cut cone, as proved in [5]. This means, roughly that the inequality does not follow from
other valid inequalities that do not contain a scalar product of the given one. For more
details, including the exact definition of the facets of the complete cut cone, a
motivation for the study of these facets as a possible approach to the study of the
max-cut problem, and several facets that arise from certain CW-Inequalities with
repeated points, see [4, 5].

As stated in Theorem 1.1, the CW-Inequalities hold even when the vectors are not
all distinct. Therefore, even if we fix the number of distinct vectors we may obtain
infinitely many valid inequalities for them by applying inequality (1.1) for sequences of
vectors obtained by repeating each of the given ones as many times as we wish.

2. The Metric Space $l_1$ and Cut Semi-metrics

It is well known that in order to prove an inequality which is linear in the distances
between vectors $v_i$ in $l_1$, it suffices to prove the same inequality for scalars
$\tilde{v}_i \in \{0, 1\}$, for which the distance is simply $d(\tilde{v}_i, \tilde{v}_j) = |\tilde{v}_i - \tilde{v}_j|$. Since this fact is simple, we include it here, for the sake of completeness. For an integer $k \geq 1$, let $l_1^k$ denote the metric
space of all real sequences of length $k$, $x = (x_1, \ldots, x_k)$, where the distance between $x$
and $y = (y_1, \ldots, y_k)$ is $d(x, y) = \sum_{i=1}^k |x_i - y_i|$. In particular, $l_1^1$ is isometric to the space
of real numbers with the usual distance function $d(x, y) = |x - y|$. Observe that each $l_1^k$
is isometrically embeddable in $l_1$, and hence every inequality that holds for vectors in $l_1$
holds for vectors in $l_1^k$ (and hence also for real numbers). A cut-semi-metric space is a
pair $(V, d)$ such that $V$ is a set of elements and $d : V \times V \to \{0, 1\}$ is a semi-metric
defined by $d(v_i, u_j) = |c(v_i) - c(u_j)|$, where $c : V \to \{0, 1\}$ is an arbitrary function.

Let $A = A(v_1, \ldots, v_n) = \sum_{1 \leq i < j \leq n} a_{ij}d(v_i, u_j)$ be an arbitrary linear combination of the
distances between $n$ vectors $v_1, \ldots, v_n$ in $l_1$. Suppose one wants to prove that
$A(v_1, \ldots, v_n) \leq 0$ for all $v_1, \ldots, v_n \in l_1$. Then it suffices to prove it for vectors in $l_1^k$
for all $k \geq 1$. Indeed, the result for $l_1^k$ implies the validity of the inequality when we
replace each $v_i \in l_1$ by the vector of its first $k$ co-ordinates, and by letting $k$ tend to
infinity the desired inequality for $l_1$ follows. Moreover, it suffices to prove the
inequality for vectors in $l_1^k$, i.e. for real numbers. This is because if $v_1 = (v_1, \ldots, v_k) \in l_1^k$
($1 \leq i \leq n$), then $A(v_1, \ldots, v_n) = \sum_{1 \leq i \leq n} A(v_i, \ldots, v_n)$ and hence
the inequality for vectors in $l_1^k$ follows from a summation of the $k$ corresponding
inequalities for real numbers. Finally, we claim that it suffices to prove the inequality
for the case where each of the real numbers is either $0$ or $1$. To see this, let
$v_1, v_2, \ldots, v_n$ be $n$ reals. By renumbering, if necessary, we may assume that
$v_1 \geq v_2 \geq \cdots \geq v_n$. For $1 \leq i < n$, define $c_i : \{v_1, \ldots, v_n\} \to \{0, 1\}$ by $c_i(v_1) = \cdots = \cdots = c_i(v_i) = 1$ and $c_i(v_{i+1}) = \cdots = c_i(v_n) = 0$. Clearly, for every $1 \leq i, j \leq n$, $|v_i - v_j| = \sum_{k=1}^n (v_k - v_{k+1})|c_k(v_i) - c_k(v_j)|$. Therefore, by linearity $A(v_1, \ldots, v_n) = \sum_{k=1}^n (v_k - v_{k+1})A(c_k(v_1), \ldots, c_k(v_n))$. Since $v_k - v_{k+1} \geq 0$ for all $k$, the validity of all the
inequalities $A(c_i(v_1), \ldots, c_i(v_n)) \leq 0$ for the $0$, $1$ variables $c_i(v_1), \ldots, c_i(v_n)$ implies
the validity of the inequality $A(v_1, \ldots, v_n) \leq 0$.

In view of the above discussion, Theorem 1.1 can be reformulated. Let $k > 0$
and $u \geq 0$ be two integers and suppose $m = k + 2u + 1$. Let $W = W(v_0, \ldots, v_{m-1}, u) = (V, E)$ be the graph defined as follows:

$V = \{v_0, \ldots, v_{m-1}\}$ and $v_i, v_j \in E$ iff there is an $s$,

$-u \leq s \leq u$, such that $i - j = s \pmod{m}$. \hspace{1cm} (2.1)

I Let $U = \{u_0, \ldots, u_{k-1}\}$ be a set, $U \cap V = \emptyset$. In this notation, Theorem 1.1 is
equivalent to the following statement.
Proposition 2.1. For every function \( c : V \cup U \to \{0, 1\} \) the inequality
\[
\sum_{v_i, v_j \in E} |c(v_i) - c(v_j)| + \sum_{1 \leq i \leq m \atop 1 \leq j < k} |c(v_i) - c(v_j)| \geq \sum_{1 \leq i \leq m - 1} |c(u_i) - c(u_j)|
\]
holds.

3. A Combinatorial Lemma

For \( u \geq 0, m \geq 2u + 1 \) and for a cyclic permutation \( (v_0, v_1, \ldots, v_{m-1}) \) of a set of \( m \) vertices let \( W = W(v_0, \ldots, v_{m-1}, u) = (V, E) \) be the graph defined by (2.1). The following lemma shows that \( W \) has an impressive connectivity property. A similar lemma has been proved, independently, by D. Macchi [7].

Lemma 3.1. Let \( S \) and \( T \) be two disjoint subsets of vertices of \( W = W(v_0, \ldots, v_{m-1}, u) \) where \( |S| = u \) and \( |T| = 2u + 1 - |S| \). Then there are \( |S| \cdot |T| \) pairwise edge-disjoint paths \( \{P_{s,t}\}_{s \in S, t \in T} \) in \( W \), where \( P_{s,t} \) is a path connecting \( s \) and \( t \).

Proof. We apply induction on \( m \). If \( m = 2u + 1 \) then \( W \) is a complete graph and the assertion of the lemma is trivial. Suppose this assertion holds for \( m - 1 \) and let us prove it for \( m \) \((m > 2u + 1)\). Given two disjoint subsets \( S \) and \( T \) of vertices of \( W(v_0, \ldots, v_{m-1}, u) \) satisfying the hypothesis of the lemma, \( |S \cup T| = 2u + 1 \) and hence there is a vertex of the graph that does not belong to \( S \cup T \). By renaming, if necessary, we may assume that this vertex is \( v_0 \). Consider the graph \( W' = W(v_1, \ldots, v_{m-1}, u) \). By the induction hypothesis, in this graph there are \( |S| \cdot |T| \) edge-disjoint paths \( \{P'_{s,t}\}_{s \in S, t \in T} \) where \( P'_{s,t} \) is a path in \( W' \) connecting \( s \) and \( t \). The only edges of \( W' \) which are not edges of \( W \) are edges of the form \( v_i, v_{i+1} \) where \( 1 \leq i \leq u \) and the indices are reduced modulo \( m \). These edges form a matching in \( W' \). Replace each such an edge \( v_{i+1} - v_i \) which appears in a path \( P'_{s,t} \) by the two edges \( v_i, v_{i+1} \) to obtain a path \( P_{s,t} \) in \( W \). Since the edges \( v_i, v_{i+1} \) form a matching, the paths \( P_{s,t} \) obtained in this manner are still pairwise edge-disjoint. This completes the induction and proves the lemma. \( \square \)

Corollary 3.2. For \( u \geq 0 \) and \( m \geq 2u + 1 \) let \( c : V \to \{0, 1\} \) be an arbitrary function from the set of vertices of \( W = W(v_0, \ldots, v_{m-1}, u) = (V, E) \) to \( \{0, 1\} \). Put \( r = |\{v_i \in V : c(v_i) = 0\}| \). Then:
(i) if \( r \leq u \) then \( \sum_{v_i \in E} |c(v_i) - c(v_j)| \geq r \cdot (2u + 1 - r) \);
(ii) if \( u \leq r \leq m/2 \) then \( \sum_{v_i \in E} |c(v_i) - c(v_j)| \geq u(u + 1) \).

Both estimates are best possible for all possible values of \( u, m \) and \( r \).

Proof. (i) Put \( S = \{v_i \in V : c(v_i) = 0\} \) and let \( T \) be an arbitrary subset of cardinality \( 2u + 1 - |S| = 2u + 1 - r \) of \( V \setminus S \). By Lemma 3.1 there are \( |S| \cdot |T| = r(2u + 1 - r) \) edge-disjoint paths \( \{P_{s,t}\}_{s \in S, t \in T} \) in \( W \), where \( P_{s,t} \) connects \( s \) and \( t \). Since \( c(s) = 0 \) and \( c(t) = 1 \) there is at least one edge \( v_i, v_j \) on \( P_{s,t} \) such that \( |c(v_i) - c(v_j)| = 1 \). However, as all the paths are edge-disjoint altogether there are at least \( r(2u + 1 - r) \) such edges of \( W \) and hence
\[
\sum_{v_i, v_j \in E} |c(v_i) - c(v_j)| \geq r(2u + 1 - r).
\]

(ii) Let \( S \) be an arbitrary subset of cardinality \( u \) of the set \( \{v_i \in V : c(v_i) = 0\} \) and let \( T \) be an arbitrary subset of cardinality \( u + 1 \) of the set \( \{v_i \in V : c(v_i) = 1\} \). The \( u(u + 1) \)
edge-disjoint paths \( \{P_{s,t}\}_{s \in S, t \in T} \) guaranteed by Lemma 3.1 supply, as before, the estimate

\[
\sum_{v \in V} |c(v_i) - c(v_j)| \geq u(u + 1).
\]

To see that both estimates are sharp observe that for all admissible \( u, m \) and \( r \), the function \( c:\ V \to \{0, 1\} \) defined by \( c(v_0) = c(v_1) = \cdots = c(v_{r-1}) = 0 \) and \( c(v_r) = c(v_{r+1}) = \cdots = c(v_{m-1}) = 1 \) satisfies the statements in the corollary with equality. \( \square \)

**Remark 3.3.** Lemma 3.1 and Corollary 3.2 are, in fact, equivalent by Menger's Theorem, or by the max-flow min-cut Theorem (cf., e.g., [2]). For our purposes here, the formulation of Corollary 3.2 is more convenient, but that of Lemma 3.1 may be useful in some other applications.

4. **The Proof of the CW-Inequalities**

We can now prove Proposition 2.1, which implies Theorem 1.1, as shown in Section 2. Let \( V = \{v_0, \ldots, v_{m-1}\} \) and \( U = \{u_0, \ldots, u_{k-1}\} \) be disjoint sets, where \( m = k + 2u + 1, \ k > 0 \) and \( u \geq 0 \), let \( W = W(v_0, \ldots, v_{m-1}, u) = (V, E) \) be the graph defined in (2.1) and let \( c:\ V \cup U \to \{0, 1\} \) be an arbitrary function. We must prove inequality (2.2). Put \( x = |\{i: 0 \leq i < m, c(v_i) = 0\}| \) and \( y = |\{j: 0 \leq j < k, c(u_j) = 0\}| \). Clearly we may assume that \( x \leq m/2 \), since otherwise we can replace the coloring \( c \) by the coloring \( c' = 1 - c \).

Define \( d = \sum_{v_i \in V, u_j \in U} |c(v_i) - c(u_j)| \) and

\[
D = d + \sum_{0 \leq i < m \atop 0 \leq j < k} |c(v_i) - c(u_j)| - \sum_{0 \leq i < j < m} |c(v_i) - c(v_j)| - \sum_{0 \leq i < j < k} |c(u_i) - c(u_j)|.
\]

Thus, inequality (2.2) is just the statement \( D \geq 0 \). By the definition of \( x \) and \( y \), and since \( m = k + 2u + 1 \),

\[
D = d + x(k - y) + (m - x)y - x(m - x) - y(k - y)
= d + x(k - y) + (k + 2u + 1 - x)y - x(k + 2u + 1 - x) - y(k - y)
= d + (x - y)^2 - (2u + 1)(x - y) = d - (x - y)(2u + 1 - (x - y)).
\]

Consider two possible cases.

**Case 1:** \( x \leq u \). In this case \( z = x - y \leq x \leq u \). The function \( f(z) = z(2u + 1 - z) \) is increasing for \( z \leq u \) and hence \( (x - y)(2u + 1 - (x - y)) \leq x(2u + 1 - x) \). However, by Corollary 3.2 part (i), \( d \geq x(2u + 1 - x) \) and hence \( D \geq 0 \), completing the proof in this case.

**Case 2:** \( x \geq u \). Recall that \( x \leq m/2 \). Therefore, by Corollary 3.2, part (ii), \( d \geq u(u + 1) \). On the other hand, the maximum value of the quadratic function \( f(z) = z(2u + 1 - z) \) over all integers \( z \) is \( u(u + 1) \) (which is attained at \( z = u \) and \( z = u + 1 \)). Since \( x - y \) is an integer \( (x - y)(2u + 1 - (x - y)) \leq x(u + 1) \) and hence in this case \( D = d - (x - y)(2u + 1 - (x - y)) = u(u + 1) - u(u + 1) \geq 0 \), as needed. This completes the proof of Proposition 2.1 and hence that of Theorem 1.1. \( \square \)

**Acknowledgements**

I would like to thank M. Deza for helpful comments. This research was supported in part by the Fund for Basic Research Administered by the Israel Academy of Sciences.
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Received 18 April 1989 and accepted in revised form 10 June 1989

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